Quaternion Solutions for the Rock’n’roller

Box & Loop Orbits

Peter Lynch & Miguel Bustamante
School of Mathematical Sciences
University College Dublin

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From Mechanical to Biological Systems
— an Integrated Approach

Abstract

We consider two types of trajectories found in a wide range of mechanical systems, viz. box orbits and loop orbits. We elucidate the dynamics of these orbits in the simple context of a perturbed nonlinear harmonic oscillator in two dimensions. We then examine the small-amplitude motion of a rigid body, the rock'n'roller, a sphere with eccentric distribution of mass. The equations of motion are expressed in quaternionic form and can be solved analytically. Both types of orbit, boxes and loops, are found, the particular form depending on the initial conditions. The phenomenon of *recession*, or reversal of precession, is associated with box orbits. The small-amplitude solutions for the symmetric case, or Routh sphere, are expressed explicitly in terms of epicycles; there is no recession in this case.

The RnR: Main Topics

- Two types of trajectories: boxs and loops.
- Simple model: Perturbed 2D harmonic oscillator.
- Small-amplitude motion of rock'n'roller.
- Equations of motion in quaternionic form.
- Recession is associated with box orbits.
- Routh sphere: epicycles; no recession.

The RnR: a Topless Bowling-ball

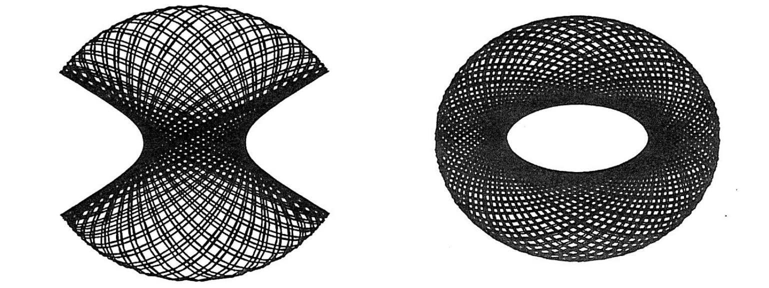
Candle-holders from Copenhagen

Fireballs (designer: Pernille Vea)

Recession I

Globular Cluster: Messier 54, NGC 6715
Class III Extragalactic Globular Cluster.

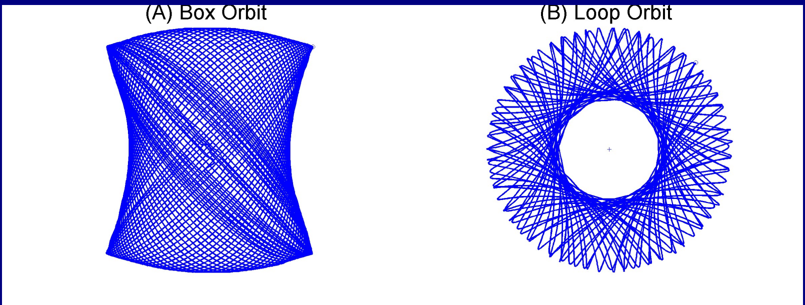
Box and Loop Orbits: Globular Cluster



Two orbits in a logarithmic gravitational potential.
Left: a box orbit. Right: a loop orbit.

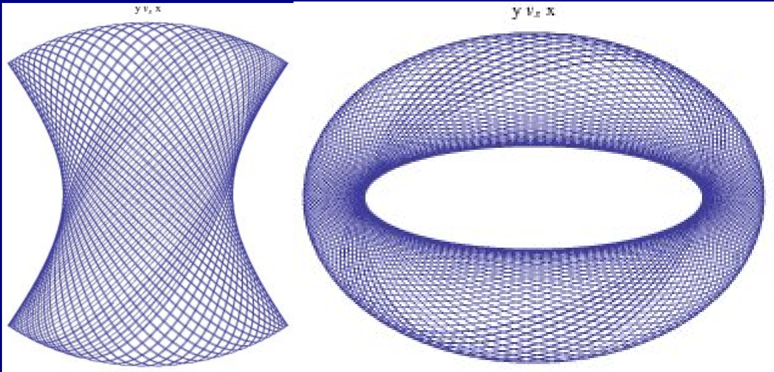
Galactic Dynamics. Binney and Tremaine (2008) [pg. 174]

Box and Loop Orbits: Rock'n'roller



Trajectory of the Rock'n'roller in θ–ϕ-plane
(θ radial, ϕ azimuthal) with $\epsilon = 0.1$.

Box and Loop Orbits: SHO



Box and Loop orbits for the perturbed SHO.

The Perturbed Harmonic Oscillator

Unperturbed system: 2D SHO with equal frequencies:

$$L_0 = \tfrac{1}{2}(\dot{x}^2 + \dot{y}^2) - \tfrac{1}{2}\omega_0^2(x^2 + y^2)$$

The perturbed system has Lagrangian:

$$L = L_0 - \delta y^2 - \epsilon r^4 \,,$$

where $\delta \ll \omega_0^2$ and $\epsilon \ll 1$.

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$$L = L_0 - \delta y^2 - \epsilon r^4 \,,$$

where $\delta \ll \omega_0^2$ and $\epsilon \ll 1$.

The δ-term breaks the $1:1$ resonance.

The ϵ-term is a radially symmetric stiffening.

To analyse the system, we assume a solution

$$x(t) = \Re\{A(t)\exp(i\omega_0 t)\} \qquad y(t) = \Re\{B(t)\exp(i\omega_0 t)\}$$

and average the Lagrangian over the fast motion. Defining new variables, we can write:

$$\begin{aligned} \frac{dW}{d\tau} &= \lambda(1 - W^2)\sin\phi\cos\phi \\ \frac{d\phi}{d\tau} &= \lambda W \sin^2\phi - 1 \end{aligned}$$

where $\lambda = 2\epsilon U/\delta$ is a non-dimensional parameter.

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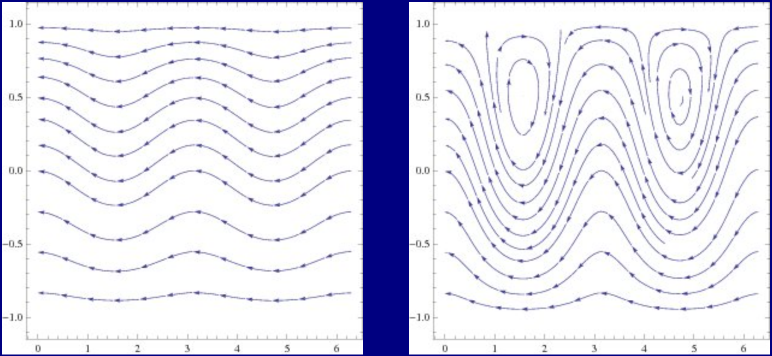
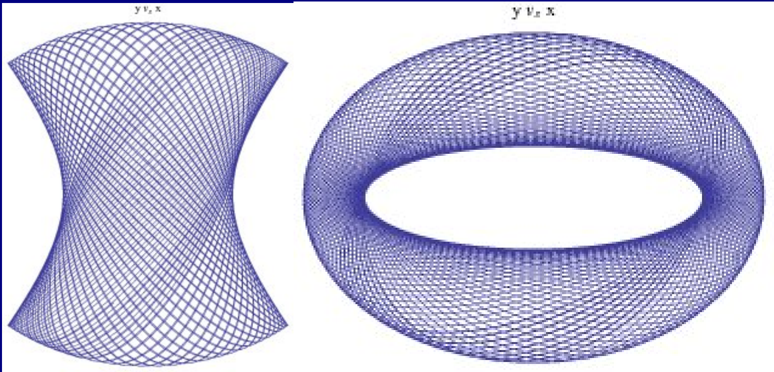
These are the canonical equations for the Hamiltonian

$$H = \tfrac{1}{2}\lambda(1 - W^2)\sin^2\phi + W\,.$$



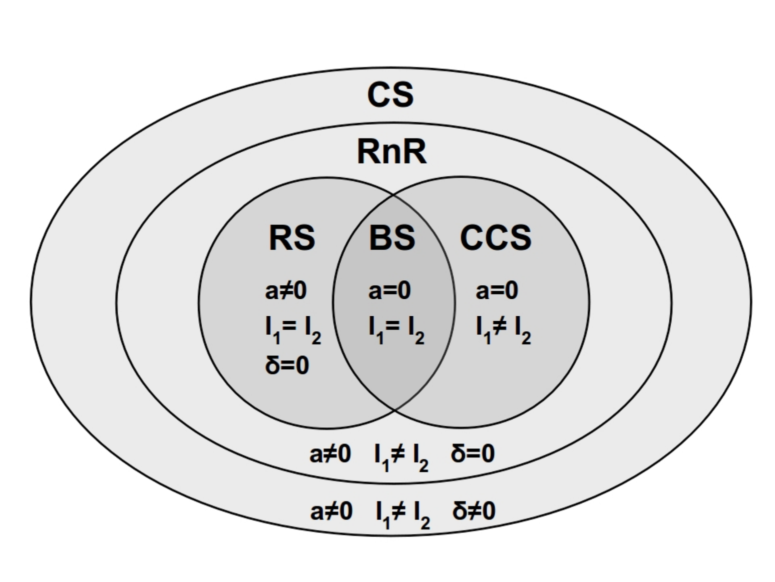
Phase portraits (*W*–*ϕ* plane) for the perturbed SHO.
Left panel: $\lambda = 0.5$. Right panel: $\lambda = 2.0$.

Box and Loop Orbits: SHO

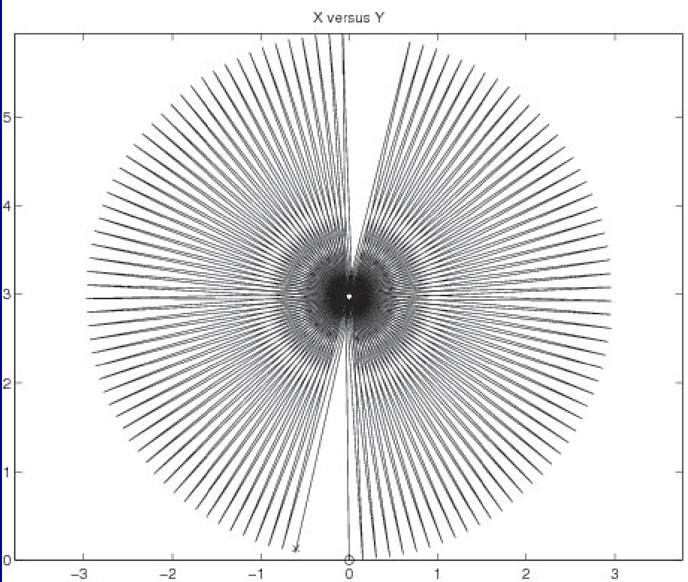


Box and Loop orbits for the perturbed SHO.

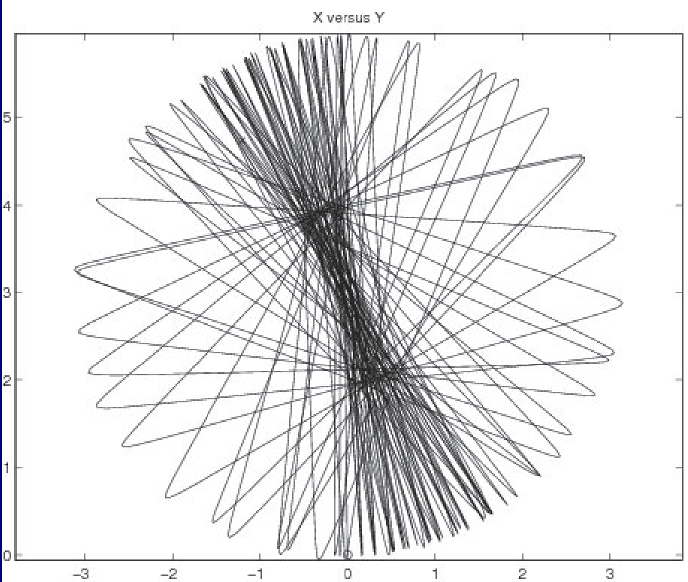
The Hierarchy of Spheres



Symmetric Case: Routh Sphere ($I_1 = I_2$)



Asymmetric Case: Rock'n'roller ($I_1 < I_2$)



Sergey Alexeyevich Chaplygin

Sergey Alexeyevich Chaplygin

Sergey Alexeyevich Chaplygin (1869–1942) was a Russian physicist, mathematician, and mechanical engineer. He is known for mathematical formulas such as Chaplygin's equation.

He graduated in 1890 from Moscow University, and later became a professor. He taught mechanical engineering at Moscow's Woman College in 1901, and applied mathematics at Moscow School of Technology, 1903.

Chaplygin was elected to the Russian Academy of Sciences in 1924. The lunar crater Chaplygin and town Chaplygin are named in his honor. His "Collected Works" in four volumes were published in 1948.

RnR: The Physical System

Consider a spherical rigid body with an asymmetric mass distribution.

Specifically, we consider a loaded sphere.

The dynamics are essentially the same as for the tippe-top, which has been studied extensively.

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The dynamics are essentially the same as for the tippe-top, which has been studied extensively.

Unit radius and unit mass.

Centre of mass off-set a distance a from the centre.

Moments of inertia I_1, I_2 and I_3, with $I_1 \approx I_2 < I_3$.

The Dynamical Equations

In an inertial frame

$$\frac{d\mathbf{v}}{dt} = \mathbf{F} \qquad\qquad \frac{d\mathbf{L}}{dt} = \mathbf{G}$$

Euler angles (θ, ϕ, ψ) **related to angular velocity**

$$\omega_1 = \dot{\theta}\,, \qquad \omega_2 = s\dot{\phi}\,, \qquad \omega_3 = c\dot{\phi} + \dot{\psi}\,.$$

where $s = \sin\theta$ **and** $c = \cos\theta$

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Rotating frame of reference: angular velocity is

$$\omega = \omega_1 \mathbf{i} + \omega_2 \mathbf{j} + \omega_3 \mathbf{k}$$

Rotating frame of reference: angular momentum is

$$\mathbf{L} = \mathbf{I_1}\omega_1 \mathbf{i} + \mathbf{I_2}\omega_2 \mathbf{j} + \mathbf{I_3}\omega_3 \mathbf{k}\,.$$

In the rotating (body) frame, the equations become

$$\frac{d\mathbf{v}}{dt} + \Omega \times \mathbf{v} = \mathbf{F}$$

and

$$\frac{d\mathbf{L}}{dt} + \Omega \times \mathbf{L} = \mathbf{G}$$

$$\begin{aligned}
\dot{v}_1 + \Omega_2 v_3 - \Omega_3 v_2 &= F_1 \\
\dot{v}_2 + \Omega_3 v_1 - \Omega_1 v_3 &= F_2 \\
\dot{v}_3 + \Omega_1 v_2 - \Omega_2 v_1 &= F_3
\end{aligned}$$

$$\begin{aligned}
\mathbf{I_1}\dot{\omega}_1 + \mathbf{I_3}\Omega_2\omega_3 - \mathbf{I_2}\Omega_3\omega_2 &= G_1 \\
\mathbf{I_2}\dot{\omega}_2 + \mathbf{I_1}\Omega_3\omega_1 - \mathbf{I_3}\Omega_1\omega_3 &= G_2 \\
\mathbf{I_3}\dot{\omega}_3 + \mathbf{I_2}\Omega_1\omega_2 - \mathbf{I_1}\Omega_2\omega_1 &= G_3
\end{aligned}$$

The Lagrangian

The Lagrangian of the system is easily written down:

$$L = \tfrac{1}{2}(\mathbf{I_1}\omega_1^2 + \mathbf{I_2}\omega_2^2 + \mathbf{I_3}\omega_3^2) + \tfrac{1}{2}(\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2) - ga(1 - \cos\theta)$$

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The equations may then be written (in vector form):

$$\boldsymbol{\Sigma}\dot{\boldsymbol{\theta}} = \boldsymbol{\omega}\,, \qquad \mathbf{K}\dot{\boldsymbol{\omega}} = \mathbf{P}_\omega$$

where the matrices $\boldsymbol{\Sigma}$ and K are known and

$$\mathbf{P}_\omega = \begin{pmatrix} -(g + \omega_1^2 + \omega_2^2)as\chi + (\mathbf{I_2} - \mathbf{I_3} - af)\omega_2\omega_3 \\ (g + \omega_1^2 + \omega_2^2)as\sigma + (\mathbf{I_3} - \mathbf{I_1} + af)\omega_1\omega_3 \\ (\mathbf{I_1} - \mathbf{I_2})\omega_1\omega_2 + as(-\chi\omega_1 + \sigma\omega_2)\omega_3 \end{pmatrix}$$

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Note that neither K nor $\mathbf{P}_\omega$ depends explicitly on ϕ.

Nonholonomic Constraints

We assume perfectly rough contact (rolling motion).

Holonomic constraints $f_k(q_\rho) = 0$ can be handled by modifying the Lagrangian:

$$L \longrightarrow L + \sum \lambda_k f_k$$

For non-holonomic constraints this doesn't work.

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Misunderstandings on non-holonomy abound:

- **Whittaker and Landau & Lifshitz get it right!**
- **Goldstein *et al.* (2002) get it wrong!**
- **See Flannery (2005) for a review.**

The enigma of nonholonomic constraints

M. R. Flannery[a)]
School of Physics, Georgia Institute of Technology, Atlanta, Georgia 30332

(Received 16 February 2004; accepted 8 October 2004)

The problems associated with the modification of Hamilton's principle to cover nonholonomic constraints by the application of the multiplier theorem of variational calculus are discussed. The reason for the problems is subtle and is discussed, together with the reason why the proper account of nonholonomic constraints is outside the scope of Hamilton's variational principle. However, linear velocity constraints remain within the scope of D'Alembert's principle. A careful and comprehensive analysis facilitates the resolution of the puzzling features of nonholonomic constraints.
[DOI: 10.1119/1.1830501]

***Am. J. Phys.*, Vol 73, 265-272 (2005)**

Nonholonomic Constraints

Assume nonholonomic constraints

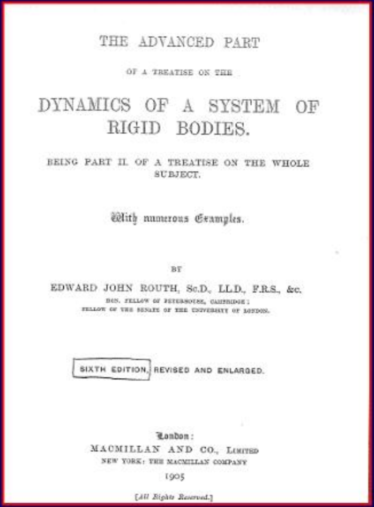
$$g_k(q_\rho, \dot{q}_\rho) = 0 \,.$$

When the constraints are linear in the velocities, we can write the equations as:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} + \sum_k \mu_k \frac{\partial g_k}{\partial \dot{q}_i} = 0 \,.$$

For the Rock'n'roller, we have one holonomic constraint and two nonholonomic constraints.

The Routh Sphere: $I_1 = I_2$



THE ADVANCED PART

OF A TREATISE ON THE

DYNAMICS OF A SYSTEM OF RIGID BODIES.

BEING PART II. OF A TREATISE ON THE WHOLE SUBJECT.

With numerous Examples.

BY

EDWARD JOHN ROUTH, Sc.D., LL.D., F.R.S., &c.

HON. FELLOW OF PETERHOUSE, CAMBRIDGE;
FELLOW OF THE SENATE OF THE UNIVERSITY OF LONDON.

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1905

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Cover of Routh's *Dynamics* Part II

In the Cambridge Mathematical Tripos Examination of 1854, James Clark Maxwell came second.

Edward John Routh came first (senior wrangler).

Constants of Motion for Routh Sphere

In case $I_1 = I_2$, there are three degrees of freedom and three constants of integration.

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The kinetic energy is

$$K = \frac{1}{2}[u^2 + v^2 + w^2] + \frac{1}{2}[\mathbf{I_1}\omega_1^2 + \mathbf{I_1}\omega_2^2 + \mathbf{I_3}\omega_3^2]$$

The potential energy is

$$V = mga(1 - \cos\theta)\,.$$

Since there is no dissipation,

$$E = K + V = \text{constant}\,.$$

Constants of Motion for Routh Sphere

Jellett's constant is the scalar product:

$$C_J = \mathbf{L} \cdot \mathbf{r} = \mathbf{I}_1 s(\sigma\omega_1 + \chi\omega_2) + \mathbf{I}_3 f\,\omega_3 = \text{constant}\,.$$

where $f = \cos\theta - a$, $\sigma = \sin\psi$ **and** $\chi = \cos\psi$**.**
S O'Brien & J L Synge first gave this interpretation.

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Routh's constant (difficult to interpret physically):

$$C_R = \left[\sqrt{\mathbf{I}_3 + s^2 + (\mathbf{I}_3/\mathbf{I}_1)f^2}\right]\omega_3 = \text{constant}\,.$$

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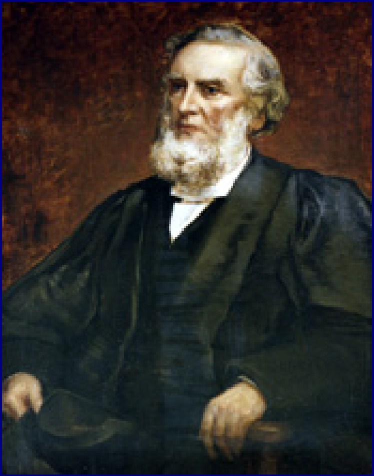
Constant C_R implies conservation of *sign* of ω_3 . . .
. . . but this does not automatically preclude recession!

Edward J Routh

John H Jellett



1831–1907

1817–1888

Edward J Routh

Edward John Routh (20 January 1831 to 7 June 1907), an English mathematician, noted as the outstanding coach of students preparing for the Mathematical Tripos examination of the University of Cambridge.

He also did much to systematize the mathematical theory of mechanics and created several ideas critical to the development of modern control systems theory.

In 1854, Routh graduated just above James Clerk Maxwell, as Senior Wrangler, sharing the Smith's prize with him. He coached over 600 pupils between 1855 and 1888, 27 of them making Senior Wrangler.

Known for: Routh-Hurwitz theorem, Routh stability criterion, Routh array, Routhian, Routh's theorem, Routh's algorithm, Kirchhoff-Routh function.

John H Jellett

J. H. Jellett was a native of Cashel, County Tipperary, the son of a clergyman. He graduated from Trinity College with honors in mathematics in 1838, and was elected to Fellowship in 1840. In 1847 he was appointed to the newly established chair of Natural Philosophy (Applied Mathematics), which he held until 1870.

Jellett was a scholar of considerable eminence and his publications cover the fields of pure and applied mathematics, notably the theory of friction and the properties of optically active solutions, as well as sermons and lectures on religious topics.

He was President of the Royal Irish Academy for five years from 1869, received the Royal Society's Medal in 1881 and an honorary degree from Oxford in 1887.

His politics were sufficiently liberal to make him an acceptable candidate to Gladstone who appointed him Provost of Trinity College Dublin in April 1881. He died in office on 19 February 1888.

Integrability of Routh Sphere

Using Routh's constant C_R, we have $\omega_3 = \omega_3(\theta)$.

Then, using Jellett's constant C_J, we have $\omega_2 = \omega_2(\theta)$.

Using the energy equation, we can now write:

$$\dot{\theta}^2 = f(\theta)\,.$$

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$$\dot{\theta}^2 = f(\theta)\,.$$

For a given θ, both ω_2 and ω_3 are fixed:
This confirms that recession is impossible.

Invariants of the Rock'n'roller

The only known constant of motion is total energy E.

There remains a symmetry: the system is unchanged under the transformation

$$\phi \longrightarrow \phi + \delta\phi$$

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There remains a symmetry: the system is unchanged under the transformation

$$\phi \longrightarrow \phi + \delta\phi$$

The spirit of Noether's Theorem would indicate another constant associated with this symmetry;

So far, we have not found a "missing constant".

Rock'n'roller

The Jellett and Routh quantities

$$Q_J = \mathbf{L}\cdot\mathbf{r} = \mathbf{I_1}\, s(\sigma\omega_1 + \chi\omega_2) + \mathbf{I_3} f\,\omega_3$$

$$Q_R = \left[\sqrt{\mathbf{I_3} + s^2 + (\mathbf{I_3}/\mathbf{I_1})f^2}\right]\omega_3$$

are no longer conserved for the Rock'n'roller.

Rock'n'roller

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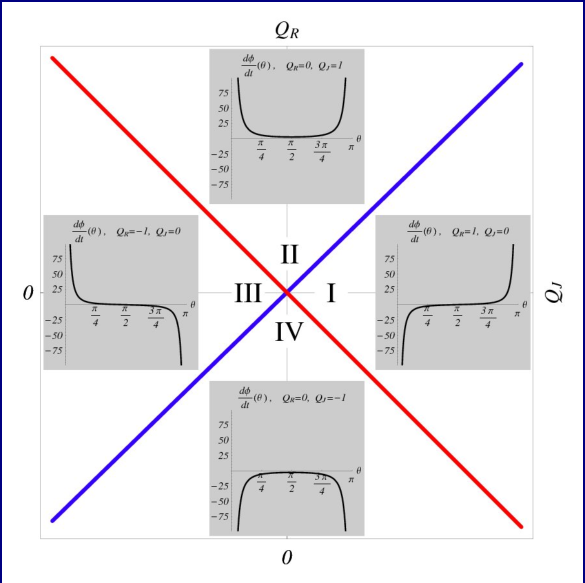
are no longer conserved for the Rock'n'roller.

We have found, analytically, that recession occurs when critical values of these quantities are crossed:

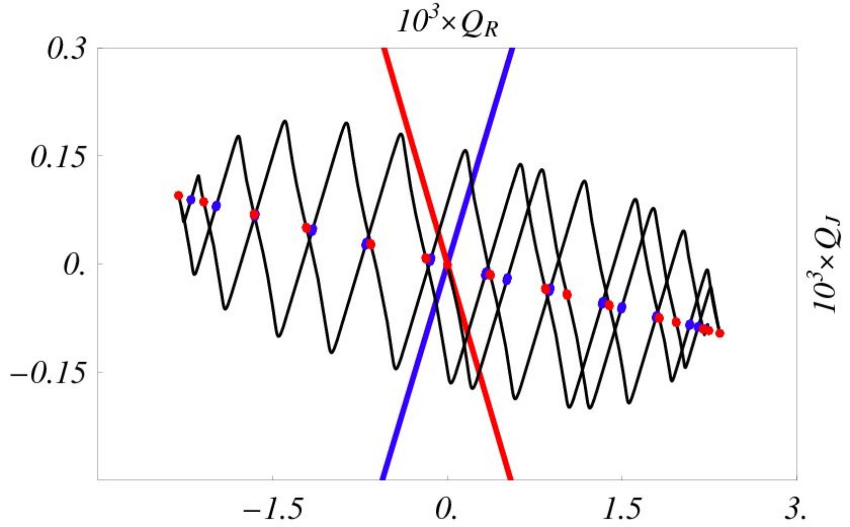
$$Q_J = Q_{J,0}^{\mathrm{crit}} \qquad \textbf{and} \qquad Q_J = Q_{J,\pi}^{\mathrm{crit}}$$

These are shown on the figure below.

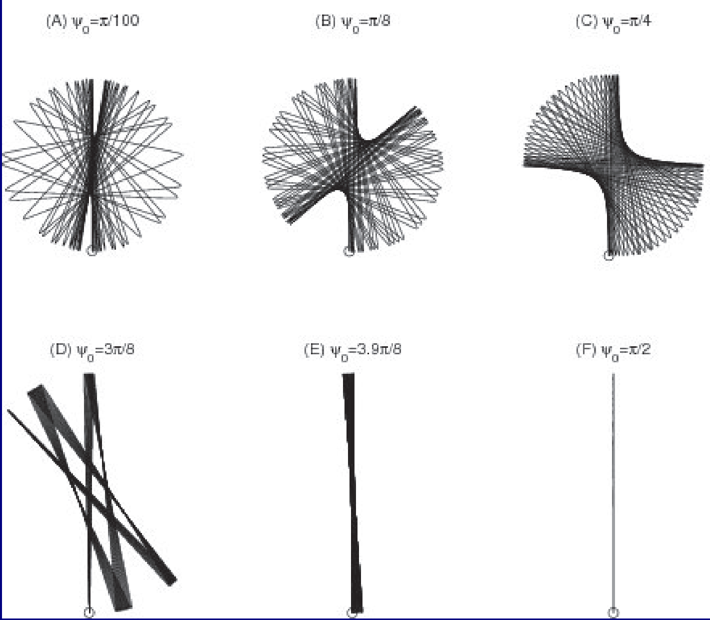


Q_R
Q_J
0
0
I
II
III
IV
$\frac{d\phi}{dt}(\theta)$, $Q_R=0$, $Q_J=1$
$\frac{d\phi}{dt}(\theta)$, $Q_R=-1$, $Q_J=0$
$\frac{d\phi}{dt}(\theta)$, $Q_R=1$, $Q_J=0$
$\frac{d\phi}{dt}(\theta)$, $Q_R=0$, $Q_J=-1$

Q_J versus Q_R



$10^3 \times Q_R$
0.3
0.15
0.
−0.15
−1.5
0.
1.5
3.
$10^3 \times Q_J$



(A) $\psi_0=\pi/100$
(B) $\psi_0=\pi/8$
(C) $\psi_0=\pi/4$
(D) $\psi_0=3\pi/8$
(E) $\psi_0=3.9\pi/8$
(F) $\psi_0=\pi/2$

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Precession and recession of the rock'n'roller

IOPSELECT

Author Peter Lynch and Miguel D Bustamante

Affiliations School of Mathematical Sciences, UCD, Belfield, Dublin 4, Ireland

E-mail Peter.Lynch@ucd.ie Miguel.Bustamante@ucd.ie

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Article | **References**

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Abstract We study the dynamics of a spherical rigid body that rocks and rolls on a plane under the effect of gra
distribution of mass is non-uniform and the centre of mass does not coincide with the geometric centre
symmetric case, with moments of inertia $I_1 = I_2 < I_3$, is integrable and the motion is completely regular.

Precession and recession of the Rock'n'roller (J.Phys.A.)

Quaternionic Formulation

The Euler angles have a singularity when $\theta = 0$
The angles ϕ and ψ are not uniquely defined there.

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We can obviate this problem by using
Euler's symmetric parameters:

$$\gamma = \cos\tfrac{1}{2}\theta \cos\tfrac{1}{2}(\phi + \psi) \qquad \xi = \sin\tfrac{1}{2}\theta \cos\tfrac{1}{2}(\phi - \psi)$$
$$\zeta = \cos\tfrac{1}{2}\theta \sin\tfrac{1}{2}(\phi + \psi) \qquad \eta = \sin\tfrac{1}{2}\theta \sin\tfrac{1}{2}(\phi - \psi)$$

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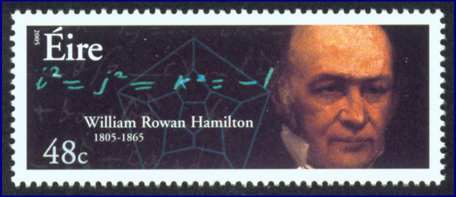
$$\zeta = \cos\tfrac{1}{2}\theta\sin\tfrac{1}{2}(\phi+\psi) \qquad \eta = \sin\tfrac{1}{2}\theta\sin\tfrac{1}{2}(\phi-\psi)$$

There are the components of a unit quaternion

$$\mathbf{q} = \gamma + \xi\mathbf{i} + \eta\mathbf{j} + \zeta\mathbf{k}$$

$$\gamma^2 + \xi^2 + \eta^2 + \zeta^2 = 1$$

William Rowan Hamilton (1805–1865)



Quaternion Equations

Euler's symmetric parameters,
or
Euler-Rodrigues parameters:

$$\gamma = \cos\tfrac{1}{2}\theta\cos\tfrac{1}{2}(\phi+\psi) \qquad \xi = \sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}(\phi-\psi)$$
$$\zeta = \cos\tfrac{1}{2}\theta\sin\tfrac{1}{2}(\phi+\psi) \qquad \eta = \sin\tfrac{1}{2}\theta\sin\tfrac{1}{2}(\phi-\psi)$$

The components of angular velocity are

$$\begin{aligned}
\omega_1 &= 2[\gamma\dot\xi - \xi\dot\gamma + \zeta\dot\eta - \eta\dot\zeta] \\
\omega_2 &= 2[\gamma\dot\eta - \eta\dot\gamma + \xi\dot\zeta - \zeta\dot\xi] \\
\omega_3 &= 2[\gamma\dot\zeta - \zeta\dot\gamma + \eta\dot\xi - \xi\dot\eta]
\end{aligned}$$

At first order in small θ: parameters as

$$\gamma = \cos\tfrac{1}{2}(\phi+\psi) = O(1) \qquad \xi = \tfrac{1}{2}\theta\cos\tfrac{1}{2}(\phi-\psi) = O(\theta)$$
$$\zeta = \sin\tfrac{1}{2}(\phi+\psi) = O(1) \qquad \eta = \tfrac{1}{2}\theta\sin\tfrac{1}{2}(\phi-\psi) = O(\theta)$$

The third equation reduces to

$$\dot{\omega}_3 = O(\theta^2)$$

so we take ω_3 to be constant.

The elements γ and ζ are

$$\gamma = \cos\tfrac{1}{2}\omega_3(t-t_{00})\,, \qquad \zeta = \sin\tfrac{1}{2}\omega_3(t-t_{00})$$

(we can choose $t_{00}=0$).

The remaining two equations are

$$\begin{aligned}
\gamma\ddot{\xi} + \zeta\ddot{\eta} - \kappa_{21}\omega_3(\zeta\dot{\xi} - \gamma\dot{\eta}) + \Omega_1^2(\gamma\xi + \zeta\eta) &= 0 \\
\zeta\ddot{\xi} - \gamma\ddot{\eta} + \kappa_{12}\omega_3(\gamma\dot{\xi} + \zeta\dot{\eta}) + \Omega_2^2(\zeta\xi - \gamma\eta) &= 0
\end{aligned}$$

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These equations may be transformed by rotation

$$\begin{pmatrix} \mu \\ \nu \end{pmatrix} = \begin{bmatrix} \gamma & \zeta \\ -\zeta & \gamma \end{bmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} .$$

The equations may now be written

$$\begin{aligned} \ddot{\mu} - 2k_2\dot{\nu} + \tilde{\Omega}_1^2\mu &= 0 \\ \ddot{\nu} + 2k_1\dot{\mu} + \tilde{\Omega}_2^2\nu &= 0 \end{aligned}$$

We seek a solution in the form

$$\mu = \mu_0 \cos \beta(t - t_0) \qquad \textbf{and} \qquad \nu = \nu_0 \sin \beta(t - t_0)$$

The system may be written

$$\begin{bmatrix} \tilde{\Omega}_1^2 - \beta^2 & -2k_2\beta \\ -2k_1\beta & \tilde{\Omega}_2^2 - \beta^2 \end{bmatrix} \begin{pmatrix} \mu_0 \\ \nu_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The determinant is a biquadratic in β with four real roots $\pm\beta_1$ and $\pm\beta_2$.

The eigenvectors are $(1, \lambda_1)^{\mathrm{T}}$ and $(1, \lambda_2)^{\mathrm{T}}$, with

$$\lambda_1 = \frac{\tilde{\Omega}_1^2 - \beta_1^2}{2k_2\beta_1} = \frac{2k_1\beta_1}{\tilde{\Omega}_2^2 - \beta_1^2}, \qquad \lambda_2 = \frac{\tilde{\Omega}_1^2 - \beta_2^2}{2k_2\beta_2} = \frac{2k_1\beta_2}{\tilde{\Omega}_2^2 - \beta_2^2}$$

Repeat: the equations for μ and ν are:

$$\begin{aligned} \ddot{\mu} - 2k_2\dot{\nu} + \tilde{\Omega}_1^2\mu &= 0 \\ \ddot{\nu} + 2k_1\dot{\mu} + \tilde{\Omega}_2^2\nu &= 0 \end{aligned}$$

The general solution is

$$\begin{aligned} \mu &= \mu_1 \cos\beta_1(t - t_1) + \mu_2 \cos\beta_2(t - t_2) \\ \nu &= \lambda_1\mu_1 \sin\beta_1(t - t_1) + \lambda_2\mu_2 \sin\beta_2(t - t_2) \end{aligned}$$

Lagrangian and Hamiltonian

The quaternion equations arise from the Lagrangian

$$L = \tfrac{1}{2}(k_1\dot{\mu}^2 + k_2\dot{\nu}^2) - \tfrac{1}{2}(k_1\tilde{\Omega}_1^2\mu^2 + k_2\tilde{\Omega}_2^2\nu^2) + k_1k_2(\mu\dot{\nu} - \nu\dot{\mu})$$

The generalized momenta are

$$p_\mu = k_1(\dot{\mu} - k_2\nu) \qquad \textbf{and} \qquad p_\nu = k_2(\dot{\nu} + k_2\mu)$$

The Hamiltonian is

$$\begin{aligned} H = \tfrac{1}{2}\left(\frac{p_\mu^2}{k_1} + \frac{p_\nu^2}{k_2}\right) \quad &- \quad [k_1\mu p_\nu - k_2\nu p_\mu] \\ &+ \quad \tfrac{1}{2}[k_1(k_1k_2 + \tilde{\Omega}_1^2)\mu^2 + k_2(k_1k_2 + \tilde{\Omega}_2^2)\nu^2] \end{aligned}$$

Constants of the Motion

The numerical value of the Hamiltonian (energy) is

$$E_{\mu+\nu} = \tfrac{1}{2}(k_1\dot{\mu}^2 + k_2\dot{\nu}^2) + \tfrac{1}{2}(k_1\tilde{\Omega}_1^2\mu^2 + k_2\tilde{\Omega}_2^2\nu^2)$$

An additional constant of the motion can be found:

$$\begin{aligned} K_1 &\equiv \left(\frac{\lambda_2\dot{\mu}+\beta_2\nu}{\beta_1\lambda_2-\beta_2\lambda_1}\right)^2 + \left(\frac{\dot{\nu}-\beta_2\lambda_2\mu}{\beta_1\lambda_1-\beta_2\lambda_2}\right)^2 &= \mu_1^2\,, \\ K_2 &\equiv \left(\frac{\lambda_1\dot{\mu}+\beta_1\nu}{\beta_1\lambda_2-\beta_2\lambda_1}\right)^2 + \left(\frac{\dot{\nu}-\beta_1\lambda_1\mu}{\beta_1\lambda_1-\beta_2\lambda_2}\right)^2 &= \mu_2^2\,. \end{aligned}$$

Numerical tests confirm that K_1 and K_2 are constant.

Epi-ellipses

The complete solution for small amplitude is:

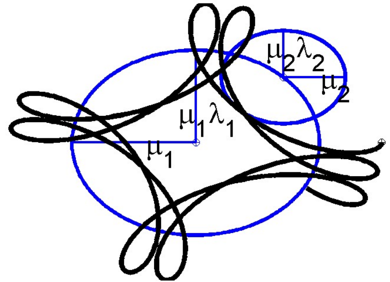
$$
\begin{aligned}
\gamma &= \cos \tfrac{1}{2}\omega_3 t \\
\zeta &= \sin \tfrac{1}{2}\omega_3 t \\
\mu &= \mu_1 \cos \beta_1 (t - t_1) + \mu_2 \cos \beta_2 (t - t_2) \\
\nu &= \lambda_1 \mu_1 \sin \beta_1 (t - t_1) + \lambda_2 \mu_2 \sin \beta_2 (t - t_2)
\end{aligned}
$$

There are two components, each an ellipse:

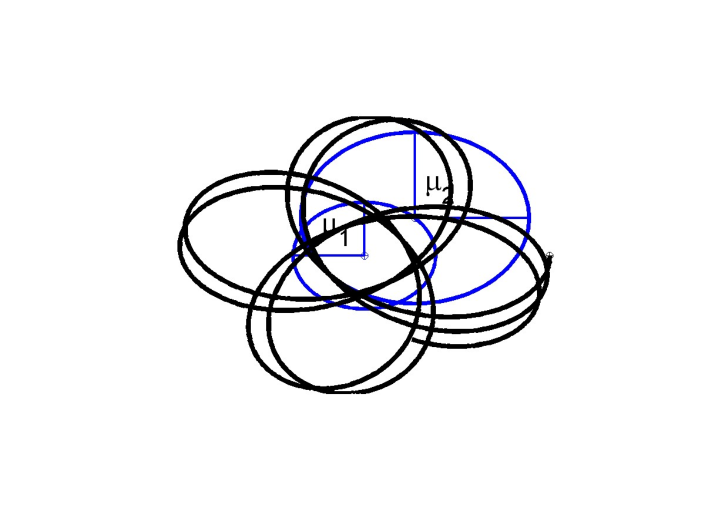
$$\mu = \mu_1 \cos[\beta_1 (t - t_1)] \,, \qquad \nu = \mu_1 \lambda_1 \sin[\beta_1 (t - t_1)] \,,$$

and

$$\mu = \mu_2 \cos[\beta_2 (t - t_2)] \,, \qquad \nu = \mu_2 \lambda_2 \sin[\beta_2 (t - t_2)] \,.$$



Rock'n'roller: epi-ellipse in the μ–ν-plane.



Rock'n'roller: epi-ellipse in the μ–ν-plane.

Criterion for Recession

The criterion for recession is:

$$(|\mu_1| - |\mu_2|)(|\lambda_1\mu_1| - |\lambda_2\mu_2|) < 0\,.$$

Routh Sphere

For the symmetric case ($\epsilon = 0$) the solution is:

$$\begin{aligned}\mu &= \mu_1 \cos \beta_1 (t - t_1) + \mu_2 \cos \beta_2 (t - t_2) \\ \nu &= \mu_1 \sin \beta_1 (t - t_1) - \mu_2 \sin \beta_2 (t - t_2)\end{aligned}$$

It follows immediately that

$$\mu^2 + \nu^2 = \mu_1^2 + \mu_2^2 + 2\mu_1\mu_2 \cos[(\beta_1 + \beta_2)t - b_{12}]$$

The absence of recession follows from:

$$||\mu_1| - |\mu_2|| \leq \sqrt{\mu^2 + \nu^2} \leq |\mu_1| + |\mu_2|$$

The accessible region is annular:
the angular momentum cannot change sign.

Constants of the motion

For the Routh Sphere there are two constants:

$$E_{\mu+\nu} = \frac{1}{2}(\dot{\mu}^2 + \dot{\nu}^2) + \frac{1}{2}\tilde{\Omega}^2(\mu^2 + \nu^2)$$

and

$$K = (\mu\dot{\nu} - \nu\dot{\mu}) + k(\mu^2 + \nu^2)$$

Constants of the motion

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$$E_{\mu+\nu} = \tfrac{1}{2}(\dot{\mu}^2 + \dot{\nu}^2) + \tfrac{1}{2}\tilde{\Omega}^2(\mu^2 + \nu^2)$$

and

$$K = (\mu\dot{\nu} - \nu\dot{\mu}) + k(\mu^2 + \nu^2)$$

Jellett's constant and Routh's constant are

$$Q_J = \mathbf{I}_1 s^2 \dot{\phi} + \mathbf{I}_3 f \omega_3 \qquad \textbf{and} \qquad Q_R = \omega_3/\rho$$

where $\rho = 1/\sqrt{\mathbf{I}_3 + s^2 + (\mathbf{I}_3/\mathbf{I}_1)f^2}$**.**

Routh and Jellett Constants

To $O(\theta^2)$, the Routh and Jellett constants are:

$$\begin{aligned}
\tilde{Q}_J &= (\mathbf{I_1}\theta^2\dot{\phi} + \mathbf{I_3} f_0\omega_3) - \tfrac{1}{2}\mathbf{I_3}\omega_3\theta^2 \\
\tilde{Q}_R &= \left[1 + \left(\frac{\mathbf{I_1} - \mathbf{I_3} f_0}{(\mathbf{I_1} + f_0^2)\mathbf{I_3}}\right)\frac{\theta^2}{2}\right]\frac{\omega_3}{\rho_0}
\end{aligned}$$

where $\rho_0 = 1/\sqrt{(\mathbf{I_1} + f_0^2)\mathbf{I_3}/\mathbf{I_1}}$.

We easily show that

$$K = \frac{1}{4\mathbf{I_1}}\left[\tilde{Q}_J - \mathbf{I_3} f_0\rho_0\tilde{Q}_R\right]$$

Epicycle character of solution

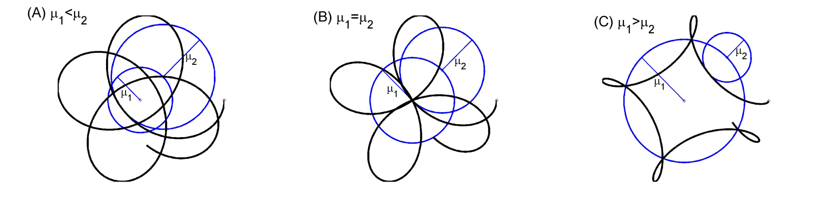
The solution has two components:

$$\textbf{(1)} \qquad \begin{aligned} \mu &= \mu_1 \cos[\beta_1(t - t_1)] \\ \nu &= \mu_1 \sin[\beta_1(t - t_1)] \end{aligned}$$

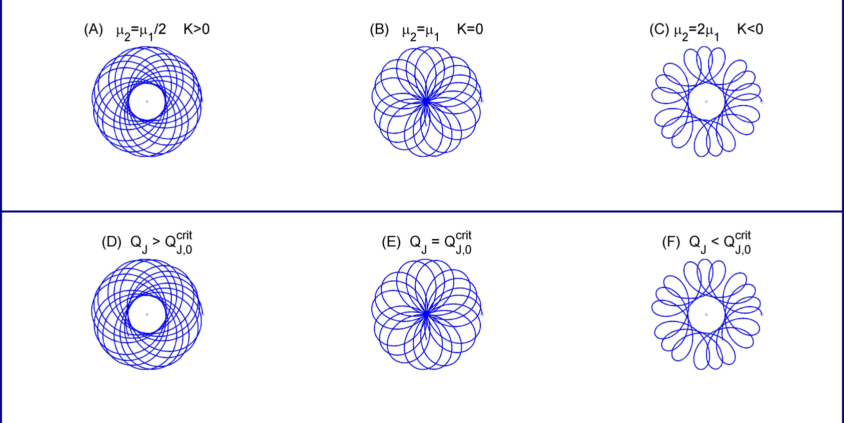
and

$$\textbf{(2)} \qquad \begin{aligned} \mu &= \mu_2 \cos[\beta_2(t - t_2)] \\ \nu &= -\mu_2 \sin[\beta_2(t - t_2)] \end{aligned}$$

The complete motion is thus an epicycle.



Routh Sphere:
Trajectories in the μ–ν-plane are epicycles.



Routh Sphere: trajectories in the θ–ϕ-plane are epicycles.
Panels (A)–(C): Analytical solutions
Panels (D)–(F): Numerical solutions.

Trajectory of Point of Contact

The movement of the geometric centre is:

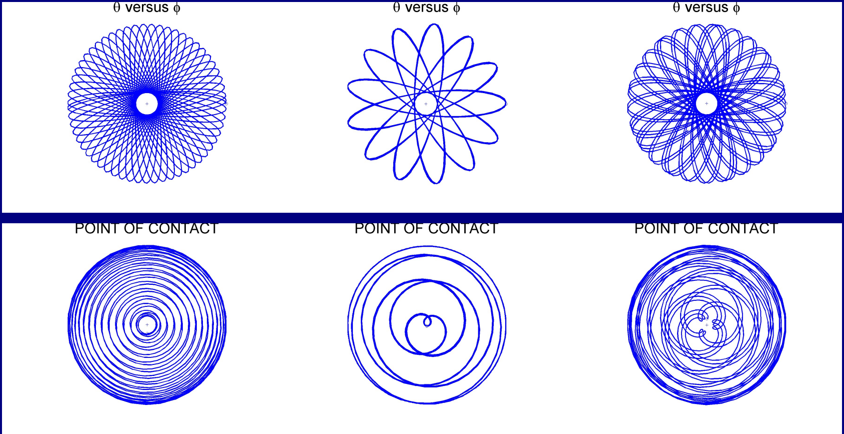
$$(\dot{X}, \dot{Y}, 0) = \omega \times \mathbf{K}$$

In terms of quaternions, this is

$$\begin{aligned} \dot{X} &= 2[\gamma\dot{\eta} - \eta\dot{\gamma} + \zeta\dot{\xi} - \xi\dot{\zeta}] \\ \dot{Y} &= 2[\xi\dot{\gamma} - \gamma\dot{\xi} + \zeta\dot{\eta} - \eta\dot{\zeta}] \end{aligned}$$

More explicitly:

$$\begin{aligned} X &= [2\mu_1\beta_1/\alpha_1]\sin(\alpha_1 t - \beta_1 t_1) - [2\mu_2\beta_2/\alpha_2]\sin(\alpha_2 t - \beta_2 t_2) \\ Y &= -[2\mu_1\beta_1/\alpha_1]\cos(\alpha_1 t - \beta_1 t_1) - [2\mu_2\beta_2/\alpha_2]\cos(\alpha_2 t - \beta_2 t_2) \end{aligned}$$



Top row: trajectories in θ–ϕ-plane.
Bottom row: plots of the point of contact.

Conclusion

Box and loop orbits are found in a wide range of physical systems. We illustrate them in the elementary context of a perturbed simple harmonic oscillator. Then, the dynamical equations for small amplitude motions of the Rock'n'roller are expressed in terms of quaternions. The complete solution is expressed as an *epi-ellipse*, a combination of two purely elliptic motions. This allows us to clarify the phenomenon of recession, and the conditions under which it occurs. In the particular case of a symmetric body ($\epsilon = 0$), the Routh Sphere, the solution reduces to an epicycle. Only loop orbits occur and there is no recession.

We have confined attention in the present study to the dynamics at first order in the polar angle θ. In an extension of this work, we will present a more detailed perturbation analysis, including a rigorous demonstration of energy conservation to second order, explicit expressions for the Routh and Jellett quantities Q_R and Q_J and a complete analysis of the recession of the Rock'n'roller.

One of the motivations for studying the Rock'n'roller is the hope of finding an invariant of the motion in addition to the energy. This expectation arises from the symmetry of the body. For the general Chaplygin Sphere, there is a finite angle δ between the principal axis corresponding to I_3 and the line joining the centres of gravity and symmetry. For the Rock'n'roller, this angle is zero and the Lagrangian is independent of the azimuthal angle ϕ. However, we have not found a second invariant and, considering the non-holonomic nature of the problem, its existence remains an open question.

Thank You