Resonant Rossby-Haurwitz Triads

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Outline

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Introduction

Barotropic Vorticity Equation (BVE)

The ENIAC Integrations

PHONIAC

Resonant Rossby-Haurwitz Triads

Forced Planetary Waves

Forced-damped Swinging Spring

Concluding Remarks

Selection of earlier publications

- Rossby et al. (1939); Haurwitz (1940)
- Charney et al. (1950): ENIAC integrations
- Fjørtoft (1953): energy/enstrophy cascade
- Lorenz (1960): “Maximum simplification”
- Platzman (1962): Spectral analysis
- Baines (1976): Resonant RH triads
- Reznik et al. (1993): More triads
- Newell et al. (2001), Chen et al (2005).

Outline

Barotropic Vorticity Equation (BVE)

- **Shallow, incompressible fluid on rotating sphere**
- **Horizontal velocity non-divergent**
- **Radius a, rotation rate Ω**
- **Longitude/latitude coordinates (λ, ϕ)**

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Dynamics governed by conservation of absolute vorticity

$$\frac{d}{dt}(\zeta + f) = 0\,.$$

$$f = 2\Omega \sin\phi \qquad \zeta = \mathbf{k} \cdot \nabla \times \mathbf{V}$$

$$f = \begin{bmatrix} \textbf{planetary} \\ \textbf{vorticity} \end{bmatrix} \qquad \zeta = \begin{bmatrix} \textbf{relative} \\ \textbf{vorticity} \end{bmatrix} \qquad f + \zeta = \begin{bmatrix} \textbf{absolute} \\ \textbf{vorticity} \end{bmatrix}$$

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Conservation of absolute vorticity:

$$\frac{d}{dt}(\zeta + f) = 0$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{u}{a\cos\phi}\frac{\partial}{\partial\lambda} + \frac{v}{a}\frac{\partial}{\partial\phi}\,.$$

Introducing a stream-function, we get:

$$\mathbf{V} = \mathbf{k} \times \nabla\psi \qquad\qquad \zeta = \nabla^2\psi$$

and the vorticity equation becomes:

$$\frac{\partial\nabla^2\psi}{\partial t} + \frac{2\Omega}{a^2}\frac{\partial\psi}{\partial\lambda} + \frac{1}{a^2}\frac{\partial(\psi,\nabla^2\psi)}{\partial(\lambda,\mu)} = 0$$

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where $\mu = \sin\phi$.

This is the non-divergent barotropic vorticity equation

The Jacobian term represents non-linear advection.

Omitting the nonlinear term, the BVE has solutions

$$\psi = \psi_0 Y_n^m(\lambda, \mu) \exp(-i\sigma t) = \psi_0 P_n^m(\mu) \exp[i(m\lambda - \sigma t)]$$

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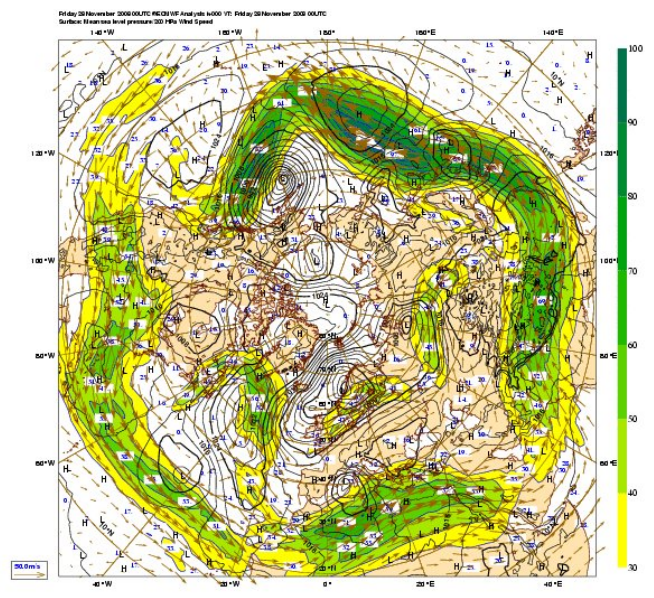
The functions $Y_n^m(\lambda,\mu)$ are eigenfunctions of the Laplacian operator on the sphere:

$$\nabla^2 Y_n^m = -\frac{n(n+1)}{a^2} Y_n^m .$$

These solutions are called Rossby-Haurwitz waves.



Friday 28 November 2008 00UTC ©ECMWF Analysis t+000 VT: Friday 28 November 2008 00UTC
Surface: Mean sea level pressure/300 HPa Wind Speed
140°W
160°W
180°
160°E
140°E
120°W
100°W
80°W
60°W
40°W
20°W
0°
20°E
40°E
50.0m/s
100
90
80
70
60
50
40
30

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This is not generally true for a combination of such waves: the velocity of one component will advect the vorticity of another so that the waves interact and their amplitudes change.

The spherical harmonics form an orthonormal basis on the sphere: Thus, the stream function has an expansion

$$\psi(\lambda, \mu, t) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \psi_n^m(t) Y_n^m(\lambda, \mu) .$$

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Defining a vector wavenumber $\gamma = (m,n)$ and its conjugate by $\bar{\gamma} = (-m,n)$. We can write

$$\psi = \sum_{\gamma} \psi_\gamma(t) Y_\gamma(\lambda,\mu) e^{-i\sigma_\gamma t} \qquad \zeta = \sum_{\gamma} \zeta_\gamma(t) Y_\gamma(\lambda,\mu) e^{-i\sigma_\gamma t}$$

with $\psi_\gamma = -a^2 \kappa_\gamma \zeta_\gamma$, where $\kappa_\gamma = 1/(n(n+1))$.

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with $\psi_\gamma = -a^2\kappa_\gamma\zeta_\gamma$, where $\kappa_\gamma = 1/(n(n+1))$.

If the nonlinear interactions are weak, the coefficients will vary slowly with time compared to $\exp(-i\sigma_\gamma t)$.

Flows governed by the BVE conserve total energy and total enstrophy:

$$E = \frac{1}{4\pi a^2}\iint \tfrac{1}{2}\mathbf{V}\cdot\mathbf{V}\,d\lambda\,d\mu = -\frac{1}{4\pi a^2}\iint \tfrac{1}{2}\psi\zeta\,d\lambda\,d\mu$$

$$S = \frac{1}{4\pi a^2}\iint \tfrac{1}{2}\zeta^2\,d\lambda\,d\mu = -\frac{1}{4\pi a^2}\iint \tfrac{1}{2}\nabla\psi\cdot\nabla\zeta\,d\lambda\,d\mu$$

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In terms of the spectral coefficients, these are:

$$E = \tfrac{1}{2}\sum_\gamma \kappa_\gamma |\zeta_\gamma|^2 , \qquad S = \tfrac{1}{2}\sum_\gamma |\zeta_\gamma|^2 .$$

The constancy of energy and enstrophy profoundly influences the energetics of solutions of the barotropic vorticity equation.

We can write equations for the evolution of the spectral coefficients:

$$\frac{d\zeta_\gamma}{dt} = \tfrac{1}{2} i \sum_{\alpha,\beta} I_{\gamma\beta\alpha}\zeta_\beta\zeta_\alpha \exp(-i\sigma t)\,,$$

where $\sigma = \sigma_\alpha + \sigma_\beta - \sigma_\gamma$ and the interaction coefficients are given by

$$I_{\gamma\beta\alpha} = (\kappa_\beta - \kappa_\alpha) K_{\gamma\beta\alpha}\,.$$

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$$I_{\gamma\beta\alpha} = (\kappa_\beta - \kappa_\alpha) K_{\gamma\beta\alpha}\,.$$

The coupling integrals $K_{\gamma\beta\alpha}$ vanish unless $m_\alpha + m_\beta = m_\gamma$, when they they are given by

$$K_{\gamma\beta\alpha} = \tfrac{1}{2} \int_{-1}^{+1} P_\gamma \left(m_\beta P_\beta \frac{dP_\alpha}{d\mu} - m_\alpha P_\alpha \frac{dP_\beta}{d\mu} \right) d\mu\,.$$

Selection rules

For non-vanishing interaction, the following selection rules must be satisfied:

$$
\begin{array}{rcl}
m_\alpha + m_\beta & = & m_\gamma \\
m_\alpha^2 + m_\beta^2 & \neq & 0 \\
n_\gamma n_\beta n_\alpha & \neq & 0 \\
n_\alpha & \neq & n_\beta \\
n_\alpha + n_\beta + n_\gamma & \text{is} & \text{odd} \\
(n_\beta - |m_\beta|)^2 + (n_\alpha - |m_\alpha|)^2 & \neq & 0 \\
|n_\alpha - n_\beta| < & n_\gamma & < n_\alpha + n_\beta \\
(m_\beta, n_\beta) \neq (-m_\gamma, n_\gamma) & \text{and} & (m_\alpha, n_\alpha) \neq (-m_\gamma, n_\gamma)
\end{array}
$$

Symmetries: $I_{\gamma\alpha\beta} = I_{\gamma\beta\alpha} \quad \text{and} \quad K_{\gamma\alpha\beta} = -K_{\gamma\beta\alpha}\,.$

Redundancy rules: $K_{\alpha\bar{\beta}\gamma} = K_{\gamma\beta\alpha} \quad \text{and} \quad K_{\beta\gamma\bar{\alpha}} = K_{\gamma\beta\alpha}\,.$

Outline

Crucial Advances, 1920–1950

- Dynamic Meteorology
  - Rossby Waves
  - Quasi-geostrophic Theory
  - Baroclinic Instability

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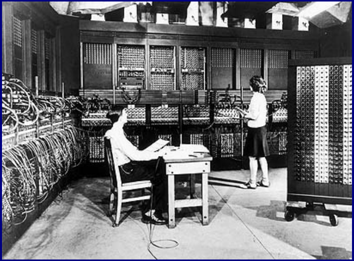
- Electronic Computing
  - ENIAC

The ENIAC



The ENIAC was the first multi-purpose programmable electronic digital computer.

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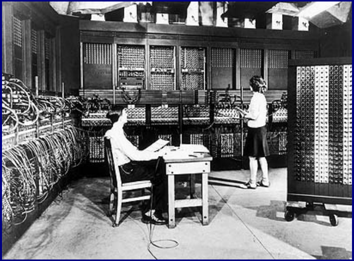

The ENIAC was the first multi-purpose programmable electronic digital computer.

It had:

- 18,000 vacuum tubes
- 70,000 resistors
- 10,000 capacitors
- 6,000 switches
- Power: 140 kWatts

Charney, et al., *Tellus*, 1950.

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- The atmosphere is treated as a single layer.
- The flow is assumed to be nondivergent.
- Absolute vorticity is conserved.

$$\boxed{\frac{\mathrm{d}(\zeta + \mathrm{f})}{\mathrm{dt}} = 0.}$$

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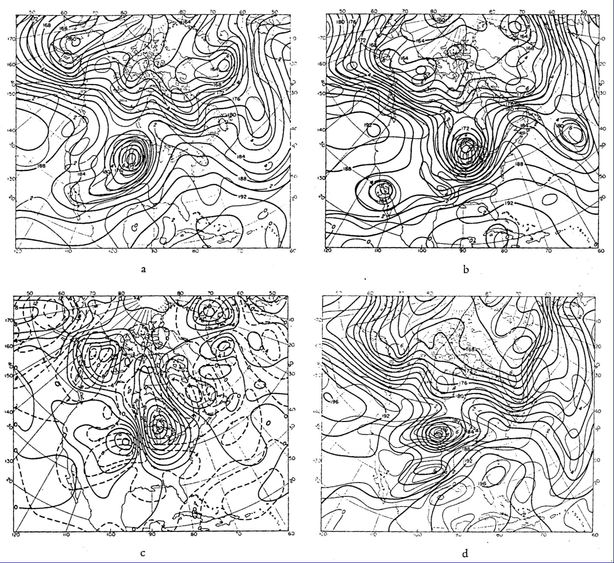
- **The flow is assumed to be nondivergent.**
- **Absolute vorticity is conserved.**

$$\boxed{\frac{\mathbf{d}(\zeta + \mathbf{f})}{\mathbf{dt}} = \mathbf{0}.}$$

This equation looks simple. But it is nonlinear:

$$\frac{\partial}{\partial t}[\nabla^2\psi] + \left\{\frac{\partial\psi}{\partial x}\frac{\partial\nabla^2\psi}{\partial y} - \frac{\partial\psi}{\partial y}\frac{\partial\nabla^2\psi}{\partial x}\right\} + \beta\frac{\partial\psi}{\partial x} = 0\,,$$

ENIAC Forecast for Jan 5, 1949



NWP Operations

The Joint Numerical Weather Prediction Unit was established on July 1, 1954:

- Air Weather Service of US Air Force
- The US Weather Bureau
- The Naval Weather Service.

Operational numerical weather forecasting began in May, 1955, using a three-level quasi-geostrophic model.

Recreating the ENIAC Forecasts

The ENIAC integrations have been recreated using:

- A **MATLAB** program to solve the BVE
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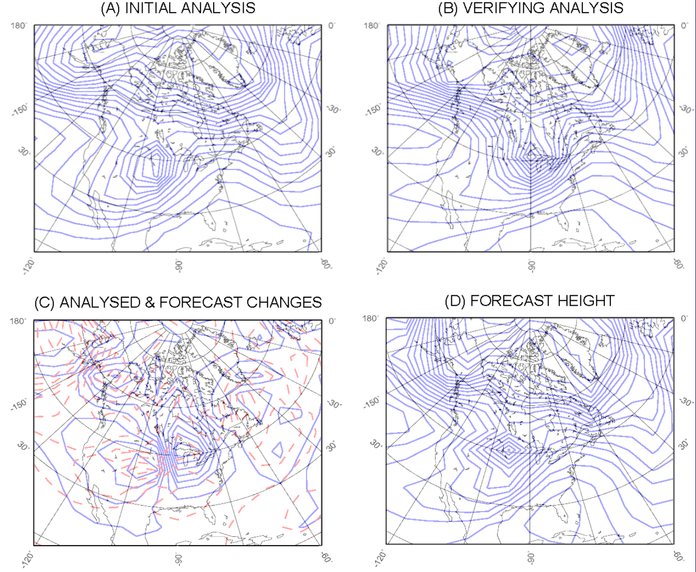
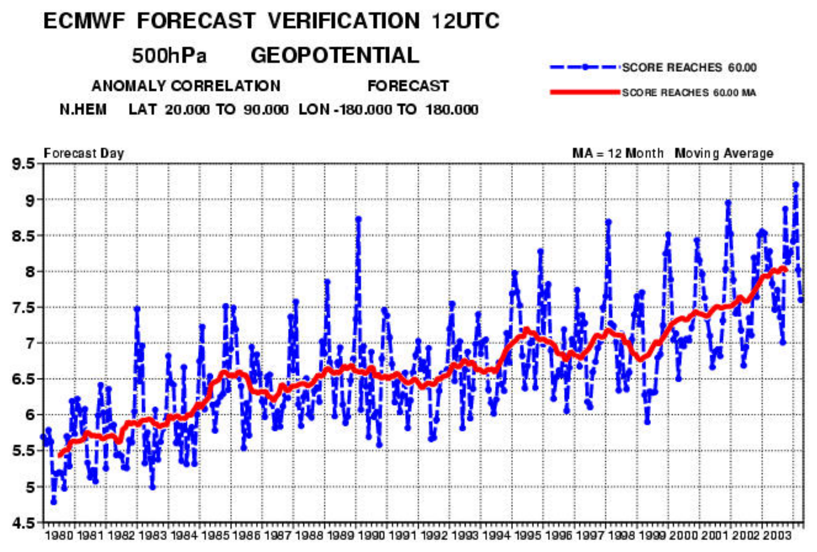
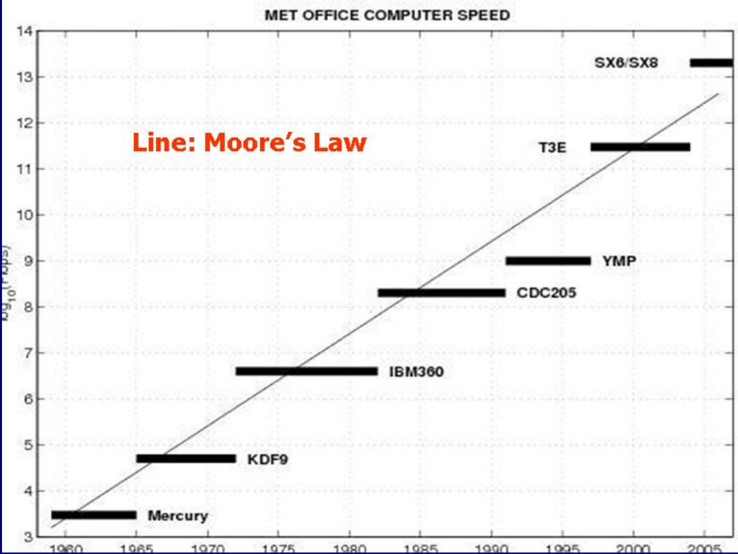
The matlab code is available on the author's website
http://maths.ucd.ie/∼plynch/eniac

Recreation of the Forecast



Increase in Forecasting Skill




MET OFFICE COMPUTER SPEED
Line: Moore's Law
SX6/SX8
T3E
YMP
CDC205
IBM360
KDF9
Mercury
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13
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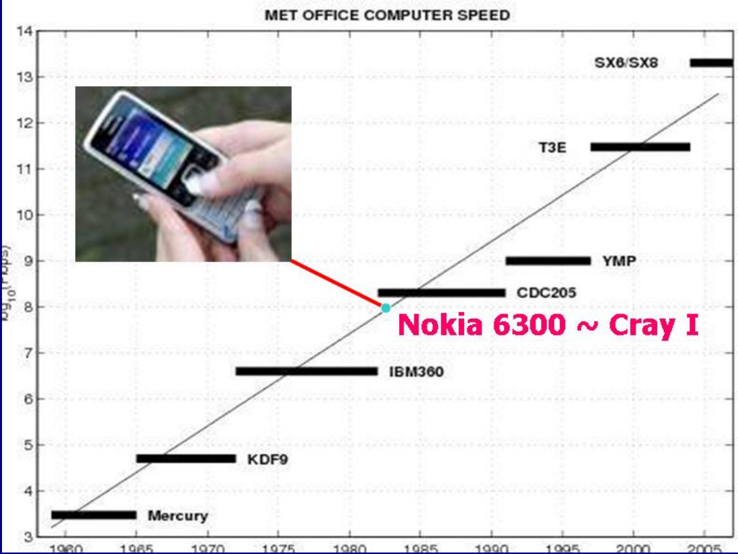
MET OFFICE COMPUTER SPEED
SX6/SX8
T3E
YMP
CDC205
Nokia 6300 ~ Cray I
IBM360
KDF9
Mercury
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1960
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Forecasts by PHONIAC

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We therefore decided to repeat the ENIAC integrations using a programmable mobile phone.

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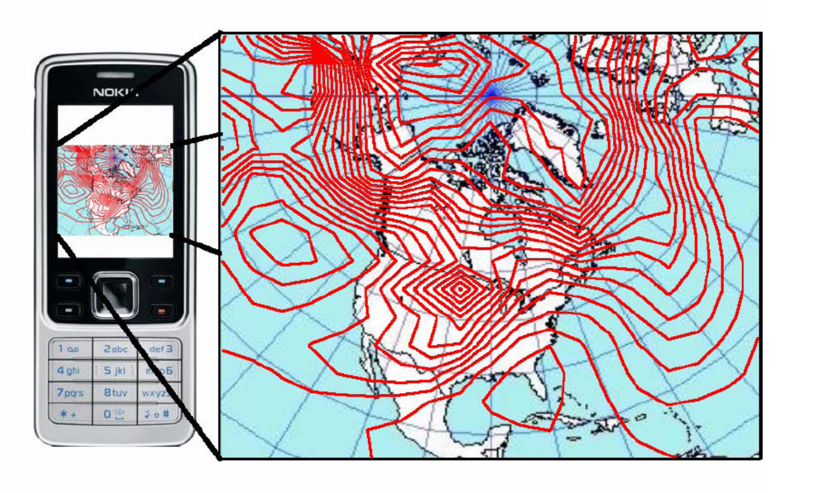
We therefore decided to repeat the ENIAC integrations using a programmable mobile phone.

We converted the program ENIAC.M **to** PHONIAC.JAR**, a J2ME application, and implemented it on a mobile phone.**

This technology has great potential for generation and delivery of operational weather forecast products.

PHONIAC: Portable Hand Operated Numerical Integrator and Computer



[See *Weather* magazine for November]

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Resonant RH triads

We now investigate truncated solutions.

Under certain circumstances, the interactions are so weak that the simple low-order structure persists.

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Under certain circumstances, the interactions are so weak that the simple low-order structure persists.

We consider the case where there are just three non-vanishing spectral components:

$$\begin{aligned}\psi &= \Re\{\psi_\alpha Y_\alpha \exp(-i\sigma_\alpha t) + \\ &\qquad \psi_\beta Y_\beta \exp(-i\sigma_\beta t) + \psi_\gamma Y_\gamma \exp(-i\sigma_\gamma t)\}\,.\end{aligned}$$

The selection rules then imply that the only non-vanishing interaction coefficients are:

$$I_{\gamma\beta\alpha} = I_{\gamma\alpha\beta} \qquad I_{\beta\bar{\alpha}\gamma} = I_{\beta\gamma\bar{\alpha}} \qquad I_{\alpha\bar{\beta}\gamma} = I_{\alpha\gamma\bar{\beta}}\,.$$

Using the symmetries and redundancy rules, all the coefficients can be expressed in terms of one, K:

$$
\begin{aligned}
i\dot{\zeta}_\alpha &= -(\kappa_\beta - \kappa_\gamma)K\zeta_\beta^*\zeta_\gamma \exp(+i\sigma t)\\
i\dot{\zeta}_\beta &= -(\kappa_\gamma - \kappa_\alpha)K\zeta_\gamma\zeta_\alpha^* \exp(+i\sigma t)\\
i\dot{\zeta}_\gamma &= +(\kappa_\alpha - \kappa_\beta)K\zeta_\alpha\zeta_\beta \exp(-i\sigma t)
\end{aligned}
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where $K = K_{\gamma\beta\alpha}$ and $\sigma = \sigma_\alpha + \sigma_\beta - \sigma_\gamma$.

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...unless $\sigma = 0$: this is the case of resonance.

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... unless $\sigma = 0$: this is the case of resonance.

We consider only the resonant case below.

The condition for resonance, $\sigma = 0$, may be written

$$m_\alpha \kappa_\alpha + m_\beta \kappa_\beta = m_\gamma \kappa_\gamma \, .$$

We consider the generic case:

$$\kappa_\alpha > \kappa_\gamma > \kappa_\beta \, .$$

Thus, $n_\alpha < n_\gamma < n_\beta$, so that the component ζ_γ is of a scale intermediate between the others (Fjørtoft, ’53).

The equations may now be written

$$\begin{aligned} i\dot{\zeta}_\alpha &= k_\alpha \zeta_\beta^* \zeta_\gamma \\ i\dot{\zeta}_\beta &= k_\beta \zeta_\gamma \zeta_\alpha^* \\ i\dot{\zeta}_\gamma &= k_\gamma \zeta_\alpha \zeta_\beta \end{aligned}$$

where, assuming $K > 0$, the coefficients

$$k_\alpha = (\kappa_\gamma - \kappa_\beta)K\,, \quad k_\beta = (\kappa_\alpha - \kappa_\gamma)K\,, \quad k_\gamma = (\kappa_\alpha - \kappa_\beta)K$$

are all positive and $k_\alpha + k_\beta = k_\gamma$.

The energy and enstrophy of the triad may be written:

$$
\begin{aligned}
E &= \tfrac{1}{2}\left(\kappa_\alpha|\zeta_\alpha|^2 + \kappa_\beta|\zeta_\beta|^2 + \kappa_\gamma|\zeta_\gamma|^2\right) \\
S &= \tfrac{1}{2}\left(|\zeta_\alpha|^2 + |\zeta_\beta|^2 + |\zeta_\gamma|^2\right).
\end{aligned}
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We now introduce the transformation

$$\eta_\alpha = \sqrt{k_\beta k_\gamma}\,\zeta_\alpha\,, \qquad \eta_\beta = \sqrt{k_\gamma k_\alpha}\,\zeta_\beta\,, \qquad \eta_\gamma = \sqrt{k_\alpha k_\beta}\,\zeta_\gamma\,,$$

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The equations then assume the standard form:

$$\begin{aligned} i\dot\eta_\alpha &= \eta_\beta^*\eta_\gamma \\ i\dot\eta_\beta &= \eta_\gamma\eta_\alpha^* \\ i\dot\eta_\gamma &= \eta_\alpha\eta_\beta \end{aligned}$$

These are the **three-wave equations.**

Energy and enstrophy are conserved for the triad.

The Manley-Rowe quantities are defined as

$$
\begin{aligned}
N_1 &= |\eta_\alpha|^2 + |\eta_\gamma|^2 \\
N_2 &= |\eta_\beta|^2 + |\eta_\gamma|^2 \\
J &= |\eta_\alpha|^2 - |\eta_\beta|^2 \,.
\end{aligned}
$$

They are all constants of the motion.

Energy and enstrophy are conserved for the triad.

The Manley-Rowe quantities are defined as

$$\begin{aligned} N_1 &= |\eta_\alpha|^2 + |\eta_\gamma|^2 \\ N_2 &= |\eta_\beta|^2 + |\eta_\gamma|^2 \\ J &= |\eta_\alpha|^2 - |\eta_\beta|^2 \,. \end{aligned}$$

They are all constants of the motion.

The system may be shown to be the canonical equations arising from the Hamiltonian $H = \Re\{\eta_\alpha\eta_\beta\eta_\gamma^*\}$

Numerical Example

We integrated the BVE with an IC dominated by mode RH(4,5).

This is the mode that Hoskins (1973) suggested was stable but that Thuburn & Li (2000) found to be unstable.

The triad $(4,5)$, $(1,3)$ $(3,7)$ comes close to satisfying the frequency criterion for resonance.

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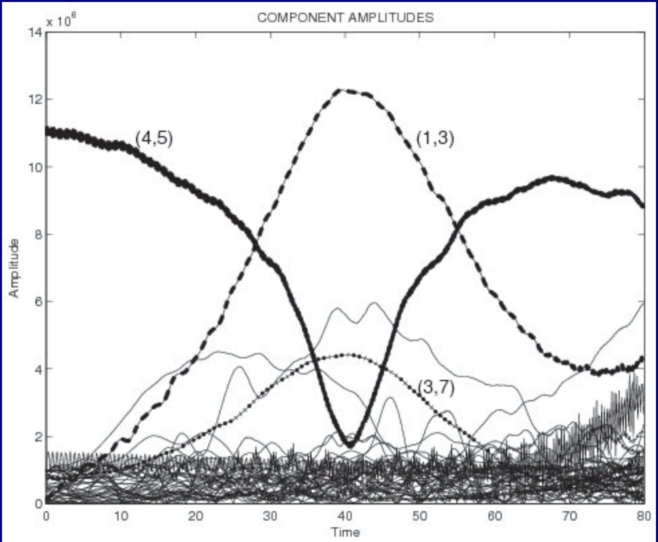
In the following figure, we show the evolution of the component amplitudes over 80 days.



Evolution of component amplitudes over 80 days.

Outline

Forced Planetary Waves

We now include forcing by orography and damping towards a reference state with potential vorticity f/H.

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The BPVE may be written

$$\frac{d}{dt}\left(\frac{\zeta+f}{H-h_0}\right) = -\nu\left(\frac{\zeta+f}{H-h_0} - \frac{f}{H}\right)$$

where H is the mean height, h_0 the elevation of the orography and ν is the damping coefficient.

The flow is separated into a zonal super-rotation $\bar{u} = a\cos\phi\,\bar{\omega}$ with constant $\bar{\omega}$, and a perturbation (u, v).

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Assuming that the orography is small, $h_0 \ll H$, we can write the equation as

$$\left(\frac{\partial}{\partial t} + \bar{\varpi}\frac{\partial}{\partial \lambda}\right)\zeta + \frac{2\Omega}{a^2}\frac{\partial \psi}{\partial \lambda} + \frac{1}{a^2}\frac{\partial(\psi, \zeta)}{\partial(\lambda, \mu)} - \frac{\bar{\varpi} f}{H}\frac{\partial h_0}{\partial \lambda} = -\nu\left(\zeta - \frac{f h_0}{H}\right)$$

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The linear normal modes have eigen-frequencies

$$\sigma_n^m = \bar{\omega} - \frac{(2\Omega + \bar{\omega})m}{n(n+1)} .$$

Bounded response to forcing

If $\bar{\omega}$ is such that σ_n^m vanishes for some (m, n), the orographic forcing leads to a solution that grows linearly with time until equilibrated by the damping.

In the absence of damping, it grows without limit.

Bounded response to forcing

If $\bar{\omega}$ is such that σ_n^m vanishes for some (m, n), the orographic forcing leads to a solution that grows linearly with time until equilibrated by the damping.

In the absence of damping, it grows without limit.

However, as the amplitude increases, nonlinear interactions transfer energy to other modes and it is possible to have a bounded response to constant orographic forcing.

This is the case we study below.

We seek a solution in the form of a resonant triad

$$\psi = \Re\{\psi_\alpha Y_\alpha \exp(-i\sigma_\alpha t) + \psi_\beta Y_\beta \exp(-i\sigma_\beta t) + \psi_\gamma Y_\gamma \exp(-i\sigma_\gamma t)\}$$

with $\sigma_\alpha + \sigma_\beta = \sigma_\gamma$**.**

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with $\sigma_\alpha + \sigma_\beta = \sigma_\gamma$**.**

Assuming that the solution is of small amplitude ϵ**, we expand the streamfunction as**

$$\psi = \epsilon\psi_1 + \epsilon^2\psi_2 + \epsilon^3\psi_3 + \ldots$$

The nonlinear term involving $J(\psi, fh_0/H)$ **does not enter at** $O(\epsilon^2)$**.**

The damping coefficient ν **is** $O(\epsilon)$**.**

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We assume that the orography (actually, fh_0) has the same spatial structure $Y_\gamma(\lambda, \phi)$ as the γ-term, and

$$\bar{\omega} = \frac{(2\Omega + \bar{\omega})m_\gamma}{n_\gamma(n_\gamma + 1)} \qquad \text{or} \qquad \bar{\omega} = \frac{2\Omega m_\gamma \kappa_\gamma}{1 - m_\gamma \kappa_\gamma} .$$

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At order ϵ, the equations are linear and unforced, so the three components evolve independently.

At order ϵ^2, the forcing, damping and nonlinearity enter, and the equations at this level are

$$\begin{aligned}
\dot{\zeta}_\alpha &= -(\kappa_\beta - \kappa_\gamma) K \zeta_\beta^* \zeta_\gamma^* - \nu \zeta_\alpha \\
\dot{\zeta}_\beta &= -(\kappa_\gamma - \kappa_\alpha) K \zeta_\gamma^* \zeta_\alpha^* - \nu \zeta_\beta \\
\dot{\zeta}_\gamma &= +(\kappa_\alpha - \kappa_\beta) K \zeta_\alpha^* \zeta_\beta^* - \nu \zeta_\gamma + F
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\end{aligned}
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where the coefficient F is a constant proportional to the magnitude of the orographic forcing.

Introducing a transformation as before, we get the **forced-damped three-wave equations:**

$$
\begin{aligned}
i\dot{\eta}_\alpha &= \eta_\beta^* \eta_\gamma - i\nu\eta_\alpha \\
i\dot{\eta}_\beta &= \eta_\gamma \eta_\alpha^* - i\nu\eta_\beta \\
i\dot{\eta}_\gamma &= \eta_\alpha \eta_\beta - i\nu\eta_\gamma + iF
\end{aligned}
$$

The quantities J, $N (= N_1 + N_2)$ and H are no longer conserved quantities, but obey the equations

$$
\begin{aligned}
\dot{J} &= -2\nu J \,, \\
\dot{N} &= -2\nu N + 2\Re\{F^* \eta_\gamma\} \,, \\
\dot{H} &= -3\nu N + 2\Re\{F^* \eta_\alpha \eta_\beta\} \,.
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Note that the energy quantity N may increase or decrease in response to the forcing F, depending on the phase relationship between F and η_γ.

Numerical Example

We integrated the BVE with orographic forcing of a single spectral component, RH(3,9).

The mean flow $\bar{\omega}$ is set so that this mode is stationary.

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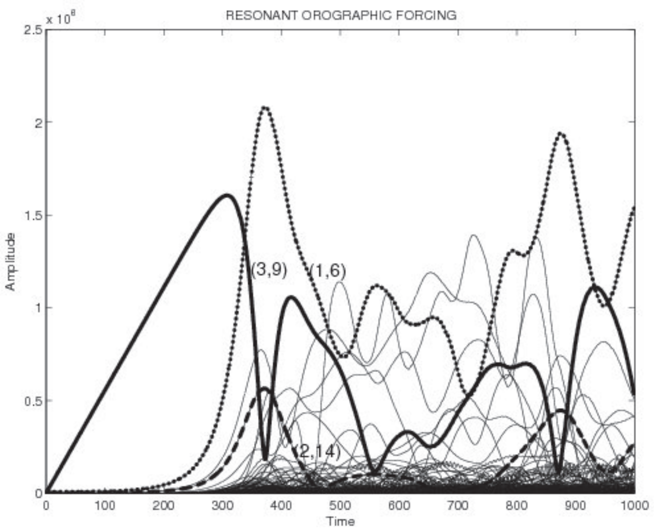
In the figure below, we show the component amplitudes for weak orographic forcing.



Evolution of component amplitudes over 80 days.

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Extended integrations confirm this behaviour.

Outline

Free Rossby wave triads in the atmosphere can be modelled by an elastic pendulum or swinging spring (Lynch, 2003).

At a certain level of approximation, the equations of the two systems are mathematically isomorphic.

Thus, behaviour such as the precession of successive horizontal excursions of the spring indicated similar behaviour in the atmosphere.

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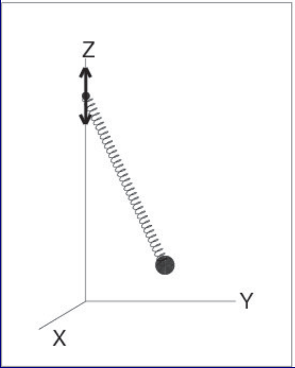
We extend this correspondence here to include forcing and damping.

Forced-damped swinging spring



We consider a swinging spring whose point of suspension oscillates vertically with period ω_Z.

ℓ_0 is unsteretched length
ℓ length at equilibrium
k is spring constant
$m = 1$ is unit mass.

The Lagrangian, approximated to cubic order, is

$$\begin{aligned}\mathcal{L} &= \tfrac{1}{2}[\dot{x}^2 + \dot{y}^2 + (\dot{z}^2 + 2\dot{z}\dot{\zeta} + \dot{\zeta}^2)] \\ &\qquad -\tfrac{1}{2}[\omega_R^2(x^2 + y^2) + \omega_Z^2 z^2] - \tfrac{1}{2}\lambda(x^2 + y^2)z\,.\end{aligned}$$

where x, y and z are Cartesian coordinates centered at the point of equilibrium.

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where x, y and z are Cartesian coordinates centered at the point of equilibrium.

$\zeta(t) = \Re\{\zeta_0 e^{i\omega_Z t}\}$ is displacement of suspension point

$\omega_R = (g/\ell)^{1/2}$ is frequency of pendular motion

$\omega_Z = (k/m)^{1/2}$ is frequency of elastic oscillations

$\lambda = \ell_0\omega_Z^2/\ell^2$ is a parameter.

Damping is introduced through a Rayleigh dissipation function

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Lagrange's equations then become

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}}\right) - \frac{\partial\mathcal{L}}{\partial q} + \frac{\partial\mathcal{F}}{\partial\dot{q}} = 0 ,$$

where q $= (x, y, z)$**.**

The motion of the suspension point introduces an inhomogeneous term $-\ddot{\zeta}$ **into the** z**-equation.**

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where $\mathbf{q} = (x, y, z)$.

The motion of the suspension point introduces an inhomogeneous term $-\ddot{\zeta}$ into the z-equation.

We employ the average Lagrangian technique to obtain an approximate solution.

We confine attention to the resonant case $\omega_Z = 2\omega_R$. The solution is assumed to be of the form

$$
\begin{aligned}
x &= \Re\{a(t)\exp(i\omega_R t)\}\,, \\
y &= \Re\{b(t)\exp(i\omega_R t)\}\,, \\
z &= \Re\{c(t)\exp(i\omega_Z t)\}\,.
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The time scale of variation of a, b and c is much longer than $\tau = 2\pi/\omega_R$.

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The time scale of variation of a, b and c is much longer than $\tau = 2\pi/\omega_R$.

If the Lagrangian and the dissipation function are averaged over time τ, the amplitude equations are

$$
\begin{aligned}
i\dot{a} &= -\mu a^* c - i\nu a \\
i\dot{b} &= -\mu b^* c - i\nu b \\
i\dot{c} &= -\tfrac{1}{4}\mu(a^2 + b^2) - i\nu c + \tfrac{1}{2}\omega_Z \zeta_0\,.
\end{aligned}
$$

where $\mu = \lambda/4\omega_R$.

Defining new variables by

$$\alpha = \tfrac{1}{2}\mu(a + ib)\,, \qquad \beta = \tfrac{1}{2}\mu(a - ib)\,, \qquad \gamma = \mu c$$

the equations for the envelope dynamics become

$$\begin{aligned} i\dot{\alpha} &= \beta^*\gamma - i\nu\alpha \\ i\dot{\beta} &= \gamma\alpha^* - i\nu\beta \\ i\dot{\gamma} &= \alpha\beta - i\nu\gamma + iF\,, \end{aligned}$$

where $F = -\frac{1}{2}i\mu\omega_Z\zeta_0$ represents the external forcing.

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where $F = -\frac{1}{2}i\mu\omega_Z\zeta_0$ represents the external forcing.

This system is isomorphic to the system for a forced-damped resonant Rossby triad.

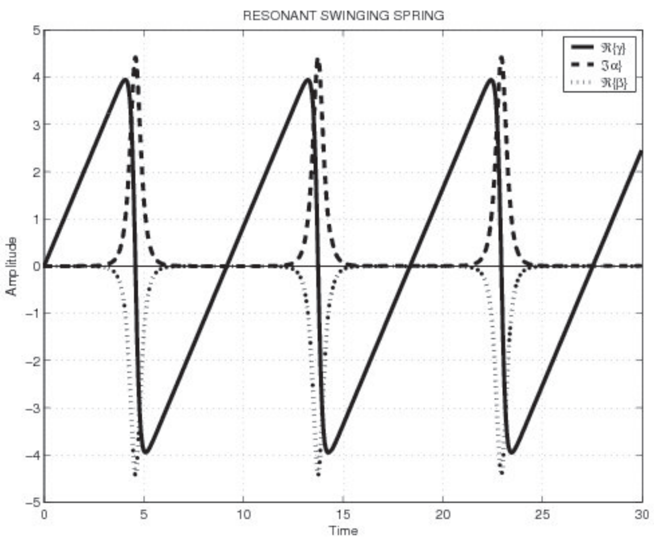
Numerical Example

We integrated the system over thirty time units, with unit forcing $F = 1$ and no damping

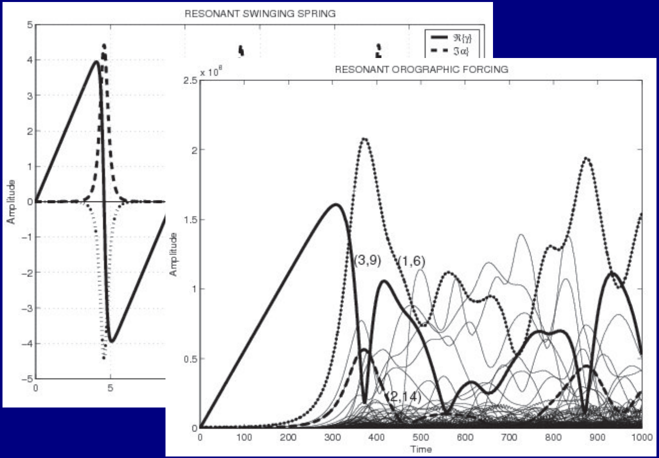
The initial conditions are

$$\begin{aligned} \alpha_0 &= (+0.0005, 0.0000)\,, \\ \beta_0 &= (-0.0005, 0.0005)\,, \\ \gamma_0 &= (+0.0000, 0.0000)\,. \end{aligned}$$

The amplitudes of the components (real and imaginary parts) are shown in the figure below. Initially, the forced component, γ, grows linearly.



Amplitudes of α, β and γ. Components $\Im\{\alpha\}$, $\Re\{\beta\}$ and $\Re\{\gamma\}$ are shown bold. Other amplitudes remain small.


RESONANT SWINGING SPRING
Amplitude
RESONANT OROGRAPHIC FORCING
x 10^8
Amplitude
(3,9)
(1,6)
(2,14)
Time

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Full account to appear in *Tellus*.

Outline

Concluding remarks

- Resonant triads can explain the instability of large-scale RH waves.
- A constant forcing can lead to a periodic response, even in the absence of damping.
- There is a mathematical equivalence between forced resonant RH triads and the forced-damped swinging spring.

Concluding remarks

- Resonant triads can explain the instability of large-scale RH waves.
- A constant forcing can lead to a periodic response, even in the absence of damping.
- There is a mathematical equivalence between forced resonant RH triads and the forced-damped swinging spring.
- Triad interactions are important in establishing and maintaining the atmospheric enegy spectrum.
- These interactions can account for quasi-periodic variations of long time-scale.
- An examination of the spectral characteristics of ERA40 (triads) data would be of great interest.

Thank You