A CORRESPONDENCE FOR CONJUGACY CLASSES IN CERTAIN EXTENSIONS OF ORDER 2 OF FINITE GROUPS OF LIE TYPE

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Abstract. Let G be a connected linear algebraic group over an algebraically closed field of prime characteristic. Let $F : G \to G$ denote a standard Frobenius mapping of G. Let τ be an involutory automorphism of G which commutes with F and let F^* denote the corresponding twisted Frobenius mapping. Let H denote either G^F or G^{F^*} and let $H\langle t\rangle$ denote the extension of H by a cyclic group of order 2, generated by t, that induces the automorphism τ on H. We show that there is a one–to–one correspondence between the conjugacy classes of $G^F\langle t\rangle \setminus G^F$ and those of $G^{F^*}\langle t\rangle \setminus G^{F^*}$. If xt and yt are elements in corresponding conjugacy classes of the two groups, then xt and yt have the same order and the centralizer of xt in G^F is isomorphic to the centralizer of yt in G^{F^*}. We also discuss numerical evidence for the existence of a related correspondence of characters of the two extension groups.

1. Introduction

Let G be a connected linear algebraic group over an algebraically closed field $\mathbb{K}$ of prime characteristic p. Let $F : G \to G$ denote a (standard) Frobenius mapping of G and let G^F denote the finite subgroup of fixed points of F in G. Suppose that G has an involutory automorphism τ which commutes with F in its action on G. We will let g^τ denote the image of $g \in G$ under τ. We may then form a twisted Frobenius mapping $F^* : G \to G$ by setting

$$F^*(g) = F(g^\tau)$$

for all g in G. Since $(F^*)^2 = F^2$, it follows that the subgroup G^{F^*} is contained in G^{F^2}.

Let x be any element of G^F. The Lang–Steinberg theorem, [2], Theorem 10.1, shows that there exists an element z in G with $x = z^{-1}F^*(z)$. Since $F(x) = x$, and F commutes with F^*, it follows that

$$F(z)^{-1}F^*(F(z)) = z^{-1}F^*(z)$$

and thus

$$zF(z)^{-1} = F^*(z)F^*(F(z)^{-1}) = F^*(zF(z)^{-1}).$$

This shows that the element $y = zF(z)^{-1}$ is in G^{F^*}. A different choice of z used to represent x according to the Lang–Steinberg theorem leads to another element in G^{F^*} which is not obviously related to the element y just obtained. The purpose of this paper is to show that the idea of associating x in G^F with y in G^{F^*} can be

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used to define a correspondence of certain conjugacy classes in extension groups of G^F and G^{F^*}, respectively, as we will now explain.

Let $G\langle t\rangle$ denote the semi–direct product of G by a cyclic group of order 2, generated by an involution t satisfying $txt = x^\tau$ for all $x \in G$. Since F commutes with τ, it follows that both G^F and G^{F^*} admit τ as an automorphism and are thus normalized by t in $G\langle t\rangle$. Let $G^F\langle t\rangle$ and $G^{F^*}\langle t\rangle$ denote the corresponding subgroups of $G\langle t\rangle$ generated by t and G^F, G^{F^*} respectively.

Such groups are relevant to the study of finite groups of Lie type. To describe a natural example, we first note that the simple algebraic groups that admit an involutory automorphism of the type described are $A_m(\mathbb{K})$, $D_m(\mathbb{K})$ and $E_6(\mathbb{K})$ (the involution being induced by a symmetry of the Dynkin diagram). We now take G to be a 2–fold cover of the simple adjoint group $D_m(\mathbb{K})$ when p is odd and $D_m(\mathbb{K})$ itself when $p = 2$. We may then identify G with the special orthogonal group $\mathrm{SO}_{2m}(\mathbb{K})$ and take τ to be the involutory automorphism induced by an involutory diagram symmetry (this is the same as the automorphism induced by conjugation by an orthogonal reflection). The group $G\langle t\rangle$ is the full orthogonal group $\mathrm{O}_{2m}(\mathbb{K})$ in this case. Moreover, if q is a power of p and F is induced by the Frobenius mapping $k \to k^q$ of $\mathbb{K}$, G^F is the split special orthogonal group $\mathrm{SO}^+_{2m}(\mathbb{F}_q)$ and G^{F^*} is the non–split special orthogonal group $\mathrm{SO}^-_{2m}(\mathbb{F}_q)$, while $G^F\langle t\rangle$ and $G^{F^*}\langle t\rangle$ are the split and non–split full orthogonal groups $\mathrm{O}^+_{2m}(\mathbb{F}_q)$ and $\mathrm{O}^-_{2m}(\mathbb{F}_q)$, respectively. Another case that arises in linear algebra occurs when we take G to be $\mathrm{GL}_{m+1}(\mathbb{K})$, a group belonging to the $A_m(\mathbb{K}))$ family. The study of the conjugacy classes in $G\langle t\rangle \setminus G$ is equivalent to the classification of non–degenerate bilinear forms.

2. The correspondence of conjugacy classes

Let H denote any of the groups G, G^F or G^{F^*}. In order to define our correspondence of conjugacy classes, we prove some elementary results relating to conjugacy of elements in $H\langle t\rangle$. As a preliminary observation, we note that elements at and bt in $H\langle t\rangle$ are conjugate if and only there exists an element c in H with $c^{-1}ac^\tau = b$.

Lemma 1. *Let x be an element of G^F and write $x = z^{-1}F^*(z)$ for some $z \in G$. Suppose that xt is conjugate in $G^F\langle t\rangle$ to wt, where $w = v^{-1}F^*(v)$ for some $v \in G$. Then*

$$zF(z)^{-1}t \quad \text{and} \quad vF(v)^{-1}t$$

are conjugate in $G^{F^*}\langle t\rangle$. Thus, if we also have $x = z_1^{-1}F^*(z_1)$ for some other element z_1 in G, the elements

$$zF(z)^{-1}t \quad \text{and} \quad z_1F(z_1)^{-1}t$$

are conjugate in $G^{F^*}\langle t\rangle$.

Proof. As we noted above, there exists $g \in G^F$ with $g^{-1}xg^\tau = w$. Moreover, as τ is involutory and $g \in G^F$, we have $F^*(g) = g^\tau$. It follows that

$$g^{-1}z^{-1}F^*(z)g^\tau = (zg)^{-1}F^*(zg) = w = v^{-1}F^*(v).$$

We deduce that $zgv^{-1} \in G^{F^*}$. We set $zgv^{-1} = u$, and then obtain $v = u^{-1}zg$, where $u \in G^{F^*}$. Since $g = F(g)$ and $u^\tau = F(u)$, it follows that

$$vF(v)^{-1} = u^{-1}zgF(g)^{-1}F(z)^{-1}F(u) = u^{-1}zF(z)^{-1}u^\tau,$$

and this equality proves that $vF(v)^{-1}t$ and $zF(z)^{-1}t$ are conjugate in $G^{F^*}\langle t\rangle$, as required. The second part is clear by taking $w = x$ and $z_1 = v$. □

Given an element x of H, we let $[xt]$ denote the conjugacy class of xt in $H\langle t\rangle$. We trust that context will make it clear which subgroup H is implied in the event of possible ambiguity. We now define a function ϕ mapping a conjugacy class $[xt]$ in $G^F\langle t\rangle$ to a conjugacy class $[yt]$ in $G^{F^*}\langle t\rangle$ in the following way. Write x in the form $z^{-1}F^*(z)$ and let $y = zF(z)^{-1} \in G^{F^*}$. Then we set

$$\phi[xt] = [yt].$$

Lemma 1 shows that the definition of ϕ does not depend on the choice of z to represent x or the choice of x to represent the conjugacy class $[xt]$. We note that ϕ is only defined at the level of conjugacy classes and is not well defined on elements.

Lemma 2. *The function ϕ defines a one–to–one correspondence between the conjugacy classes of the form $[xt]$ in $G^F\langle t\rangle$ and the conjugacy classes of the form $[yt]$ in $G^{F^*}\langle t\rangle$.*

Proof. We first show that ϕ is injective. Suppose then that $\phi[xt] = \phi[x_1t]$. Write

$$x = z^{-1}F^*(z), \quad x_1 = z_1^{-1}F^*(z_1)$$

where z and z_1 are appropriate elements of G. Then there exists some $u \in G^{F^*}$ with

$$u^{-1}zF(z)^{-1}u^\tau = z_1F(z_1)^{-1}.$$

Since $u^\tau = F(u)$, this implies that $z^{-1}uz_1 \in G^F$. We set $g = z^{-1}uz_1$. Then, since $g^\tau = F^*(g)$, we have

$$g^{-1}z^{-1}F^*(z)g^\tau = (zg)^{-1}F^*(zg) = (uz_1)^{-1}F^*(uz_1) = z_1^{-1}F^*(z_1)$$

and this implies that g conjugates xt into x_1t. Thus $[xt] = [x_1t]$ and it follows that ϕ is injective.

Next, we show that ϕ is surjective. Let u be any element of G^{F^*}. The Lang–Steinberg theorem implies that $u = zF(z)^{-1}$ for some $z \in G$. Since $F^*(u) = u$, we readily check that $z^{-1}F^*(z)$ is in G^F. Thus if we put $x = z^{-1}F^*(z)$, we have $\phi[xt] = [ut]$, which implies that ϕ is surjective, as required. □

We now show that if the order of an element in $[xt]$ is r, the order of an element in $\phi[xt]$ is also r. Thus ϕ preserves the order of the elements in a conjugacy class.

Lemma 3. *Given $x \in G^F$, let $[yt] = \phi[xt]$. Then xt and yt have the same (finite) multiplicative order in $G\langle t\rangle$.*

Proof. As xt and yt have finite even order, it suffices to show that $(xt)^2$ and $(yt)^2$ have the same order. As usual, we write $x = z^{-1}F^*(z)$ and set $y = zF(z)^{-1}$. Then we have

$$(xt)^2 = xx^\tau = z^{-1}F^*(z)(z^\tau)^{-1}F(z)$$
$$(yt)^2 = yy^\tau = zF(z)^{-1}z^\tau F^*(z)^{-1}.$$

It follows that

$$z^{-1}(yt)^2z = (xt)^{-2},$$

and this implies that $(xt)^{-2}$ and $(yt)^2$ have the same order, since they are conjugate in G. Thus $(xt)^2$ and $(yt)^2$ also have the same order, as required. □

The mapping ϕ has an additional useful property, since the centralizer of xt in G^F is conjugate in G to the centralizer of yt in G^{F^*}, where $yt \in \phi[xt]$, as we now show.

Lemma 4. *Let x be an element of G^F, with $x = z^{-1}F^*(z)$ for some $z \in G$. Let $y = zF(z)^{-1}$. Then the centralizer of xt in G^F is $z^{-1}G^{F^*}z \cap G^F$ and the centralizer of yt in G^{F^*} is $zG^F z^{-1} \cap G^{F^*}$. Thus, since these are conjugate subgroups, the centralizer of xt in G^F is isomorphic to the centralizer of yt in G^{F^*}.*

Proof. An element $u \in G^F$ commutes with xt if and only if $u^{-1}xtu = xt$. This occurs if and only if $u^{-1}xu^\tau = x$. Since $u^\tau = F*(u)$, u commutes with xt if and only if zuz^{-1} is in G^{F^*}. Thus the centralizer of xt in G^F is $z^{-1}G^{F^*}z \cap G^F$, and a similar argument shows that the centralizer of yt in G^{F^*} is $zG^F z^{-1} \cap G^{F^*}$. Since

$$z(z^{-1}G^{F^*}z \cap G^F)z^{-1} = zG^F z^{-1} \cap G^{F^*},$$

the two centralizers are conjugate in G^F and hence isomorphic. □

We sum up our findings related to ϕ in the following theorem, which amalgamates the various lemmas we have proved.

Theorem 1. *Let G be a connected linear algebraic group over an algebraically closed field of prime characteristic. Let $F : G \to G$ denote a standard Frobenius mapping of G. Suppose that G has an involutory automorphism τ which commutes with F and let F^* denote the corresponding twisted Frobenius mapping. Let H denote either G^F or G^{F^*} and let $H\langle t\rangle$ denote the extension of H by a cyclic of order 2 that induces the automorphism τ on H. Given $h \in H$, let $[ht]$ denote the conjugacy class of ht in $H\langle t\rangle$. Given $x \in G^F$, write $x = z^{-1}F^*(z)$ for some $z \in G$ and set $y = zF(z)^{-1} \in G^{F^*}$.*

Then the function ϕ defined by $\phi[xt] = [yt]$ is a one–to–one correspondence between the conjugacy classes of $G^F\langle t\rangle \backslash G^F$ and those of $G^{F^*}\langle t\rangle \backslash G^{F^*}$. The elements xt and yt have the same order and the centralizer of xt in G^F is isomorphic to the centralizer of yt in G^{F^*}.

We mention the following special case of our theorem, which does not appear to be obvious from the standpoint of linear algebra.

Corollary 1. *There is a one–to–one correspondence between the conjugacy classes of $\mathrm{O}^+_{2m}(\mathbb{F}_q) \setminus \mathrm{SO}^+_{2m}(\mathbb{F}_q)$ and those of $\mathrm{O}^-_{2m}(\mathbb{F}_q) \setminus \mathrm{SO}^-_{2m}(\mathbb{F}_q)$ which preserves the order of the elements in corresponding conjugacy classes. Under this correspondence, the centralizer in $\mathrm{SO}^+_{2m}(\mathbb{F}_q)$ of an element in a conjugacy class in $\mathrm{O}^+_{2m}(\mathbb{F}_q) \setminus \mathrm{SO}^+_{2m}(\mathbb{F}_q)$ is isomorphic to the centralizer in $\mathrm{SO}^-_{2m}(\mathbb{F}_q)$ of an element in the corresponding conjugacy class of $\mathrm{O}^-_{2m}(\mathbb{F}_q) \setminus \mathrm{SO}^-_{2m}(\mathbb{F}_q)$.*

If $[xt]$ is a real class in $G^F\langle t\rangle$, it is not necessarily the case that $\phi[xt]$ is also a real class. Examples of this phenomenon occur when G is the general linear group $\mathrm{GL}_n(\mathbb{K})$ of degree $n \geq 2$ over $\mathbb{K}$. Taking τ to be the transpose-inverse automorphism and F to be the Frobenius mapping already defined, G^F is the general linear group $\mathrm{GL}_n(\mathbb{F}_q)$ and G^{F^*} is the general unitary group $\mathrm{U}_n(\mathbb{F}_q)$. It is known that all classes of $G^F\langle t\rangle$ are real, whereas, certainly for odd q, there are classes of the form $[yt]$ in $G^{F^*}\langle t\rangle$ that are not real. This point arises in the discussion of characters in the final section.

3. Possible character correspondences

Let H now denote either of the groups G^F or G^{F^*} and let χ be an irreducible complex character of H. Since H is a normal subgroup of index 2 in $H\langle t\rangle$, we may consider the action of t on χ. If χ is not fixed by t, χ induced to $H\langle t\rangle$ is an irreducible character that vanishes on all elements of $H\langle t\rangle \setminus H$. We will therefore assume that χ is fixed by t. It then extends to an irreducible character χ', say, of $H\langle t\rangle$. There is a second extension of χ obtained by multiplying χ' by the linear character of order 2 of $H\langle t\rangle$ whose kernel is H. When H is a quasi–simple group included in the ATLAS of Finite Groups, [1], and the character table of $H\langle t\rangle$ is presented, only one of the extensions of χ is displayed. Assuming that t is an involution of $H\langle t\rangle \setminus H$ whose centralizer has largest order, the convention adopted is that, provided $\chi'(t) \neq 0$, there is a unique extension of χ that takes a positive integer value on t, and it is this extension that is displayed. In the great majority of cases listed in the ATLAS, $\chi'(t) \neq 0$ for all fixed χ but there are exceptions to this pattern.

The theory that we have outlined shows that the number of irreducible characters of H fixed by t is the same for either isomorphism type of H and this number is the number of conjugacy classes in $H\langle t\rangle \setminus H$. While the correspondence ϕ does not preserve all aspects of conjugacy classes relating to character theory, it is interesting to see if there is any discernible correspondence in the values of the extended characters on the elements of $H\langle t\rangle \setminus H$ in the two cases. We remark that we cannot deduce the exact form of the correspondence ϕ using only the data displayed in the ATLAS, since there may exist several conjugacy classes of elements having the same order and centralizers of the same size. Thus any correspondence of character values we discover by inspection is not necessarily one associated to ϕ, but the existence of ϕ certainly provides the motivation for looking for any possible correspondence.

We find a complete correspondence for the groups $\mathrm{GL}_4(\mathbb{F}_2)$ and $\mathrm{U}_4(\mathbb{F}_2)$, where there are 10 extended characters, and the two parts of the character tables relating to the extended groups are identical up to permutations of the 10 rows and 10 columns. For other groups, there is no complete correspondence but nonetheless remarkable duplications may be observed. In the case of the groups $\mathrm{SO}_8^+(\mathbb{F}_2)$ and $\mathrm{SO}_8^-(\mathbb{F}_2)$ there are 27 extended characters, and 23 of these display complete numerical correspondence in the two extension groups. For the groups $\mathrm{SO}_{10}^+(\mathbb{F}_2)$ and $\mathrm{SO}_{10}^-(\mathbb{F}_2)$ there are 71 extended characters, and 63 of these display complete numerical correspondence in the two extension groups.

What appears to be most fascinating near–correspondence occurs for the groups $\mathrm{GL}_n(\mathbb{F}_q)$ and $\mathrm{U}_n(\mathbb{F}_q)$. We remarked above on a complete correspondence of characters when $n = 4$ and $q = 2$. When $n = 5$ and $q = 2$, there are 13 extended characters, 10 of which display complete correspondence. In the case of $\mathrm{U}_5(\mathbb{F}_2)$, the three characters that do not admit a correspondence of values are precisely those that have Frobenius–Schur indicator equal to -1. These characters extend to characters that are not real–valued, whereas, for $\mathrm{GL}_5(\mathbb{F}_2)$, all the extensions are real–valued, and so there can clearly be no correspondence. Furthermore, these three extensions that are not real–valued are precisely those that vanish on t. Even in the case of non–correspondence, there are still similarities, as there are precisely three extended characters of $\mathrm{GL}_5(\mathbb{F}_2)$ that vanish on t, and their values on elements of the form xt are largely 0, as is the case for the three exceptional characters of $\mathrm{U}_5(\mathbb{F}_2)$.

While the numerical evidence is limited, this pattern of behaviour suggests the following conjectures. We first note that the irreducible characters fixed by t are precisely the real–valued characters for each type of group. Suppose that χ is an irreducible character of $\mathrm{U}_n(\mathbb{F}_q)$ that has Frobenius–Schur indicator equal to 1. Then χ should have a real–valued extension and the values of the extended character should correspond to the values of an extension of a real–valued character of $\mathrm{GL}_n(\mathbb{F}_q)$. Suppose, on the other hand, that χ is an irreducible character of $\mathrm{U}_n(\mathbb{F}_q)$ that has Frobenius–Schur indicator equal to -1. Then the extensions of χ are not real–valued and hence there is no correspondence with extended characters of $\mathrm{GL}_n(\mathbb{F}_q)$. We remark that we can prove this assertion when q is odd. More speculatively, the extensions that vanish on t are precisely those that are not real–valued, and there are equal numbers of extensions of characters of $\mathrm{GL}_n(\mathbb{F}_q)$ that vanish on t. Finally, the number of irreducible characters of $\mathrm{U}_n(\mathbb{F}_q)$ that have Frobenius–Schur indicator equal to 1 should equal the number of real classes of the form $[yt]$ in the extension group, where y runs over $\mathrm{U}_n(\mathbb{F}_q)$. Here again, we remark that we have proved this last conjecture when $n = 3$ and $n = 4$.

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