First Arts Modular Degree

Mathematical Studies 2004–2005

Combinatorics and Number Theory Problem Sheet 4

1. In problem sheet 2, you were asked to prove that

$$m\binom{n}{m} = n\binom{n-1}{m-1}$$

when m and n are integers with $0 \leq m \leq n$. Use this result to prove that under these circumstances, if m and n are *relatively prime*, then n divides $\binom{n}{m}$.

2. Find the smallest positive integer x that satisfies $31x \equiv 3 \bmod 41$.

3. Find the smallest positive integer x that satisfies $317x \equiv 3 \bmod 409$.

4. Let b and c be relatively prime integers. Prove that for any positive integers m and n, b^m and c^n are also relatively prime.

5. Prove that the three binomial coefficients $\binom{n}{r-1}$, $\binom{n}{r}$ and $\binom{n+1}{r}$ cannot all be odd integers.

6. Find the smallest positive integer x that satisfies $2^{20} \equiv x \bmod 47$.

7. Find the order of 3 modulo 31.

8. Let p be a prime and let a be an integer not divisible by p satisfying $a \not\equiv 1 \bmod p$. Use Fermat's Little Theorem to show that

$$1 + a + a^2 + \cdots + a^{p-2} \equiv 0 \bmod p.$$

9. Let p be an odd prime. Show that the order of 4 modulo p is a divisor of $(p-1)/2$. Hint: $4 = 2^2$.

10. Let p be an odd prime and let n be an integer not divisible by p. Suppose that n has order 2 modulo p. Prove that p divides $n+1$.