ON TWO FIFTH ORDER MOCK THETA FUNCTIONS

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Abstract. We consider the fifth order mock theta functions χ_0 and χ_1, defined by Ramanujan, and find identities for these functions, which relate them to indefinite theta functions. Similar identities have been found by Andrews for the other fifth order mock theta functions and the seventh order functions.

1. Introduction and statement of the result

The mock theta functions were discovered by Ramanujan and are the subject of the last letter he sent to Hardy (see [5, p. 127–131]), in 1920. In [1] we can find the mathematical content of this letter and an account of the mock theta functions.

In [2] Andrews gave identities which relate a lot of the mock theta functions from Ramanujan's letter to indefinite theta functions. For example for the fifth order function

$$f_0(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1+q)(1+q^2)\cdots(1+q^n)}$$

he found (slightly rewritten)

$$f_0(q) = \frac{1}{(q)_\infty}\Big(\sum_{n+j\geq 0, n-j\geq 0} - \sum_{n+j<0, n-j<0}\Big)(-1)^j q^{\frac{5}{2}n^2+\frac{1}{2}n-j^2},$$

with $(q)_\infty := \prod_{k=1}^{\infty}(1-q^k)$. He found similar identities for 7 other fifth order functions and for all 3 seventh order functions. The only fifth order functions that he didn't deal with are χ_0 and χ_1, which are defined by

$$\chi_0(q) := \sum_{n=0}^{\infty} \frac{q^n}{(q^{n+1})_n},$$
$$\chi_1(q) := \sum_{n=1}^{\infty} \frac{q^{n-1}}{(q^n)_n}.$$

Here we use the standard notation $(x)_n := \prod_{k=0}^{n-1}(1-xq^k)$.

Recently, an intrinsic characterization of the mock theta functions has been found: they can be seen as the holomorphic parts of certain non-holomorphic modular forms. For a description see [8], [4] and [6] (in chronological order). In [7] (see also [9]) this is shown for most of the fifth and all seventh order functions using the before mentioned identities by Andrews.

For completeness, we give similar identities for χ_0 and χ_1, which relate them to indefinite theta functions. In fact, we will prove

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Theorem 1. *We have*

$$\chi_0(q) = 2 - \frac{1}{(q)_\infty^2}\Big(\sum_{k,l,m\geq 0} + \sum_{k,l,m<0}\Big)(-1)^{k+l+m} q^{Q(k,l,m)+\frac{1}{2}(k+l+m)},$$
$$\chi_1(q) = \frac{1}{(q)_\infty^2}\Big(\sum_{k,l,m\geq 0} + \sum_{k,l,m<0}\Big)(-1)^{k+l+m} q^{Q(k,l,m)+\frac{3}{2}(k+l+m)},$$

with $Q(k,l,m) = \frac{1}{2}k^2 + \frac{1}{2}l^2 + \frac{1}{2}m^2 + 2kl + 2km + 2lm$*.*

Note that the quadratic form Q is of type $(1,2)$, while in the 11 identities that Andrews found for the fifth and seventh order mock theta functions, the quadratic form is always of type $(1,1)$. This means that we can't use the methods from [7] to find the nature of the modularity of χ_0 and χ_1 from these identities. An extension of the theory of indefinite theta functions is required for that.

In the next section we will state and prove two necessary lemmas. In Section 3 we will then prove Theorem 1.

2. Some lemmas

To simplify the notation we introduce

Definition 2. For $r, s \in \mathbf{Z}$ we define

$$\rho_{r,s} = \begin{cases} 1 & \text{if } r,s \geq 0, \\ -1 & \text{if } r,s < 0, \\ 0 & \text{otherwise,} \end{cases}$$

and for $n \in \mathbf{Z}$

$$\delta_n = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

The next lemma is basically a rewritten version of a result by Andrews, Lemma 12 in [2].

Lemma 3. *For* $n \geq 0$ *we have*

$$\frac{1}{(q^{n+1})_n} = 2\delta_n - \sum_{r,s\in\mathbf{Z},\ |r+s|\leq n} \rho_{r,s}\ \frac{(-1)^{r+s} q^{\frac{1}{2}r^2+2rs+\frac{1}{2}s^2-\frac{1}{2}r-\frac{1}{2}s}}{(q)_{n+r+s}(q)_{n-r-s}},$$

and for $n \geq 1$

$$\frac{1}{(q^n)_n} = -\sum_{r,s\in\mathbf{Z},\ |r+s+1|\leq n} \rho_{r,s}\ \frac{(-1)^{r+s} q^{\frac{1}{2}r^2+2rs+\frac{1}{2}s^2+\frac{5}{2}r+\frac{5}{2}s+2}}{(q)_{n+1+r+s}(q)_{n-1-r-s}}.$$

Proof of Lemma 3. Andrews in [2] defines for $n \geq 0$

$$\mathscr{B}_n(0) = \frac{1}{(q^{n+1})_n},$$
$$\mathscr{B}_n(1) = \begin{cases} \frac{1}{(q^n)_n} & \text{if } n > 0, \\ 0 & \text{if } n = 0, \end{cases}$$

and

$$\mathscr{A}_{2n}(0) = q^{3n^2+n} \sum_{|j|\le n} q^{-j^2} - q^{3n^2-n} \sum_{|j|<n} q^{-j^2},$$

$$\mathscr{A}_{2n+1}(0) = -2q^{3n^2+4n+1} \sum_{j=0}^{n} q^{-j^2-j} + 2q^{3n^2+2n} \sum_{j=0}^{n-1} q^{-j^2-j},$$

$$\mathscr{A}_{2n}(1) = -2q^{3n^2-2n}(1-q^{4n}) \sum_{j=0}^{n-1} q^{-j^2-j},$$

$$\mathscr{A}_{2n+1}(1) = q^{3n^2+n}(1-q^{4n+2}) \sum_{|j|\le n} q^{-j^2}.$$

According to Lemma 12 $(\mathscr{A}_n(0), \mathscr{B}_n(0))$ and $(\mathscr{A}_n(1), \mathscr{B}_n(1))$ are Bailey pairs for $a = 1$. This means

$$\mathscr{B}_n(i) = \sum_{k=0}^{n} \frac{\mathscr{A}_k(i)}{(q)_{n-k}(q)_{n+k}} \qquad \text{with } i = 0, 1. \tag{2.1}$$

We have for $n > 0$

$$\sum_{|j|\le n} q^{-j^2} = \sum_{j=-n}^{n} q^{-j^2} = q^{-n^2} \sum_{j=0}^{2n} q^{j(2n-j)} = q^{-n^2} \sum_{r,s\in\mathbf{Z},\ r+s=2n} \rho_{r,s}\ q^{rs},$$

$$\sum_{|j|<n} q^{-j^2} = \sum_{j=-n+1}^{n-1} q^{-j^2} = q^{-n^2} \sum_{j=-2n+1}^{-1} q^{j(-2n-j)} = -q^{-n^2} \sum_{r,s\in\mathbf{Z},\ r+s=-2n} \rho_{r,s}\ q^{rs},$$

and similarly

$$2\sum_{j=0}^{n} q^{-j^2-j} = \sum_{j=-n-1}^{n} q^{-j^2-j} = q^{-n^2-n} \sum_{j=0}^{2n+1} q^{j(2n+1-j)}$$
$$= q^{-n^2-n} \sum_{r,s\in\mathbf{Z},\ r+s=2n+1} \rho_{r,s}\ q^{rs},$$

$$2\sum_{j=0}^{n-1} q^{-j^2-j} = \sum_{j=-n}^{n-1} q^{-j^2-j} = q^{-n^2-n} \sum_{j=-2n}^{-1} q^{j(-2n-1-j)}$$
$$= -q^{-n^2-n} \sum_{r,s\in\mathbf{Z},\ r+s=-2n-1} \rho_{r,s}\ q^{rs}.$$

If we put this in the definition of $\mathscr{A}_n(0)$ we find

$$\mathscr{A}_n(0) = \begin{cases} a_n + a_{-n} & \text{if } n > 1, \\ a_0 & \text{if } n = 0, \end{cases}$$

with

$$a_n = (-1)^n q^{\frac{1}{2}n^2+\frac{1}{2}n} \sum_{r,s\in\mathbf{Z},\ r+s=n} \rho_{r,s}\ q^{rs}.$$

Using this in equation (2.1) with $i = 0$, we get

$$\begin{aligned}\frac{1}{(q^{n+1})_n} &= \sum_{k=0}^{n} \frac{\mathscr{A}_k(0)}{(q)_{n-k}(q)_{n+k}} = \sum_{k=-n}^{n} \frac{a_k}{(q)_{n-k}(q)_{n+k}} \\ &= \sum_{k=-n}^{n} \sum_{r,s\in\mathbf{Z},\ r+s=k} \rho_{r,s} \frac{(-1)^k q^{\frac{1}{2}k^2+\frac{1}{2}k+rs}}{(q)_{n-k}(q)_{n+k}} \\ &= \sum_{r,s\in\mathbf{Z},\ |r+s|\le n} \rho_{r,s} \frac{(-1)^{r+s} q^{\frac{1}{2}r^2+2rs+\frac{1}{2}s^2+\frac{1}{2}r+\frac{1}{2}s}}{(q)_{n-r-s}(q)_{n+r+s}} \\ &= \sum_{r,s\in\mathbf{Z},\ |r+s|\le n} \rho_{-r,-s} \frac{(-1)^{r+s} q^{\frac{1}{2}r^2+2rs+\frac{1}{2}s^2-\frac{1}{2}r-\frac{1}{2}s}}{(q)_{n-r-s}(q)_{n+r+s}},\end{aligned}$$

where we have replaced (r, s) by $(-r, -s)$ in the last step. We can easily check that $\rho_{-r,-s} = \delta_r + \delta_s - \rho_{r,s}$, which gives

$$\frac{1}{(q^{n+1})_n} = 2 \sum_{l\in\mathbf{Z},\ |l|\le n} \frac{(-1)^l q^{\frac{1}{2}l^2-\frac{1}{2}l}}{(q)_{n-l}(q)_{n+l}} - \sum_{r,s\in\mathbf{Z},\ |r+s|\le n} \rho_{r,s} \frac{(-1)^{r+s} q^{\frac{1}{2}r^2+2rs+\frac{1}{2}s^2-\frac{1}{2}r-\frac{1}{2}s}}{(q)_{n-r-s}(q)_{n+r+s}}.$$

If we proof that

$$\sum_{l\in\mathbf{Z},\ |l|\le n} \frac{(-1)^l q^{\frac{1}{2}l^2-\frac{1}{2}l}}{(q)_{n-l}(q)_{n+l}} = \delta_n,$$

then we get the desired identity for $\frac{1}{(q^{n+1})_n}$. This equation is equivalent to the statement that (α_n, δ_n) is a Bailey pair for $a = 1$, with

$$\alpha_n = \begin{cases} (-1)^n q^{\frac{1}{2}n^2-\frac{1}{2}n}(1+q^n) & \text{if } n \ge 1, \\ 1 & \text{if } n = 0, \end{cases}$$

which is proved in [3, p. 27–28].

The proof of the identity for $\frac{1}{(q^n)_n}$ is similar, so we omit some of the details. By rewriting the definition of $\mathscr{A}_n(1)$ we find

$$\mathscr{A}_n(1) = a_{n-1} + a_{-n-1}.$$

If we put this in equation (2.1) we find that if $n \ge 1$

$$\begin{aligned}\frac{1}{(q^n)_n} &= \sum_{k=0}^{n} \frac{\mathscr{A}_k(1)}{(q)_{n-k}(q)_{n+k}} = \sum_{k=-n}^{n} \frac{a_{k-1}}{(q)_{n-k}(q)_{n+k}} = \sum_{k=-n-1}^{n-1} \frac{a_k}{(q)_{n-1-k}(q)_{n+1+k}} \\ &= \sum_{r,s\in\mathbf{Z},\ |r+s+1|\le n} \rho_{r,s} \frac{(-1)^{r+s} q^{\frac{1}{2}r^2+2rs+\frac{1}{2}s^2+\frac{1}{2}r+\frac{1}{2}s}}{(q)_{n-1-r-s}(q)_{n+1+r+s}},\end{aligned}$$

where we have to note that $a_{-1} = 0$ (if $r + s = -1$ then $\rho_{r,s} = 0$). If we replace (r, s) by $(-r-1, -s-1)$ in the last sum and use $\rho_{-r-1,-s-1} = -\rho_{r,s}$, we get the desired result. □

Lemma 4. *For $m \in \mathbf{Z}$ we have*

$$\sum_{n\in\mathbf{Z},\ n\geq |m|} \frac{q^n}{(q)_{n-m}(q)_{n+m}} = \frac{1}{(q)_\infty^2}\sum_{n\in\mathbf{Z}} \rho_{n,m}\ (-1)^n q^{\frac12 n^2+\frac12 n+2nm+m}$$
$$= \frac{q^{-m}}{(q)_\infty^2} - \frac{1}{(q)_\infty^2}\sum_{n\in\mathbf{Z}} \rho_{n,m}\ (-1)^n q^{\frac12 n^2-\frac12 n+2nm-m}$$

Proof of Lemma 4. We have the partial fraction decomposition

$$\frac{(q)_\infty^2}{(x)_\infty(x^{-1}q)_\infty} = \sum_{n\in\mathbf{Z}} \frac{(-1)^n q^{\frac12 n^2+\frac12 n}}{1-xq^n}, \tag{2.2}$$

found in [3, p. 42]. If $|q|<|x|<1$, then

$$\frac{1}{1-xq^n} = \sum_{k\in\mathbf{Z}} \rho_{n,k}\ x^k q^{nk},$$

so the right hand side of equation (2.2) can be written as

$$\sum_{n,k\in\mathbf{Z}} \rho_{n,k}\ (-1)^n x^k q^{\frac12 n^2+\frac12 n+nk}.$$

Since

$$\frac{1}{(x)_\infty} = \sum_{k=0}^\infty \frac{x^k}{(q)_k}$$

for $|x|<1$ (see equation (2.9) in [3] with $a=0$ and $z=x$), we see that for $|q|<|x|<1$ we can write the left hand side of equation (2.2) as

$$(q)_\infty^2 \sum_{k=0}^\infty\sum_{l=0}^\infty \frac{x^{k-l}q^l}{(q)_k(q)_l} = (q)_\infty^2 \sum_{l=0}^\infty\sum_{k=-l}^\infty \frac{x^k q^l}{(q)_{k+l}(q)_l} = (q)_\infty^2 \sum_{k\in\mathbf{Z}}\sum_{l=\max(0,-k)}^\infty \frac{x^k q^l}{(q)_{k+l}(q)_l}.$$

If we compare the coefficient of x^k on both side of the equation (2.2) we see

$$(q)_\infty^2 \sum_{l=\max(0,-k)}^\infty \frac{q^l}{(q)_{k+l}(q)_l} = \sum_{n\in\mathbf{Z}} \rho_{n,k}\ (-1)^n q^{\frac12 n^2+\frac12 n+nk}.$$

If we take $k=2m$, replace l by $l-m$ in the sum on the left, multiply both sides by $q^m/(q)_\infty^2$ and note that $\rho_{n,2m}=\rho_{n,m}$, we get

$$\sum_{l\in\mathbf{Z},\ l\geq|m|} \frac{q^l}{(q)_{l-m}(q)_{l+m}} = \frac{1}{(q)_\infty^2}\sum_{n\in\mathbf{Z}} \rho_{n,m}\ (-1)^n q^{\frac12 n^2+\frac12 n+2nm+m} \tag{2.3}$$

This is the first part of the lemma.

If we replace (n,m) by $(-n,-m)$ in (2.3) and use $\rho_{-n,-m}=-\rho_{n,m}+\delta_n+\delta_m$ and $\sum_{n\in\mathbf{Z}}(-1)^n q^{\frac12 n^2-\frac12 n}=0$ we get the second part. □

3. Proof of Theorem 1

Proof of Theorem 1. Using Lemma 3 and the first part of Lemma 4 with $m = r+s$, we see

$$\begin{aligned}\chi_0(q) = \sum_{n=0}^{\infty} \frac{q^n}{(q^{n+1})_n} &= 2 - \sum_{n=0}^{\infty} \sum_{r,s\in\mathbf{Z},\ |r+s|\leq n} \rho_{r,s}\ \frac{(-1)^{r+s} q^{\frac{1}{2}r^2+2rs+\frac{1}{2}s^2-\frac{1}{2}r-\frac{1}{2}s+n}}{(q)_{n+r+s}(q)_{n-r-s}}\\
&= 2 - \sum_{r,s\in\mathbf{Z}} \rho_{r,s}\ (-1)^{r+s} q^{\frac{1}{2}r^2+2rs+\frac{1}{2}s^2-\frac{1}{2}r-\frac{1}{2}s} \sum_{n\in\mathbf{Z},\ n\geq|r+s|} \frac{q^n}{(q)_{n+r+s}(q)_{n-r-s}}\\
&= 2 - \frac{1}{(q)_\infty^2} \sum_{r,s,n\in\mathbf{Z}} \rho_{r,s}\ \rho_{n,r+s}\ (-1)^{r+s+n} q^{Q(r,s,n)+\frac{1}{2}(r+s+n)},\end{aligned}$$

If we use

$$\rho_{r,s}\ \rho_{n,r+s} = \begin{cases} 1 & \text{if } r,s,n\geq 0 \text{ or } r,s,n<0,\\ 0 & \text{otherwise,}\end{cases}$$

we get the first part of the theorem. The proof of the second part is similar

$$\begin{aligned}\chi_1(q) = \sum_{n=1}^{\infty} \frac{q^{n-1}}{(q^n)_n} &= -\sum_{n=1}^{\infty} \sum_{r,s\in\mathbf{Z},\ |r+s+1|\leq n} \rho_{r,s}\ \frac{(-1)^{r+s} q^{\frac{1}{2}r^2+2rs+\frac{1}{2}s^2+\frac{5}{2}r+\frac{5}{2}s+n+1}}{(q)_{n+1+r+s}(q)_{n-1-r-s}}\\
&= -\sum_{r,s\in\mathbf{Z}} \rho_{r,s}\ (-1)^{r+s} q^{\frac{1}{2}r^2+2rs+\frac{1}{2}s^2+\frac{5}{2}r+\frac{5}{2}s+1} \sum_{n\in\mathbf{Z},\ n\geq|r+s+1|} \frac{q^n}{(q)_{n+1+r+s}(q)_{n-1-r-s}},\end{aligned}$$

where we have to note that $\rho_{r,s} = 0$ if $r+s+1 = 0$. If we use the second part of Lemma 4 with $m = r+s+1$ we get

$$\begin{aligned}\chi_1(q) = \frac{1}{(q)_\infty^2} \Bigg\{ &-\sum_{r,s\in\mathbf{Z}} \rho_{r,s}\ (-1)^{r+s} q^{\frac{1}{2}r^2+2rs+\frac{1}{2}s^2+\frac{3}{2}r+\frac{3}{2}s}\\
&+ \sum_{r,s,n\in\mathbf{Z}} \rho_{r,s}\ \rho_{n,r+s+1}\ (-1)^{r+s+n} q^{Q(r,s,n)+\frac{3}{2}(r+s+n)} \Bigg\}.\end{aligned}$$

The second part of the theorem follows if we use

$$\rho_{r,s}\ \rho_{n,r+s+1} = \begin{cases} 1 & \text{if } r,s,n\geq 0 \text{ or } r,s,n<0,\\ 0 & \text{otherwise,}\end{cases}$$

and

$$\sum_{r,s\in\mathbf{Z}} \rho_{r,s}\ (-1)^{r+s} q^{\frac{1}{2}r^2+2rs+\frac{1}{2}s^2+\frac{3}{2}r+\frac{3}{2}s} = 0.$$

The last equation follows from the fact that $(-1)^{r+s} q^{\frac{1}{2}r^2+2rs+\frac{1}{2}s^2+\frac{3}{2}r+\frac{3}{2}s}$ is invariant if we replace (r,s) by $(-r-1,-s-1)$, while $\rho_{-r-1,-s-1} = -\rho_{r,s}$. □

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