First Arts Modular Degree

Mathematical Studies 2004–2005

Combinatorics and Number Theory Problem Sheet 1

1. Prove by induction on n that for all integers $n \geq 1$,

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

2. Prove by induction on n that for all integers $n \geq 1$,

$$1 + x + x^2 + \cdots + x^n = \frac{x^{n+1} - 1}{x - 1},$$

provided $x \neq 1$.

3. The sequence $a_1, a_2, \ldots, a_n$ is defined by

$$a_n = \frac{(2n)!}{2^n n!}.$$

Show that

$$\frac{a_{r+1}}{a_r} = 2r + 1.$$

Hence show by induction on n that a_n is an odd positive integer.

4. Prove by induction on n that

$$(1 - \frac{1}{4}) \times (1 - \frac{1}{9}) \times \cdots \times (1 - \frac{1}{n^2}) = \frac{n+1}{2n}$$

for all integers $n \geq 2$.

5. If $a_1, a_2, \ldots, a_n$ is a sequence of numbers, with $a_1 = 1$, and if the relation $a_{n+1} = 3a_n - 1$ holds for all $n \geq 1$, then $a_n = \frac{1}{2}(3^{n-1} + 1)$.

6. Prove by induction on n that 9 divides $2^{2n} - 3n - 1$ for all integers $n \geq 1$.

7. How many different integers can be formed by permuting the 7 digits 3, 4, 5, 6, 7, 8, 9? How many of the integers thus formed lie between 5,000,000 and 8,700,000?

8. In how many permutations of the letters a, b, c, d, e, f and g do the letters a, b, c, d occur in a single group in this order? In how many of these permutations do the letters $abcd$ occur in a single group but in any order within this group?

9. How many different integers can be formed by permuting the digits 1234. What is the sum of all the integers thus formed (try to work this out theoretically rather than by adding together several numbers).