Resonant Rossby Wave Triads and the Swinging Spring

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Department of Meteorology, University of Reading,

2nd February, 2004.

In a Nutshell

A mathematical equivalence with a
simple mechanical system
sheds light on the dynamics of
resonant Rossby waves
in the atmosphere.

The Swinging Spring

Two distinct oscillatory modes with two distinct restoring forces:

- ***Elastic or springy modes***
- ***Pendular or swingy modes***

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Take a peek at the Java Applet

In a paper in 1981, *Breitenberger and Mueller* **made the following comment:**

**This simple system looks like a toy at best,
but its behaviour is astonishingly complex,
with many facets of more than academic lustre.**

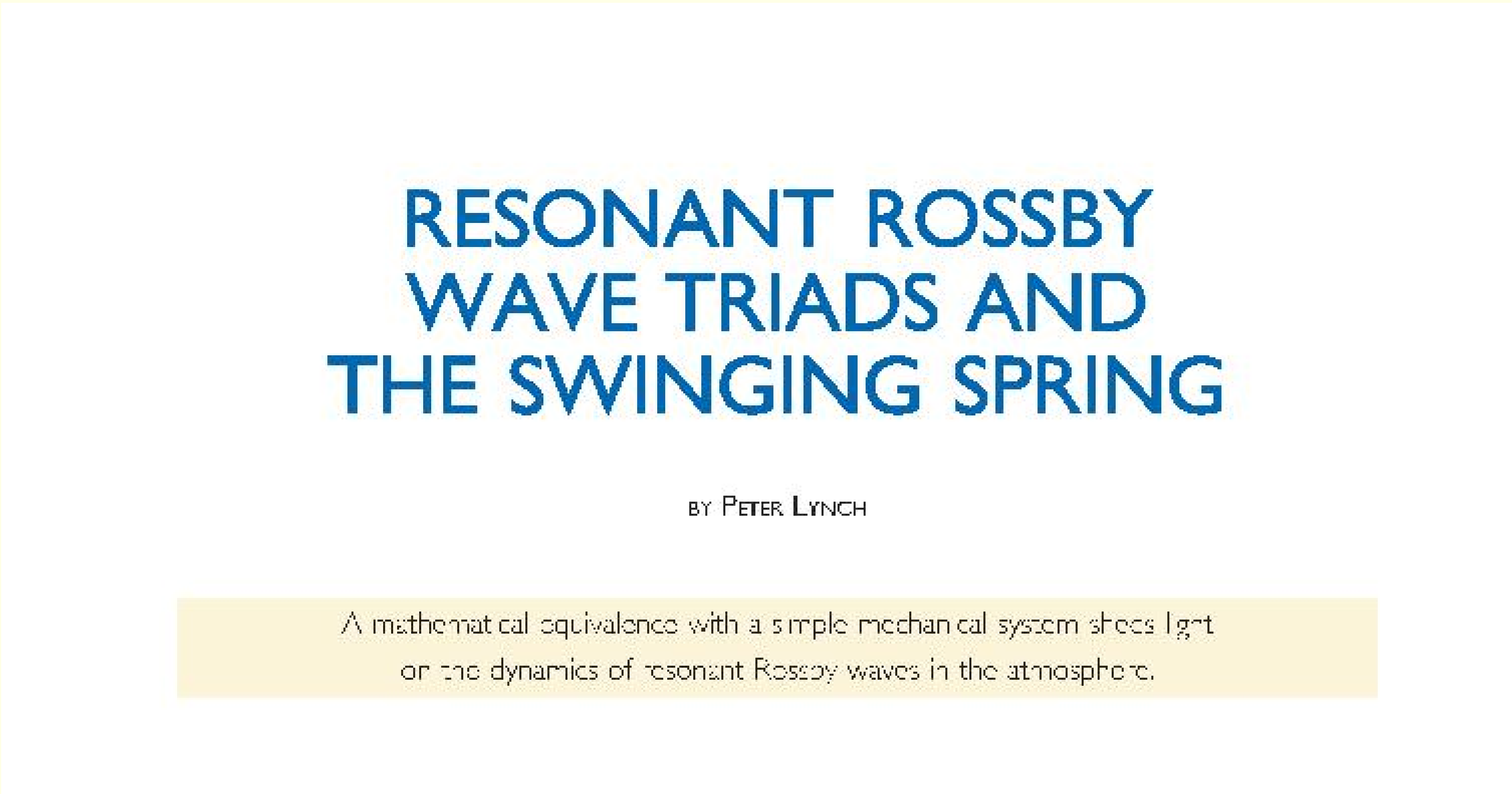
I hope to convince you of the validity of this remark.

Information and Resources

http://www.maths.tcd.ie/~plynch

- ***Papers* on Spring and Triads:**
  Click on 'Publications'.
- ***Java Applet* on Swinging Spring:**
  Click on 'The Swinging Spring'.
- ***Slides* of this talk (Sapporo Version):**
  Click on 'Talks'.
- ***Matlab Code* for Spring and Triads:**
  Click on 'Rossby Wave Triads'.

Bulletin of the AMS, May, 2003



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Printed version for dilettantes and *smatterers in mathematics.*

Full mathematical details are in the Electronic Supplement.

Motivation

The linear normal modes of the atmosphere fall into two categories, the low frequency Rossby waves and the high frequency gravity waves.

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The linear normal modes of the atmosphere fall into two categories, the low frequency Rossby waves and the high frequency gravity waves.

The Swinging Spring is a simple mechanical system having low frequency and high frequency oscillations.

The elastic oscillations of the spring are analogues of the high frequency gravity waves in the atmosphere.

The low frequency pendular motions correspond to the rotational or Rossby-Haurwitz waves.

Analogy and Equivalence

Analogies are interesting.
Equiavlences are useful.

Value of the Analogy

1. **Wave-like motions**
2. **Initialization**
3. **Filtered Equations**
4. **Slow manifold theory**

Power of the Equivalence

1. **Triad Resonance**
2. **Rossby Wave Precession**
3. **Predictability**

Original Reference

First comprehensive analysis of the elastic pendulum:

Oscillations of an Elastic Pendulum as an Example of the Oscillations of Two Parametrically Coupled Linear Systems.

Vitt and Gorelik (1933).

Inspired by analogy with Fermi resonance of CO_2.

Translation of this paper available as

Historical Note #3 (1999)

Published by Met Éireann.

Available at URL:

http://www.maths.tcd.ie/~plynch

The Exact Equations of Motion

In Cartesian coordinates the Lagrangian is

$$L = T - V = \underbrace{\frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{Z}^2\right)}_{K.E} - \underbrace{\frac{1}{2}k(r - \ell_0)^2}_{E.P.E} - \underbrace{mgZ}_{G.P.E}$$

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The equations of motion are (with $\omega_Z^2 \equiv k/m$):

$$\begin{aligned}
\ddot{x} &= -\omega_Z^2\left(\frac{r-\ell_0}{r}\right)x \\
\ddot{y} &= -\omega_Z^2\left(\frac{r-\ell_0}{r}\right)y \\
\ddot{Z} &= -\omega_Z^2\left(\frac{r-\ell_0}{r}\right)Z - g
\end{aligned}$$

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There are two constants, the energy and the angular momentum:

$$E = T + V \qquad h = x\dot{y} - y\dot{x}\,.$$

The system is <u>not integrable</u> (two invariants, three DOF).

The Hamiltonian

At equilibrium, the elastic restoring force is balanced by the weight

$$k(\ell - \ell_0) = mg \qquad \textbf{or} \qquad \ell = \ell_0 \left(1 + \frac{mg}{k\ell_0} \right) .$$

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For the time being, we consider motion in a plane.

The total energy is sum of kinetic, elastic potential and gravitational potential energy.

$$\begin{aligned} \textbf{Radial Momentum}: \quad p_r &= m\dot{r} \\ \textbf{Angular Momentum}: \quad p_\theta &= mr^2\dot{\theta} \end{aligned}$$

$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} \right) + \tfrac{1}{2}k(r - \ell_0)^2 - mgr\cos\theta$$

The Canonical Equations

The canonical equations of motion are:

$$\begin{aligned}
\dot{\theta} &= p_\theta / mr^2 \\
\dot{p}_\theta &= -mgr \sin\theta \\
\dot{r} &= p_r / m \\
\dot{p}_r &= p_\theta^2 / mr^3 - k(r - \ell_0) + mg\cos\theta
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These equations may also be written symbolically in vector form

$$\dot{\mathbf{X}} + \mathbf{L}\mathbf{X} + \mathbf{N}(\mathbf{X}) = 0$$

The state vector $\mathbf{X}$ specifies a point in 4-dimensional phase space.

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If Hamiltonian methods are unfamiliar, the equations may be derived using standard Newtonian dynamics.

Linear Normal Modes

Suppose that amplitude of motion is small:

$$\frac{d}{dt}\begin{pmatrix}\theta\\ p_\theta\\ r'\\ p_r\end{pmatrix} = \begin{pmatrix} 0 & 1/m\ell^2 & 0 & 0\\ -mg\ell & 0 & 0 & 0\\ 0 & 0 & 0 & 1/m\\ 0 & 0 & -k & 0\end{pmatrix}\begin{pmatrix}\theta\\ p_\theta\\ r'\\ p_r\end{pmatrix}$$

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The matrix is block-diagonal:

$$\mathbf{X} = \begin{pmatrix}\mathbf{Y}\\ \mathbf{Z}\end{pmatrix}, \qquad \mathbf{Y} = \begin{pmatrix}\theta\\ p_\theta\end{pmatrix}, \qquad \mathbf{Z} = \begin{pmatrix}r'\\ p_r\,.\end{pmatrix}$$

Linear dynamics evolve independently:

$$\dot{\mathbf{Y}} = \begin{pmatrix}0 & 1/m\ell^2\\ -mg\ell & 0\end{pmatrix}\mathbf{Y}\,, \qquad \dot{\mathbf{Z}} = \begin{pmatrix}0 & 1/m\\ -k & 0\end{pmatrix}\mathbf{Z}\,.$$

SLOW **FAST**

The motion described by Y is the rotational component:

$$\ddot{\theta} + (g/\ell)\theta = 0$$

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Eigenfrequencies, or normal mode frequencies:

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Ratio of rotational and elastic frequencies:

$$\epsilon \equiv \left(\frac{\omega_R}{\omega_Z}\right) = \sqrt{\frac{mg}{k\ell}}\,.$$

Frequency Ratio ϵ

Strange-but-True:

Rotational frequency is always less than elastic!

$$\epsilon^2 = \left(1 - \frac{\ell_0}{\ell}\right) < 1\,, \qquad \textbf{so} \qquad |\omega_R| < |\omega_Z|\,.$$

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If the frequencies are commensurate (ϵ rational) the linear motion is periodic. If ϵ is irrational, the variables never return simultaneously to their starting values, but come arbitrarily close; the motion is then said to be quasi-periodic.

Large Frequency Separation: $\epsilon \ll 1$

We will assume (for the moment) parameters such that:

$$\epsilon \equiv \left(\frac{\omega_{\mathrm{R}}}{\omega_{\mathrm{E}}}\right) = \sqrt{\frac{mg}{k\ell}} \ll 1\,.$$

Linear normal modes are clearly distinct: the rotational mode has low frequency (LF) and the elastic mode has high frequency (HF).

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Analogy:

$$\left(\begin{array}{c}\textbf{Gravity Wave}\\ \textbf{Frequency}\end{array}\right) \ll \left(\begin{array}{c}\textbf{Rossby Wave}\\ \textbf{Frequency}\end{array}\right)$$

For $\epsilon = 0$, there is no coupling between the modes.

For $\epsilon \ll 1$ the coupling is weak.
We can apply classical *Hamiltonian perturbation theory.*

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The consequences of the Kolmogorov-Arnol'd-Moser (KAM) Theorem for the swinging spring are discussed in:

Lynch, Peter, 2002: *The Swinging Spring: A Simple Model for Atmospheric Balance.*
At URL:

`http://www.maths.tcd.ie/~plynch`

Only a bare outline will be given here.

Outline of KAM Theory

I. Completely Integrable Hamiltonian Systems

The key to integrating a Hamiltonian system with n degrees of freedom is to find n independent constants of motion.

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Suppose that we have found n such constants, I_k. We define a canonical transformation to new coordinates, treating I_k as the new momenta, and denoting the new conjugate position coordinates as ϕ_k.

Since the I_x's are constant, the trajectories are confined to an n-dimensional manifold $\mathcal{M}$ of the $2n$-dimensional phase space. For bounded motion the manifold $\mathcal{M}$ may be shown to have the topology of an $\underline{n\text{-torus}}$, that is, the cartesian product of n circles.

II. Perturbed (Non-Integrable) Hamiltonian Systems

QUESTION:

What happens when a completely integrable system is slightly perturbed in such a way that integrability no longer holds? Are the toroidal structures simply disturbed slightly or do they disintegrate completely?

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This fundamental question was resolved in the 1960's:

The Kolmogorov-Arnol'd-Moser Theorem

Most of the original tori persist in the case of small perturbations. They are topologically distorted but not destroyed.

Poincaré Sections

To visualise the motion, we choose a two-dimensional surface and plot the intersection of a trajectory each time it passes through the surface in a particular direction. This is called a Poincaré section.

Two especially convenient choices are

- ***The 'slow-plane'*** $(\rho = 0, p_\rho > 0)$
- ***The 'fast-plane'*** $(\vartheta = 0, p_\vartheta > 0)$

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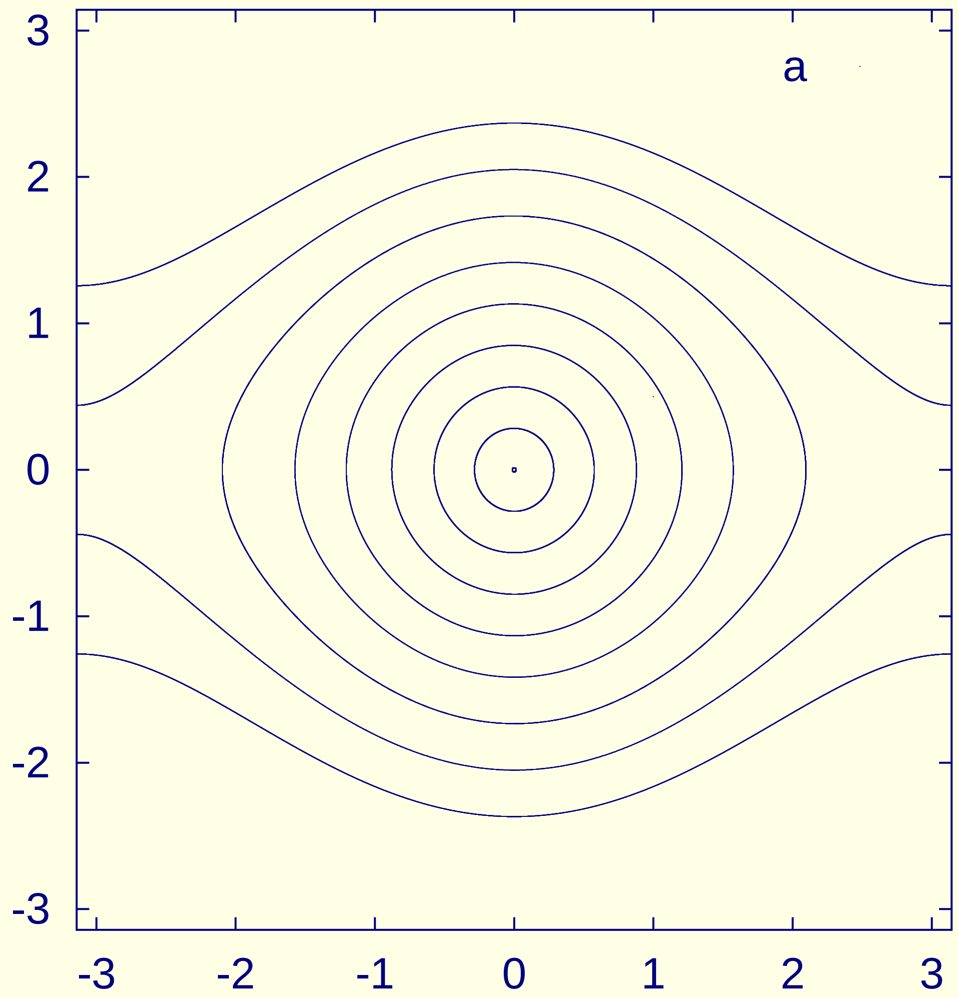
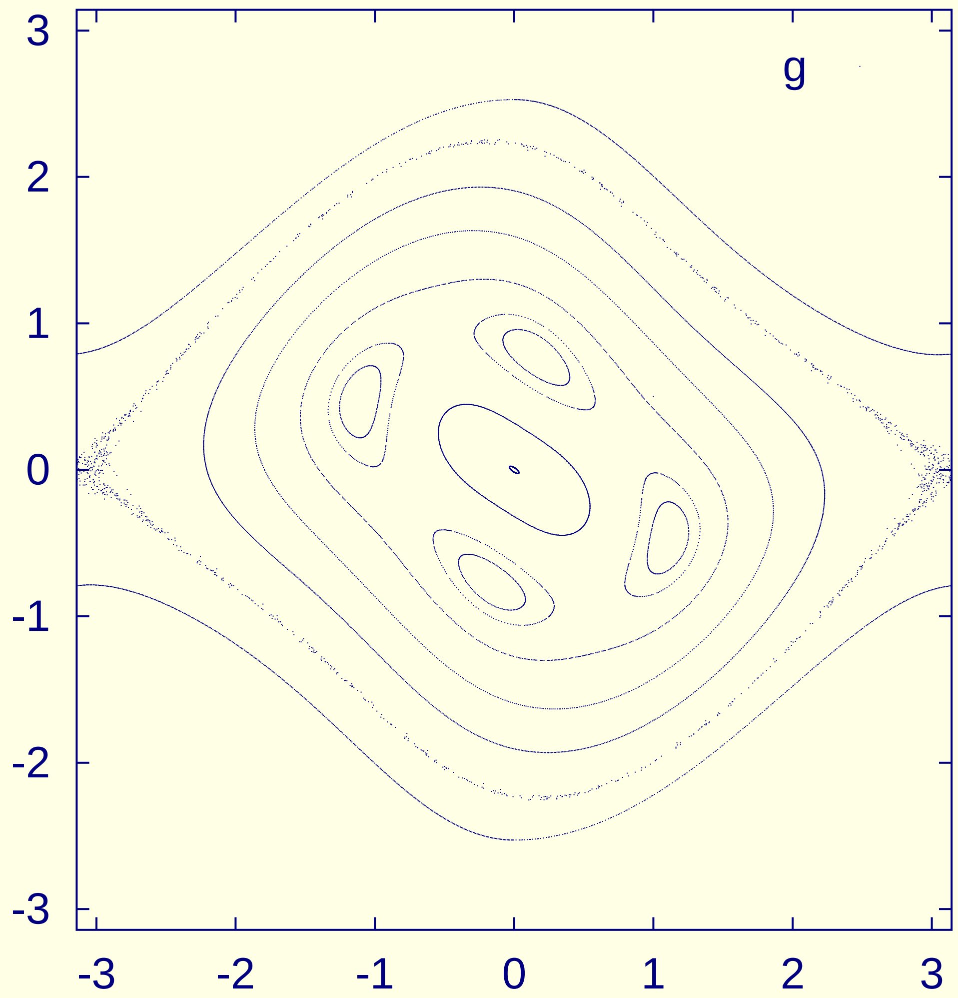
If the motion is integrable, the trajectory lies on a torus which cuts the section in a smooth curve.

For non-integrable motion, the system explores a three-dimensional region of the energy level, whose intersection with a plane is an area rather than a curve.

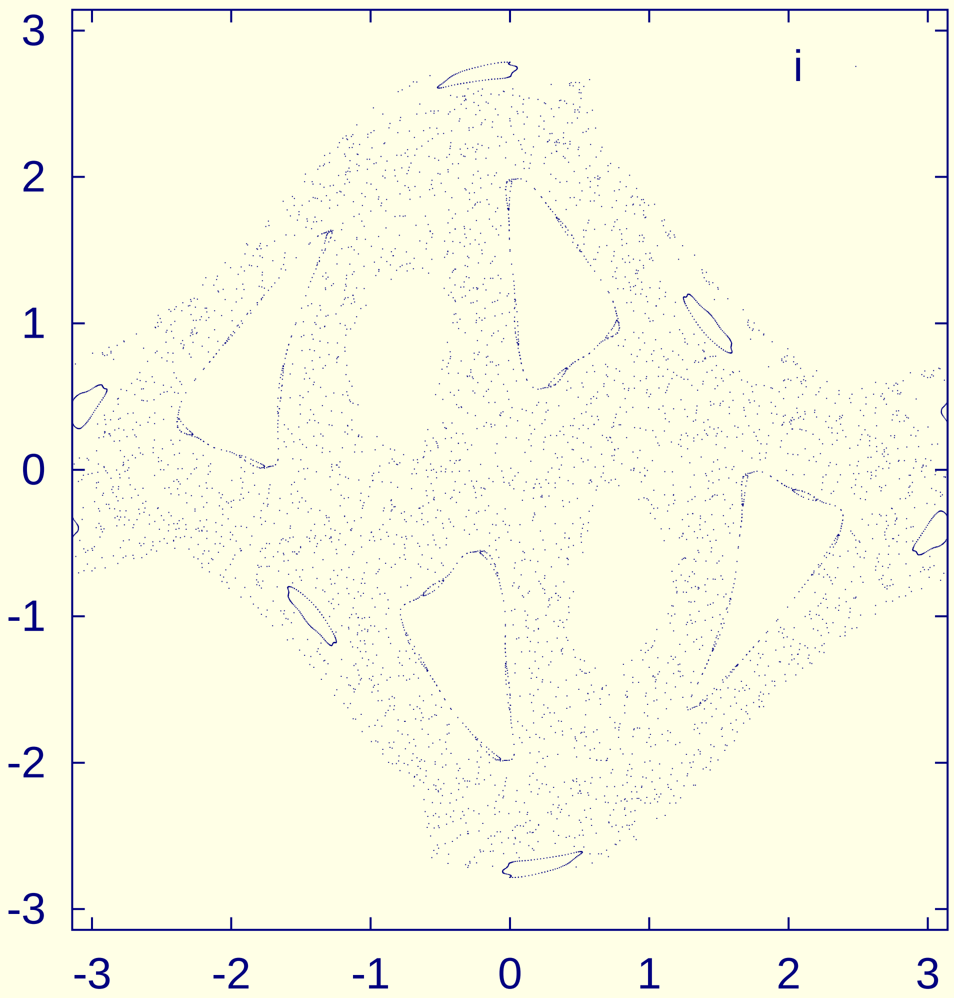
Poincaré Section for total energy $E = 1.8$. $\epsilon = 0.025$.



Poincaré Section for total energy $E = 1.8$. $\epsilon = 0.25$.



Poincaré Section for total energy $E = 1.8$. $\epsilon = 0.40$.



Regular and Chaotic Motion

We wish to discuss the phenomenon of *Resonance* for the spring, and its *Pulsation* and *Precession*.

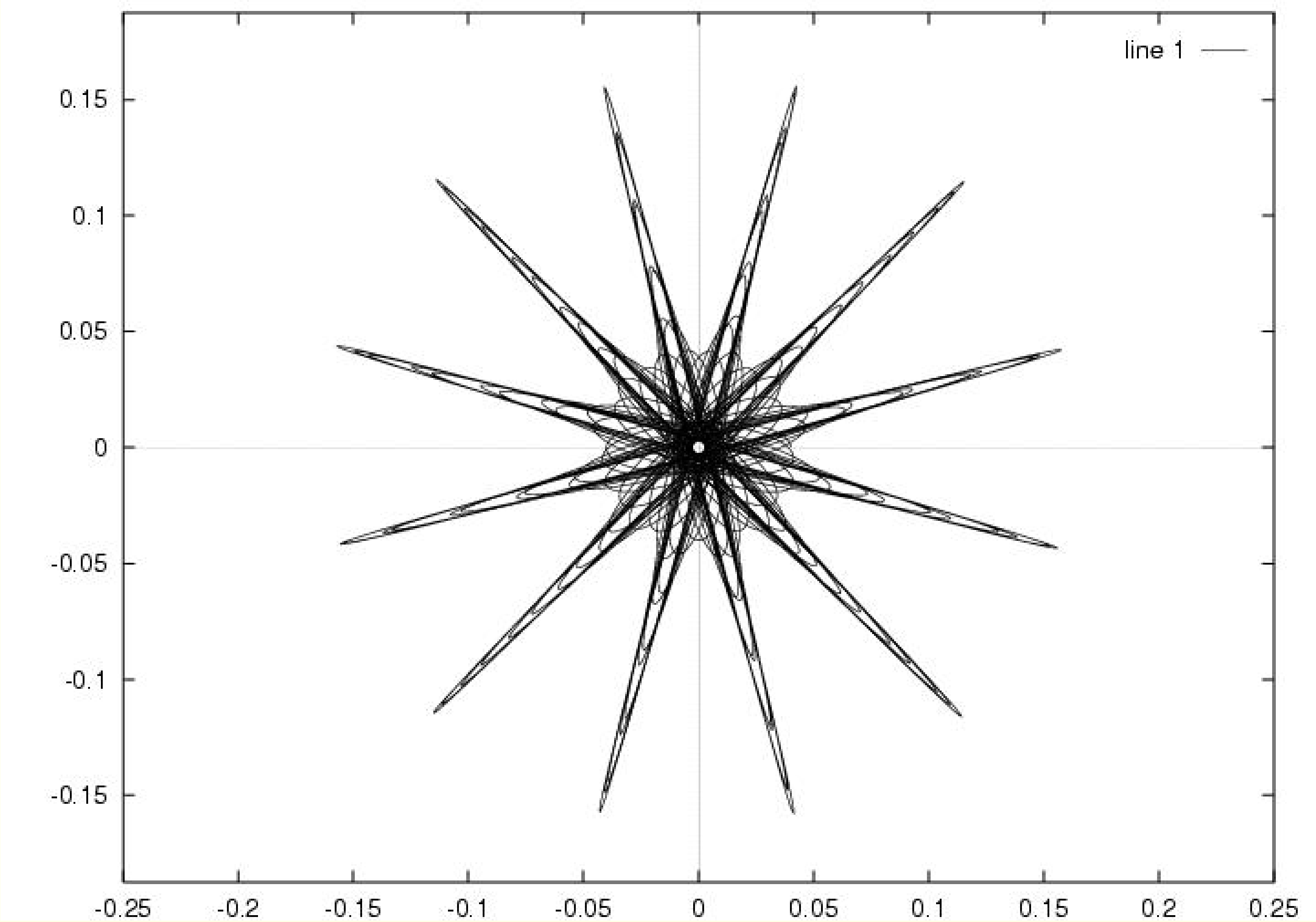
Resonance occurs for

$$\epsilon = \frac{1}{2}\,.$$

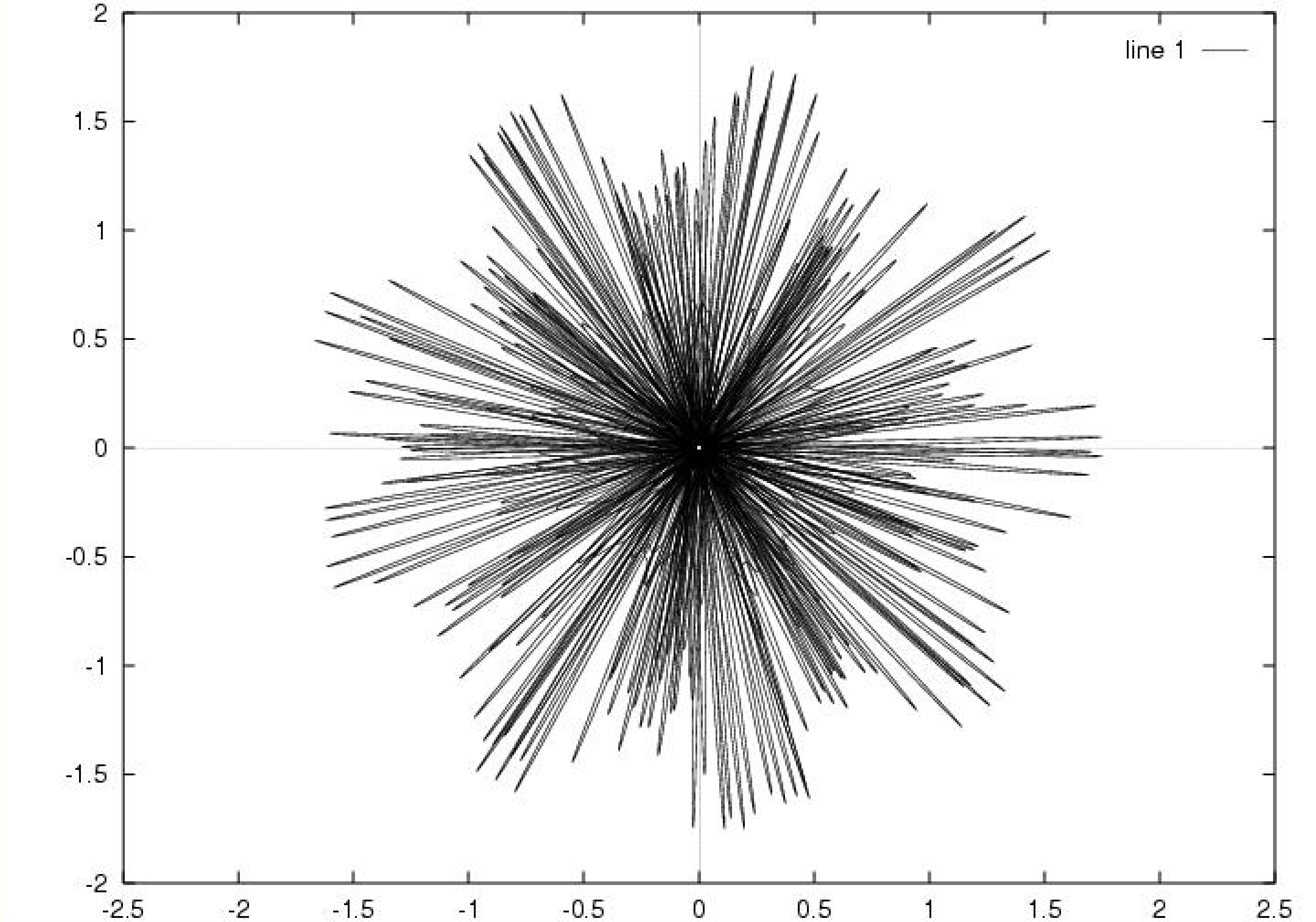
Clearly, this is far from the quasi-integrable case of small ϵ.

However, for *small amplitudes*, the motion is also quasi-integrable. We look at two numerical solutions, one with small amplitude, one with large.

Horizontal plan: Low energy case



Horizontal plan: High energy case



The Phenomenon of Precession

- **When started with almost vertical springing motion, the movement gradually develops into an essentially horizontal swinging motion.**
- **This does not persist, but is soon replaced by springy oscillations similar to the initial motion.**
- **Again a horizontal swing develops, but now in a different direction.**
- **This variation between springy and swingy motion continues indefinitely.**
- **The change in direction of the swing plane from one horizontal excursion to the next is difficult to predict:**
- **The plane of swing precesses in a manner which is quite sensitive to the initial conditions.**

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- The plane of swing precesses in a manner which is quite sensitive to the initial conditions.

Demo: Precession of Swinging Spring.

or Peek at the Applet.

The Resonant Case

The Lagrangian (to cubic order) is

$$L = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - \frac{1}{2}\left(\omega_R^2(x^2 + y^2) + \omega_Z^2 z^2\right) + \frac{1}{2}\lambda(x^2 + y^2)z\,,$$

We study the <u>resonant case:</u>

$$\omega_Z = 2\omega_R\,.$$

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The equations of motion are

$$\begin{aligned}\ddot{x} + \omega_R^2 x &= \lambda x z\\ \ddot{y} + \omega_R^2 y &= \lambda y z\\ \ddot{x} + \omega_Z^2 x &= \tfrac{1}{2}\lambda(x^2+y^2)\,.\end{aligned}$$

The system is <u>not integrable.</u>

Averaged Lagrangian technique

We seek a solution of the form:

$$
\begin{aligned}
x &= \Re[a(t)\exp(i\omega_R t)]\,, \\
y &= \Re[b(t)\exp(i\omega_R t)]\,, \\
z &= \Re[c(t)\exp(2i\omega_R t)]
\end{aligned}
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Coefficients $a(t)$, $b(t)$ and $c(t)$ vary slowly.

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Coefficients $a(t)$, $b(t)$ and $c(t)$ vary slowly.

The Lagrangian is <u>averaged over fast time</u>:

$$\langle L\rangle = \left(\frac{\omega_R}{2}\right)\left[\Im(a\dot{a}^* + b\dot{b}^* + 2c\dot{c}^*) + \kappa\,\Re(a^2+b^2)c^*)\right]$$

where $\kappa = \lambda/(4\omega_R)$ (we absorb κ in t).

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$$

where $\kappa = \lambda/(4\omega_R)$ (we absorb κ in t).

We then derive the Euler-Lagrange equations resulting from this averaged Lagrangian.

The Three-wave Equations

The Euler-Lagrange equations are:

$$
\begin{aligned}
i\dot{a} &= a^* c\,, \\
i\dot{b} &= b^* c\,, \\
i\dot{c} &= \tfrac{1}{4}(a^2 + b^2)
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We transform to new dependent variables:

$$A = \tfrac{1}{2}(a + ib)\,, \qquad B = \tfrac{1}{2}(a - ib)\,, \qquad C = c\,.$$

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The equations become:

$$\begin{aligned} i\dot{A} &= B^* C\,, \\ i\dot{B} &= C A^*\,, \\ i\dot{C} &= AB\,, \end{aligned}$$

These are the *three-wave interaction equations*.

Invariants

The three-wave equations conserve the following three quantities,

$$\begin{aligned} H &= \tfrac{1}{2}(ABC^* + A^*B^*C) \\ N &= |A|^2 + |B|^2 + 2|C|^2 \\ J &= |A|^2 - |B|^2 \,. \end{aligned}$$

The Three-wave equations are completely integrable.

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\end{aligned}
$$

The Three-wave equations are completely integrable.

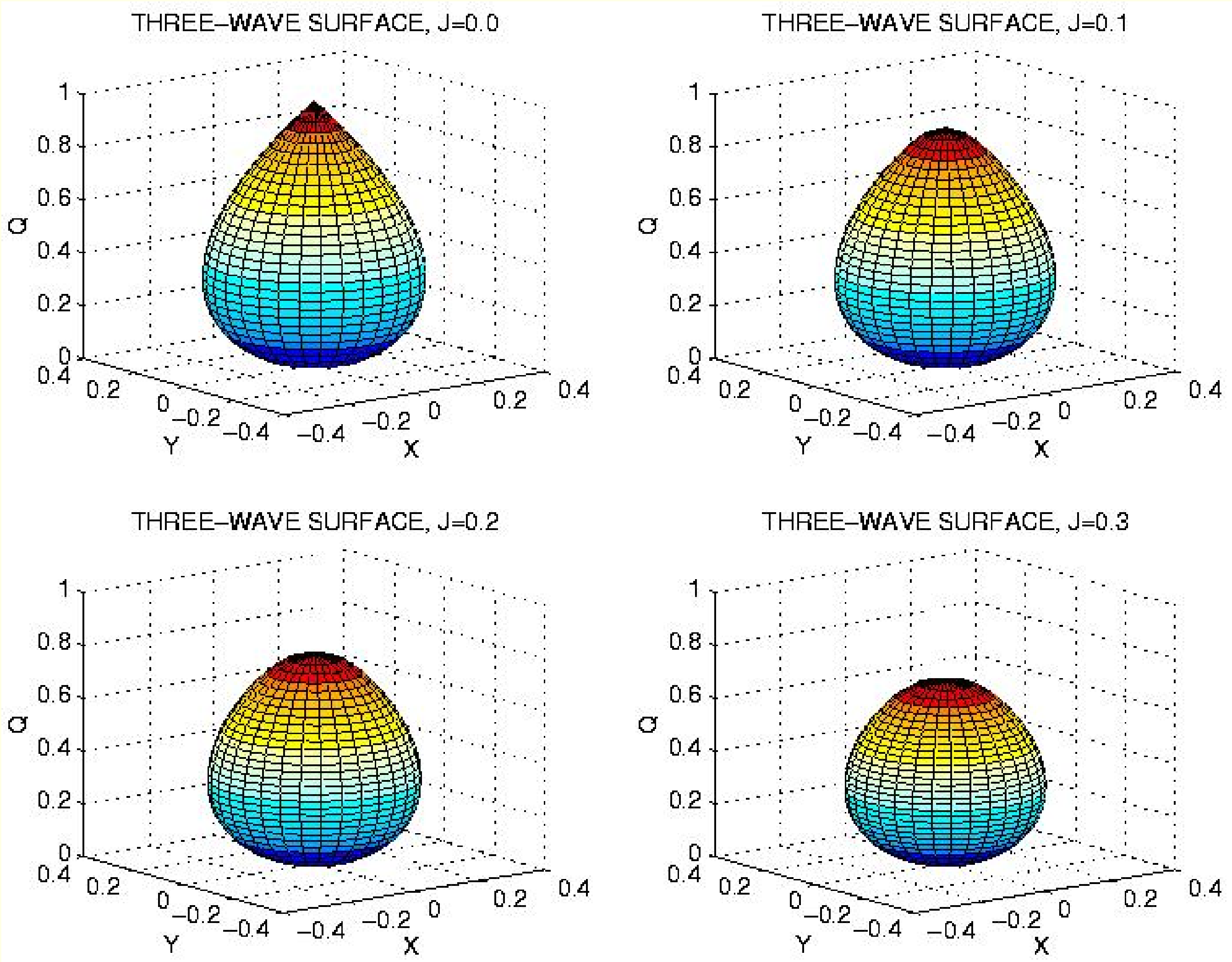
Physically significant combinations of N and J:

$$
N_+ \equiv \tfrac{1}{2}(N + J) = |A|^2 + |C|^2 \,,
$$

$$
N_- \equiv \tfrac{1}{2}(N - J) = |B|^2 + |C|^2 \,.
$$

These are the *Manley-Rowe relations.*
H, N_+ and N_- provide three independent constants of the motion. Constant N_+ or N_- correspond to orthogonal families of circular cylinders in phase-space.

Surfaces of Revolution



Motion is on the intersection with plane of constant X**.**

Ubiquity of Three-Wave Equations

- ***Modulation equations for wave interactions in fluids and plasmas.***
- ***Three-wave equations govern envelop dynamics of light waves in an inhomogeneous material; and phonons in solids.***
- ***Maxwell-Schrödinger envelop equations for radiation in a two-level resonant medium in a microwave cavity.***
- ***Euler's equations for a freely rotating rigid body (when $H = 0$).***

Precession of the Swing Plane

By transforming to rotating co-ordinates, we can derive an expression for the slow rotation Ω once the reduced system is solved for the vertical amplitude.

The averaged Lagrangian in the rotating frame is

$$\langle L \rangle = \left(\frac{\omega_R}{2}\right)\left[\Im\{a\dot{a}^* + b\dot{b}^* + 2c\dot{c}^*\} + \Re\{(a^2+b^2)c^*\} + 2\Omega J\right]$$

where $J = \Im\{ab^*\}$ is the angular momentum.

The Euler-Lagrange equations are:

$$\begin{aligned} i\dot{a} &= a^*c + i\Omega b \\ i\dot{b} &= b^*c - i\Omega a \\ i\dot{c} &= \tfrac{1}{4}(a^2+b^2) \end{aligned}$$

We now introduce the pattern evocation assumption:

Precession Rate

The angle between the amplitudes a and b is constant. This implies

$$\frac{d}{dt}|ab|^2 = 2\Re\{ab^*\}\left[2\Im\{abc^*\} + \Omega(|a|^2 - |b|^2)\right] = 0\,.$$

Assume the second factor vanishes:

$$\Omega = -\frac{2\Im\{abc^*\}}{|a|^2 - |b|^2} = -\frac{|abc|\sin(\alpha + \beta - \gamma)}{|a|^2 - |b|^2}\,. \tag{1}$$

The precession angle Θ can be ascertained by integrating Ω over the time interval of the motion.

In the special case $\alpha - \beta = \frac{\pi}{2} (\mathrm{mod}\,\pi)$, we get

$$\Omega = -\frac{2JH}{(N^2 - 4|c|^2)^2 - 4J^2}\,. \tag{2}$$

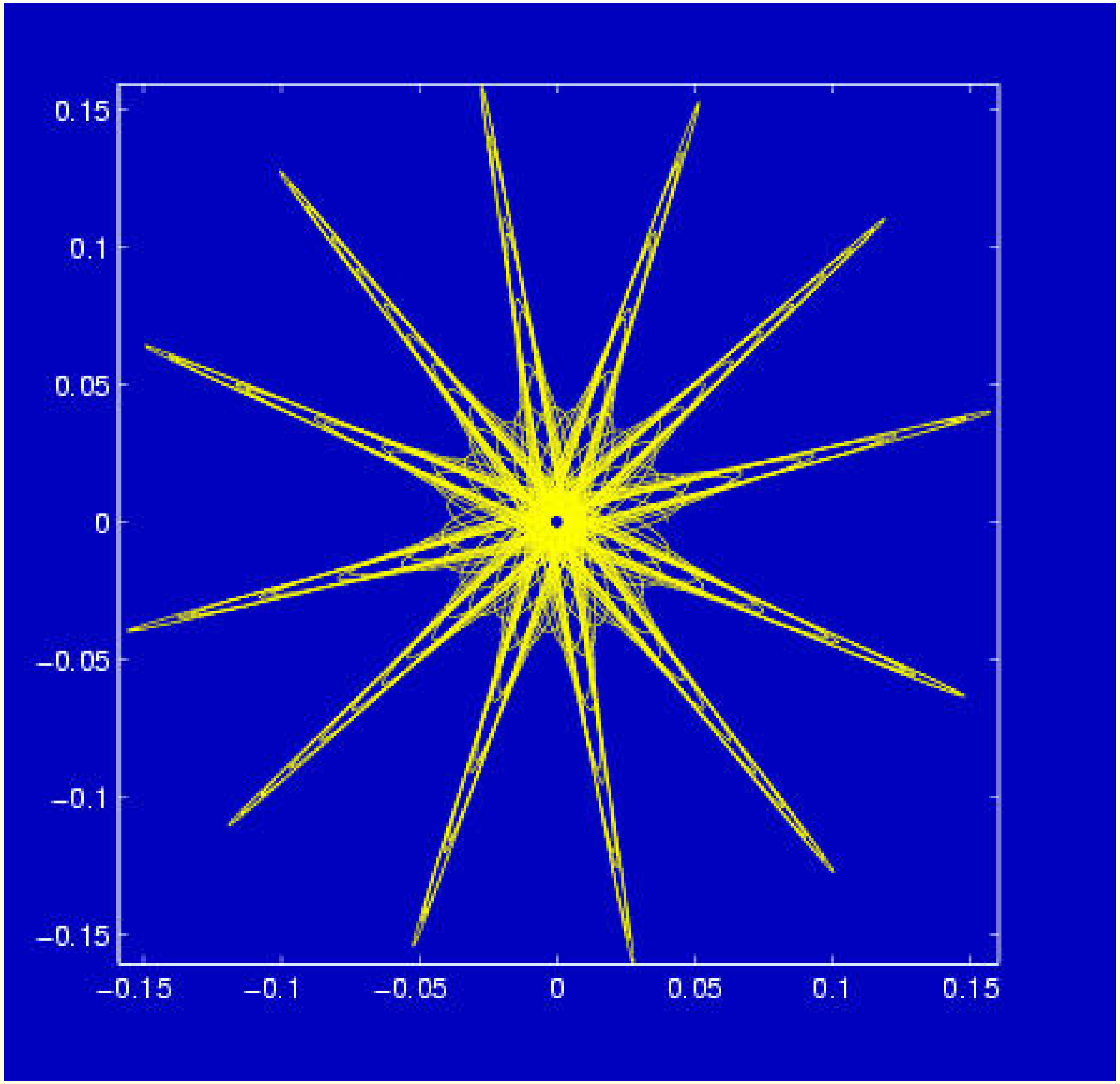
In this case, Ω can be computed as soon as $|c|$ is known.

Numerical Results

We present results of numerical solutions of the modulation equations, and compare them to the solutions of the exact equations.

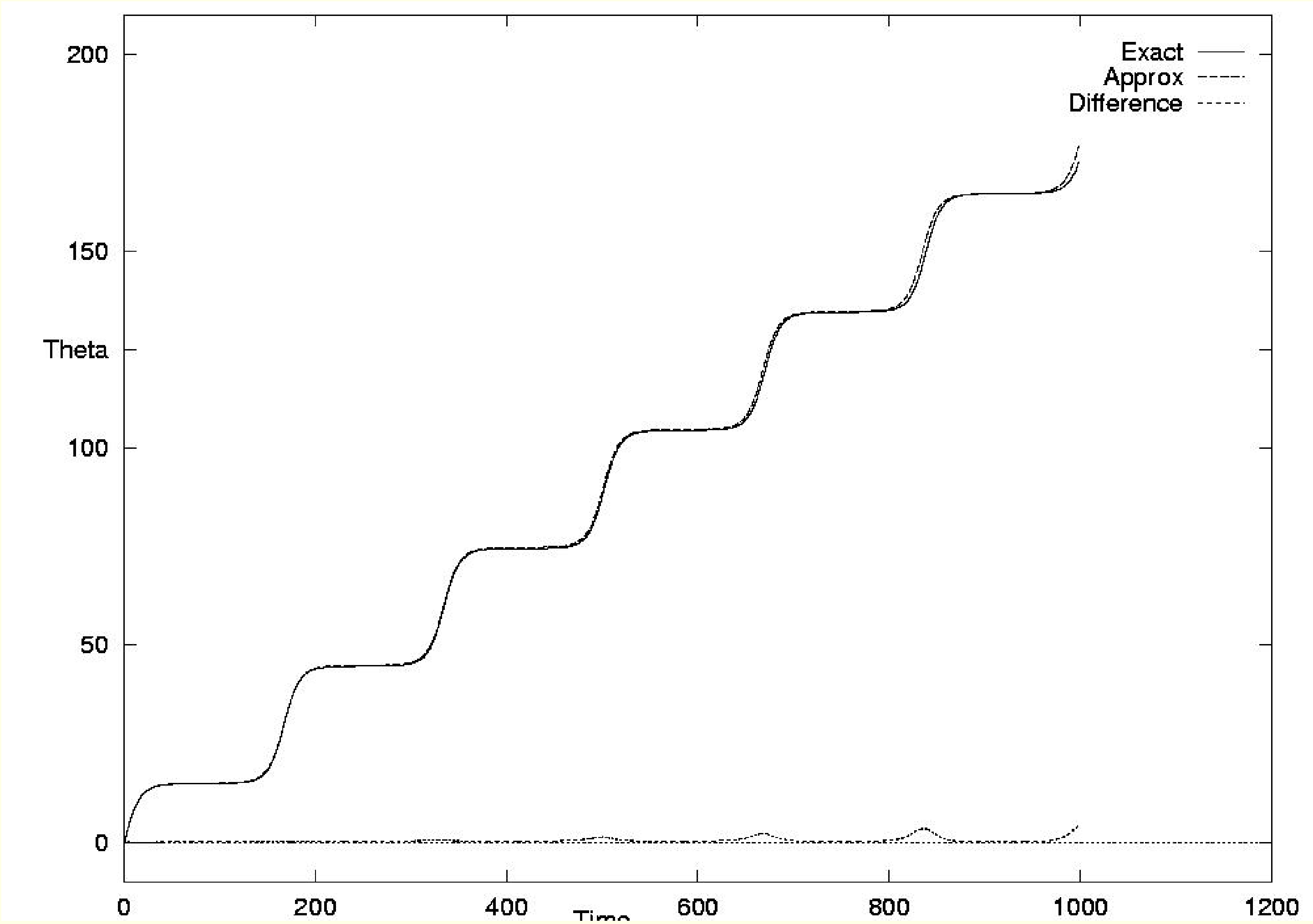
It will be seen that the modulation equations provide an excellent description of the envelop of the rapidly varying solution of the full equations.

We then compare the stepwise precession angle formula determined from constancy of the projected elliptical area with the numerical simulation of this quantity and show that the two values track each other essentially exactly.



Horizontal projection of **spring solution,** y **vs.** x**.**

Step-wise Precession



Azimuth for ‘exact’ and ‘approximate’ solutions.
Difference plotted as dotted line.

Potential Vorticity Conservation

From the *Shallow Water Equations*, we derive the principle of conservation of potential vorticity:

$$\frac{d}{dt}\left(\frac{\zeta + f}{h}\right) = 0\,.$$

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$$\frac{\partial}{\partial t}[\nabla^2\psi - F\psi] + \left\{\frac{\partial\psi}{\partial x}\frac{\partial\nabla^2\psi}{\partial y} - \frac{\partial\psi}{\partial y}\frac{\partial\nabla^2\psi}{\partial x}\right\} + \beta\frac{\partial\psi}{\partial x} = 0$$

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This is the barotropic quasi-geostrophic potential vorticity equation (BQGPVE).

The equation is occasionally called Charney's Equation. However, this is not historically accurate. A more appropriate equation to bear his name is the three-dimensional QG potential vorticity equation.

Rossby Waves

Wave-like solutions of BQGPV Equation:

$$\psi = A\cos(kx + \ell y - \sigma t)$$

satisfies the equation provided

$$\sigma = -\frac{k\beta}{k^2 + \ell^2 + F}\,.$$

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A pure Rossby wave is solution of nonlinear equation.

When there is more than one wave present, this is no longer true: the components interact with each other through the nonlinear terms.

Resonant Rossby Wave Triads

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Nonlinear interaction essentially confined to three components. These three waves are called a resonant triad.

$$\psi = \sum_{n=1}^{3} \Re\{a_n(t) \exp[i(k_n x + \ell_n y - \sigma_n t)]\}$$

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Resonant triad must satisfy selection conditions:

$$k_1 + k_2 + k_3 = 0$$
$$\ell_1 + \ell_2 + \ell_3 = 0$$
$$\sigma_1 + \sigma_2 + \sigma_3 = 0$$

Pedlosky (1987) used two-timing perturbation approach to study resonant triads. The amplitudes of a resonant triad satisfy

$$\begin{aligned}\kappa_1^2\dot{a}_1 + B_1 a_2^* a_3 &= 0\\ \kappa_2^2\dot{a}_2 + B_2 a_3 a_1^* &= 0\\ \kappa_3^2\dot{a}_3 + B_3 a_1 a_2 &= 0\end{aligned}$$

Here $\kappa_n^2 = K_n^2 + F$ and $K_n^2 = k_n^2 + \ell_n^2$, and the interaction coefficients are

$$B_1 = \frac{1}{2}(k_2\ell_3 - k_3\ell_2)(K_2^2 - K_3^2)\,, \mathrm{etc.}$$

Conserved Quantities

The solutions of the Quasigeostrophic Equation conserve not only the total energy, but also the potential enstrophy. For a wave triad, E and S are constant:

$$E = \frac{1}{4}\left(\kappa_1^2 a_1^2 + \kappa_2^2 a_2^2 + \kappa_3^2 a_3^2\right)$$

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Wave components ordered so that

$$K_1 < K_3 < K_2$$

This is consistent with arrangment in order of increasing frequency,

$$|\sigma_1| < |\sigma_2| < |\sigma_3|$$

Defining

$$A_n = \mu_1 \mu_2 \mu_3 \left(\frac{a_n}{\mu_n} \right) , \qquad n = 1, 2, 3$$

where $\mu_n^2 = |B_n/\kappa_n^2|$**, the modulation equations are:**

$$\dot{A}_1 = -A_2^* A_3$$
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The Spring Equations

and the

Triad Equations are

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Mathematically Identical!

The three-wave equatons are the canonical equations resulting from the Hamiltonian $H = \Im\{A_1 A_2 A_3^*\}$, which is a constant of the motion (Holm and Lynch, 2001).

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The energy and enstrophy may be linearly combined to give two constants known as the *Manley-Rowe* quantities:

$$N_1 = |A_1|^2 + |A_3|^2 , \qquad N_2 = |A_2|^2 + |A_3|^2 ,$$

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This is remarkable:
If the energy of the wave with intermediate scale grows, the smaller and larger waves must lose energy.
And, of course, vice-versa.

Numerical Example of Resonance

Important scaling property of the three-wave equations:
If the amplitudes are scaled by a constant γ and the time is contracted by a similar factor, the form of the equations is unchanged. Thus, if

$$\mathbf{A}(t) = (A_1(t), A_2(t), A_3(t))$$

is a solution, then so is

$$\gamma\mathbf{A}(\gamma t)\,.$$

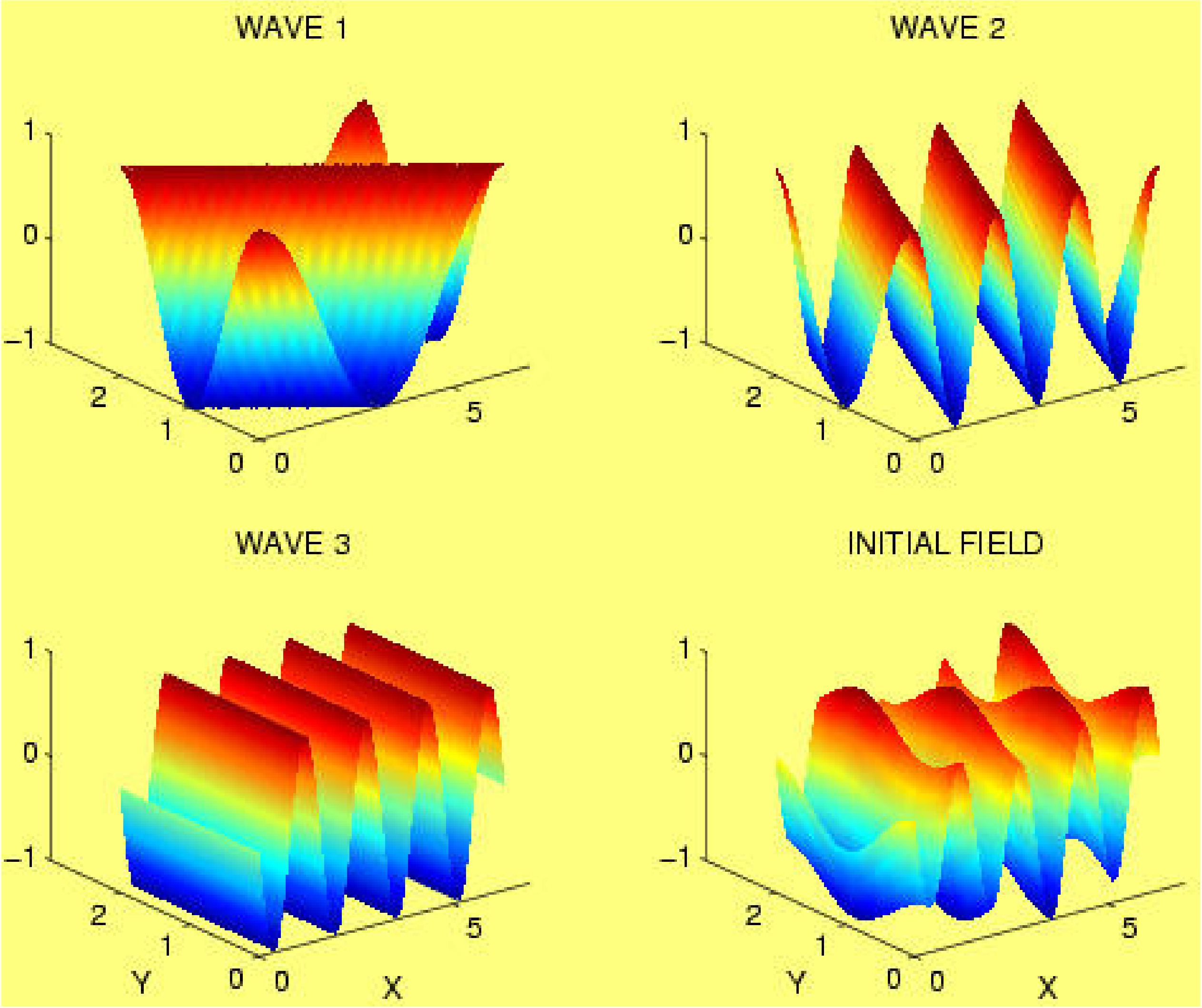
Thus, the period of the modulation envelop will vary inversely with its amplitude.

Parameters for Numerical Experiments

$a = 4 \times 10^7/2\pi\,\mathbf{m}$, $L_x = a$, $L_y = \frac{1}{3}a$, $N_x = 61$, $N_y = 21$, $H_0 = 10\,\mathbf{km}$, $g = \pi^2\,\mathbf{m\,s^{-2}}$, $\phi_0 = 45^\circ$ **and** $\Omega = 2\pi\,\mathbf{rad/day}$. **Thus** $f_0 \approx 10^{-4}\,\mathbf{s}^{-1}$, $\beta \approx 1.6{\times}10^{-11}\,\mathbf{m}^{-1}\mathbf{s}^{-1}$, $L_R \approx 3 \times 10^6\,\mathbf{m}$ **and** $F \approx 10^{-13}\,\mathbf{m}^{-2}$.

The means of defining the wavenumbers $\mathbf{K}_n$ and frequencies σ_n so that the conditions for resonance obtain are discussed by Pedlosky.

The wavenumbers (k_n, ℓ_n) of the three components are given in the paper. We refer to Wave 3 as the *primary component*.



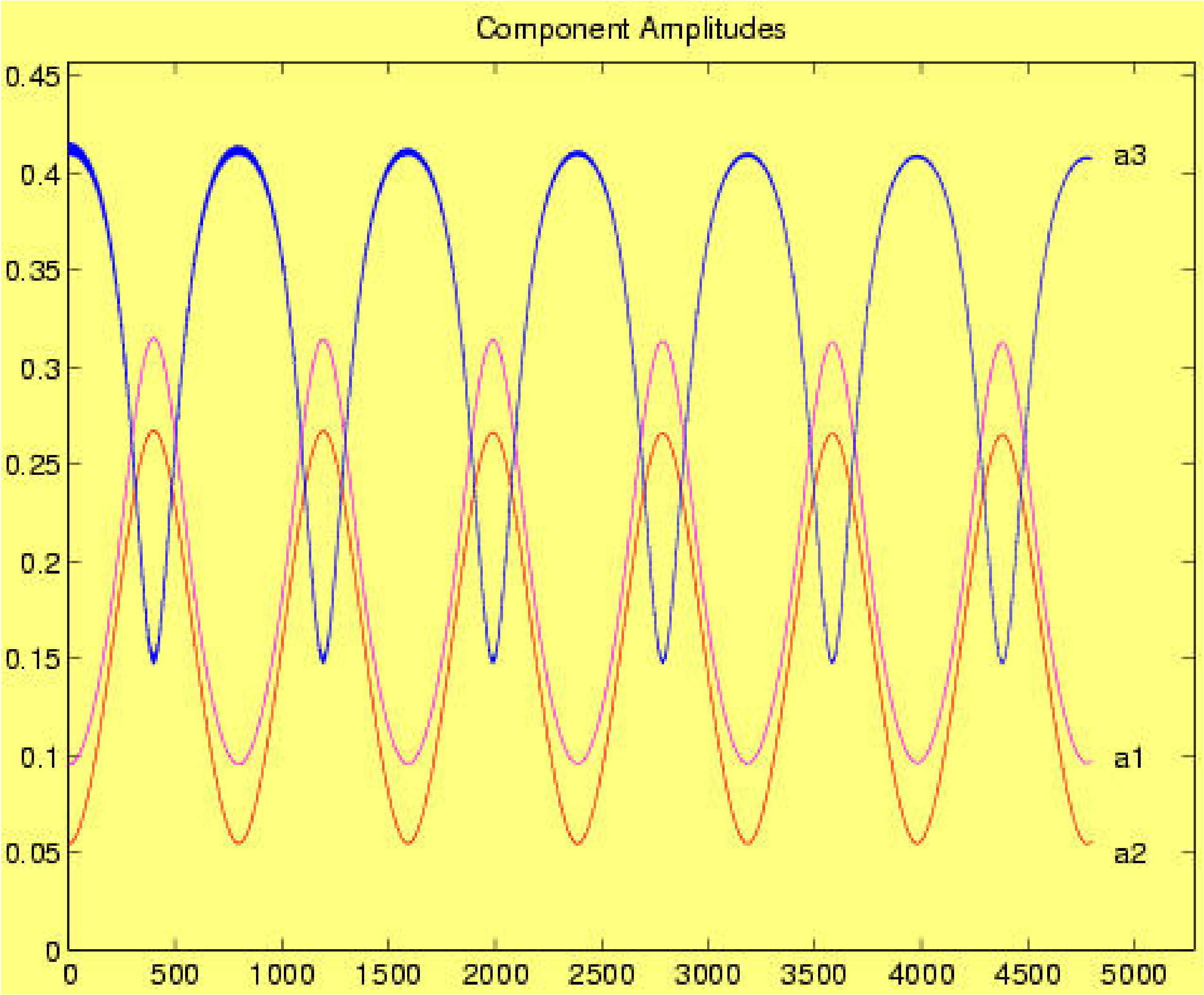
Components of a resonant Rossby wave triad
All fields are scaled to have unit amplitude.

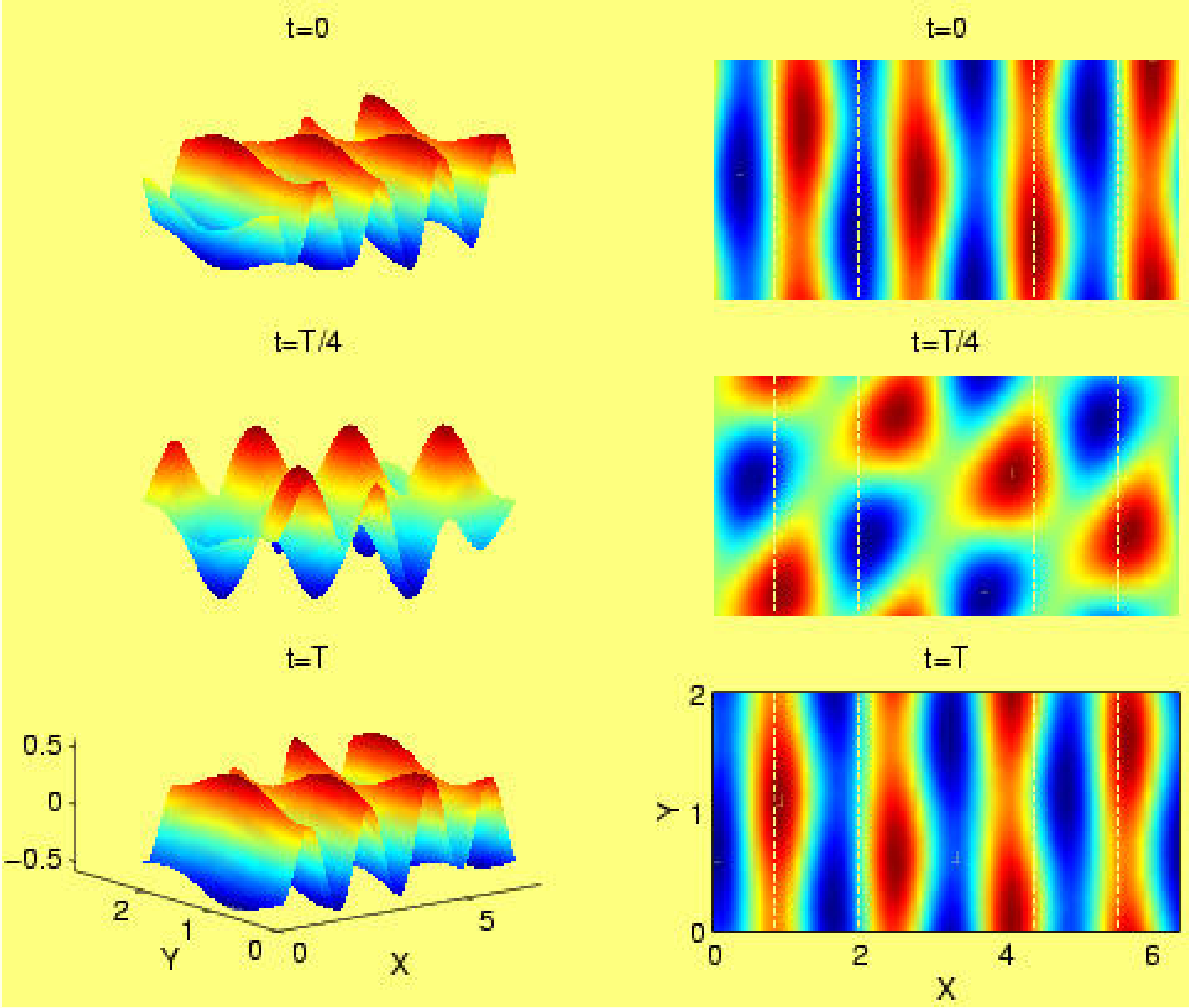
Numerical solution of PDE

$$\frac{\partial}{\partial t}[\nabla^2\psi - F\psi] + \left\{\frac{\partial\psi}{\partial x}\frac{\partial\nabla^2\psi}{\partial y} - \frac{\partial\psi}{\partial y}\frac{\partial\nabla^2\psi}{\partial x}\right\} + \beta\frac{\partial\psi}{\partial x} = 0$$

- **Potential vorticity, $q = [\nabla^2\psi - F\psi]$ is stepped forward (using a leap-frog method)**
- **ψ is obtained by solving a Helmholtz equation with periodic boundary conditions**
- **The Jacobian term is discretized following Arakawa (to conserve energy and enstrophy)**
- **Amplitude is chosen very small. Therefore, interaction time is very long.**

Variation with time of the amplitudes of three components of the stream function.





Stream function at three times during an integration of duration $T = 4800$ days.

Precession of Triads

- **Analogies are Interesting — Equivalences are Useful!**

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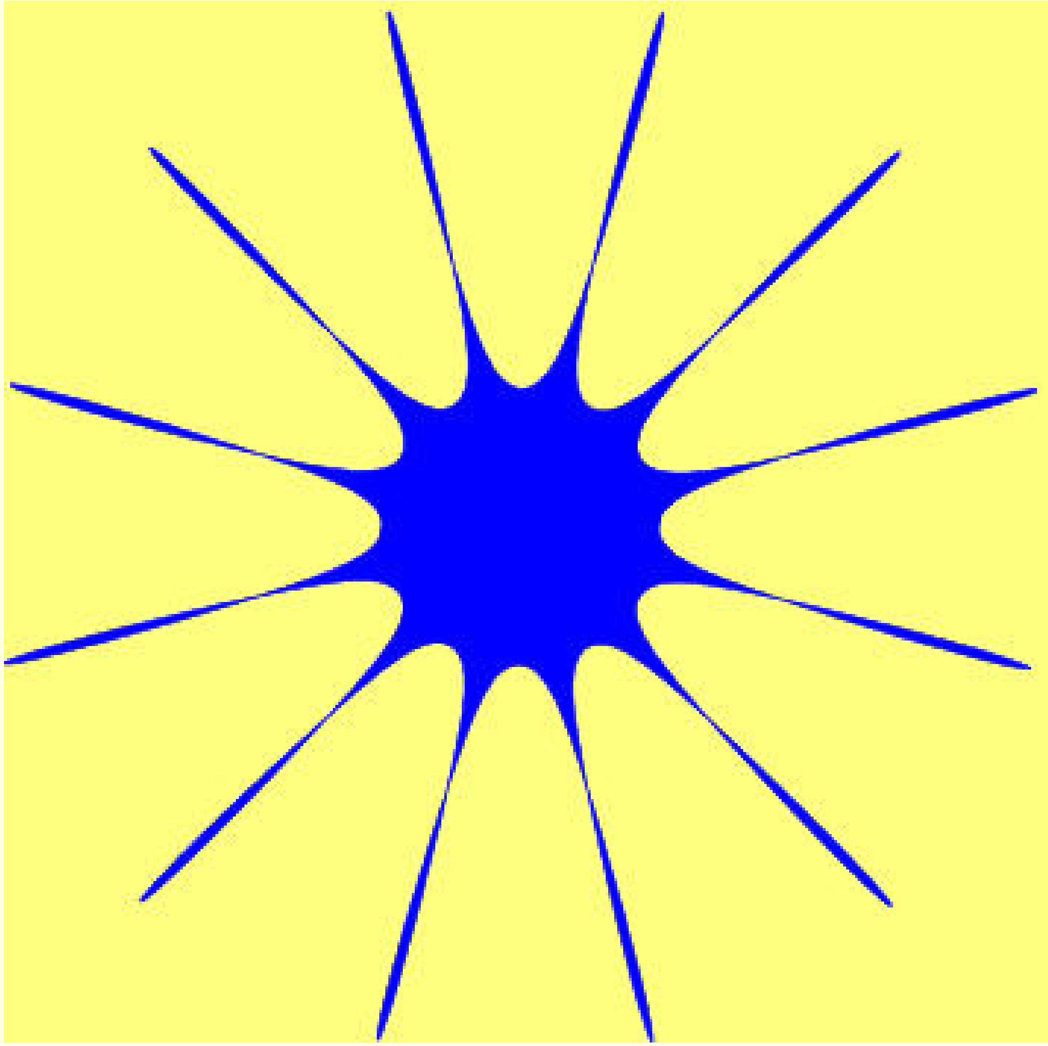
Since the same equations apply to both the spring and triad systems, the stepwise precession of the spring must have a counterpart for triad interactions.
In terms of the variables of the three-wave equations, the semi-axis major and azimuthal angle θ are

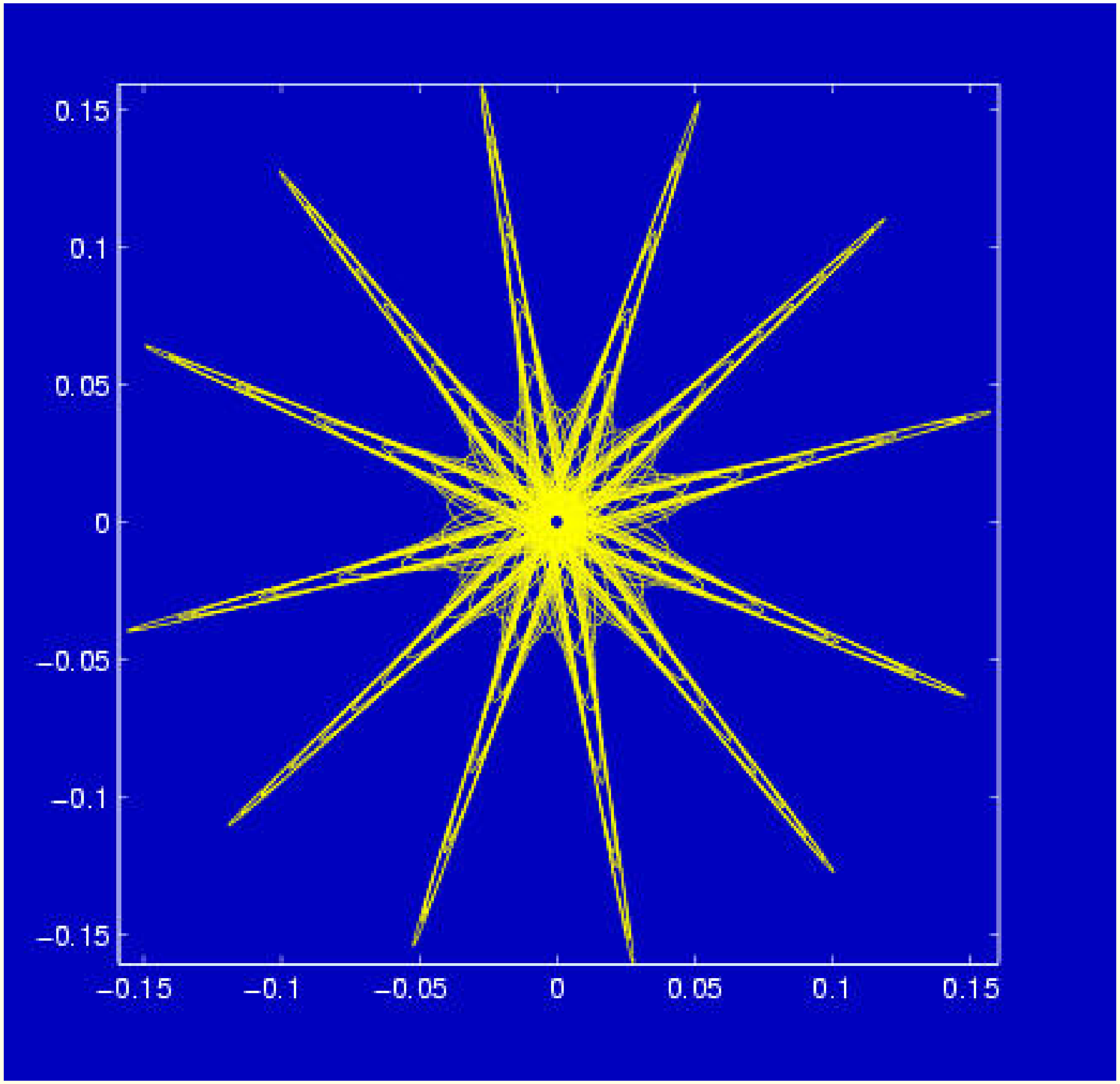
$$A_{\rm maj} = |A_1| + |A_2|\,, \qquad \theta = \frac{1}{2}(\varphi_1 - \varphi_2)\,.$$

Initial conditions chosen as for the spring (by means of the transformation relations).
Initial field scaled to ensure that small amplitude approximation accurate



Polar plot of A_{maj} versus θ for resonant triad.

Take a peek at the Applet, if available!



Horizontal projection of spring solution, y vs. x.

Polar plots of A_{maj} versus θ.

(These are the quantities for the Triad, which correspond to the horizontal projection of the swinging spring.)

- **The Star-like pattern is immediately evident.**
- **Precession angle again about 30°.**

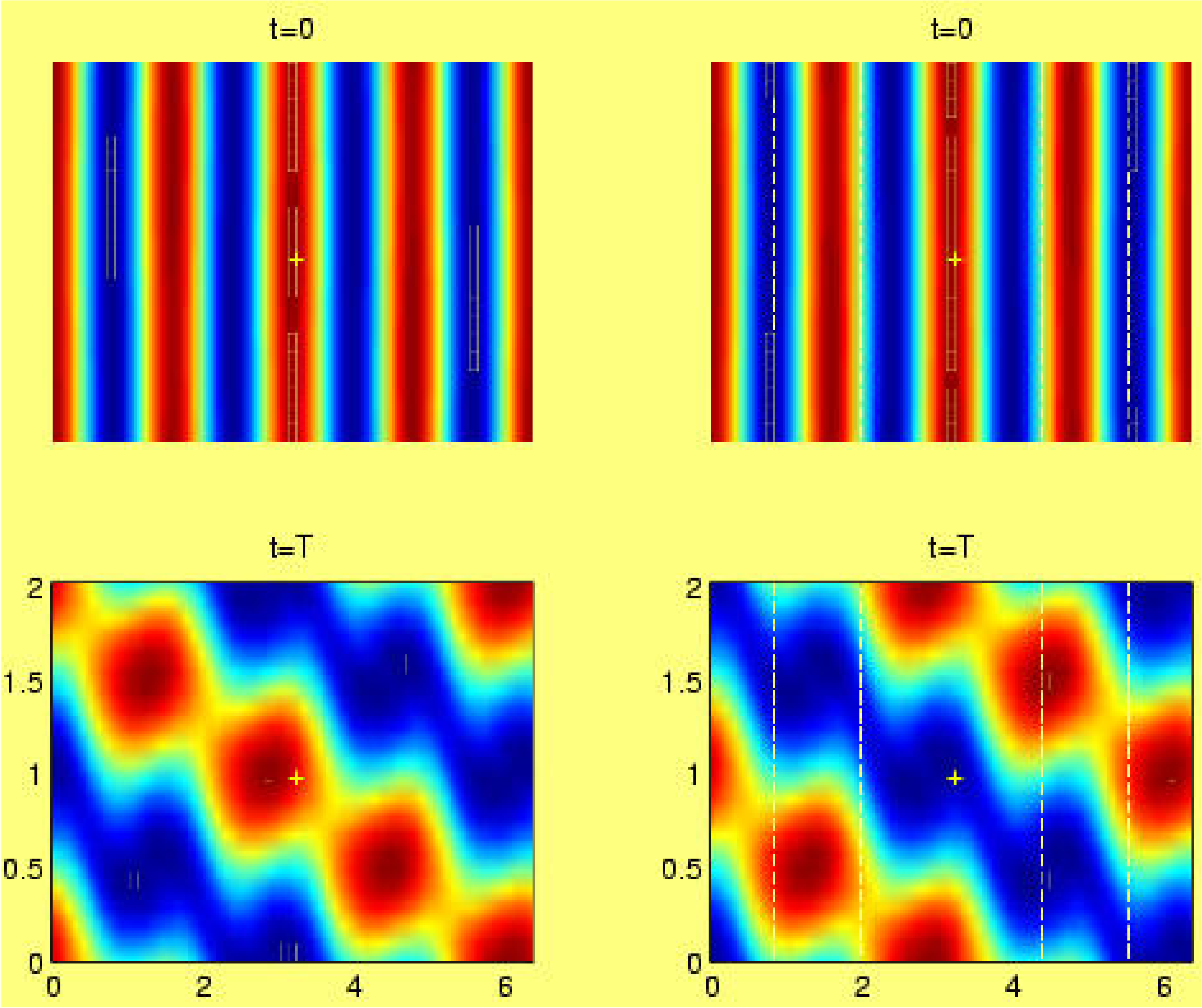
This is remarkable, and illustrates the value of the equivalence:

Phase precession for Rossby wave triads has not been noted before.

Take a peek at the Applet if available!

Rossby Wave Breakdown

- **Precession has implications for predictability**
- **A single Rossby wave may be unstable**
- **Triad resonance is a mechanism for breakdown**
- **Highly sensitive to details of minute perturbations**
- **These are impossible to determine accurately.**



Initial and final fields for two four-day integrations.
Initial fields differ only in the sign of the small perturbation.

Predictability

Drastically different patterns can result from states which are initially very similar!

This sensitivity is in an integrable system
No appeal is made to chaos theory!

Refer to BAMS Article (May, 2003) for details.

Conclusion

I hope I have convinced you that:

> **This simple system looks like a toy at best, but its behaviour is astonishingly complex, with many facets of more than academic lustre . . .**
> (*Breitenberger and Mueller, 1981*)

. . . and that the Swinging Spring is a valuable model of some important aspects of atmospheric dynamics.

The End

Typesetting Software: TEX, *Textures*, LATEX, `hyperref`, `texpower`, Adobe Acrobat 4.05
Graphics Software: Adobe Illustrator 9.0.2
LATEX Slide Macro Packages: Wendy McKay, Ross Moore