Parity and Partition of the Rational Numbers

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Irish Mathematical Society Meeting
Technical University of Dublin, 2 September 2022

Outline

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Synopsis

The concept of parity is extended from the integers to the rational numbers. Three parity classes are found — odd, even and 'none'.

Using the 2-adic valuation, we partition the rationals into subgroups with a rich algebraic structure.

The Calkin-Wilf tree has a remarkably simple parity pattern, with the sequence odd/none/even repeating indefinitely.

A similar result holds for the Stern-Brocot tree.

Outline

Partitioning the Rational Numbers

The natural numbers $\mathbb{N}$ split nicely into two subsets:

$$\begin{aligned} \mathbb{N}_{\mathrm{O}} &= \{1, 3, 5, 7, \dots\} \\ \mathbb{N}_{\mathrm{E}} &= \{2, 4, 6, 8, \dots\}. \end{aligned}$$

The odd and even numbers are equinumerous.

A similar split applies to the integers $\mathbb{Z}$:

$$\begin{aligned} \mathbb{Z}_{\mathrm{O}} &= \{\dots -3, -1, +1, +3, +5, \dots\} \\ \mathbb{Z}_{\mathrm{E}} &= \{\dots -4, -2, 0, +2, +4, \dots\}. \end{aligned}$$

The integers form an abelian group $(\mathbb{Z}, +)$.
$\mathbb{Z}_{\mathrm{E}}$ is an additive subgroup of $(\mathbb{Z}, +)$.
It is of index 2, with cosets $\mathbb{Z}_{\mathrm{E}}$ and $\mathbb{Z}_{\mathrm{E}} + 1$.

Parity

The distinction between odd and even is called parity.

Parity is defined for the integers (whole numbers).
Can we extend the concept of parity to the rationals?

The usual ‘rules’ of parity might be required:

1. Sum of even numbers is even; product is even.
2. Sum of odd numbers is even; product is odd.
3. Sum of even and odd is odd; product is even.
4. Odd number plus 1 is even; even plus 1 is odd.

Rules of Parity

Table: Addition (left) and multiplication (right) tables for $\mathbb{Z}$.

$+$	**even**	**odd**
even	*even*	*odd*
odd	*odd*	*even*

$\times$	**even**	**odd**
even	*even*	*even*
odd	*even*	*odd*

Even and Uneven

For $\mathbb{Q}$, we could define a number $q = m/n$ to be even if the numerator m is even and odd if m is odd.

But then $\frac{1}{4} + \frac{1}{4} = \frac{1}{2}$, meaning that two odd rationals might add to yield another odd one.

We distinguish between 'odd' and 'uneven':

$$\text{For } q = m/n\,, \qquad \begin{cases} q \text{ is even} & \text{if } m \text{ is even}, \\ q \text{ is uneven} & \text{if } m \text{ is odd}\,. \end{cases}$$

Numerical Evidence

A MATHEMATICA program was written to count the number of even and uneven rationals in $(0, 1)$.

We can list all rationals in $(0, 1)$ in a sequence:

$$\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{6}, \frac{5}{6}, \frac{1}{7}, \frac{2}{7}, \frac{3}{7}, \frac{4}{7}, \frac{5}{7}, \frac{6}{7}, \frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}, \ldots$$

As $n \to \infty$, the proportion of even numbers tends to $\frac{1}{3}$.
The proportion of uneven numbers tends to $\frac{2}{3}$.

Colloquially, there are:

"twice as many uneven as even rationals".

A Three-way Split

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In fact, there is:

$$\text{For } q = \frac{m}{n}\,, \quad \begin{cases} q \text{ has even parity if } m \text{ is even}, \\ q \text{ has odd parity if } m \text{ is odd and } n \text{ is odd}, \\ q \text{ has none if } m \text{ is odd and } n \text{ is even}\,. \end{cases}$$

The term none is an acronym:

none = 'neither odd nor even'

A Three-way Split

Let e be an even integer and o an odd one.

$$\text{Even: } \frac{e}{o} \qquad \text{Odd: } \frac{o}{o} \qquad \text{None: } \frac{o}{e} \,.$$

We define three subsets of the rational numbers:

$$\begin{aligned}
\text{Even: } \mathbb{Q}_{\mathrm{E}} &= \{q \in \mathbb{Q} : q = \tfrac{2m}{2n+1} \text{ for some } m, n \in \mathbb{Z}\} \\
\text{Odd: } \mathbb{Q}_{\mathrm{O}} &= \{q \in \mathbb{Q} : q = \tfrac{2m+1}{2n+1} \text{ for some } m, n \in \mathbb{Z}\} \\
\text{None: } \mathbb{Q}_{\mathrm{N}} &= \{q \in \mathbb{Q} : q = \tfrac{2m+1}{2n} \text{ for some } m, n \in \mathbb{Z}\} \,.
\end{aligned}$$

These three sets are disjoint: $\mathbb{Q} = \mathbb{Q}_{\mathrm{E}} \uplus \mathbb{Q}_{\mathrm{O}} \uplus \mathbb{Q}_{\mathrm{N}}$.

Addition and multiplication tables for $\mathbb{Q}$.

$+$	**even**	**odd**	**none**
even	*even*	*odd*	*none*
odd	*odd*	*even*	*none*
none	*none*	*none*	*any*

$\times$	**even**	**odd**	**none**
even	*even*	*even*	*any*
odd	*even*	*odd*	*none*
none	*any*	*none*	*none*

Note that the first two rows and columns are identical to the tables for the integers.

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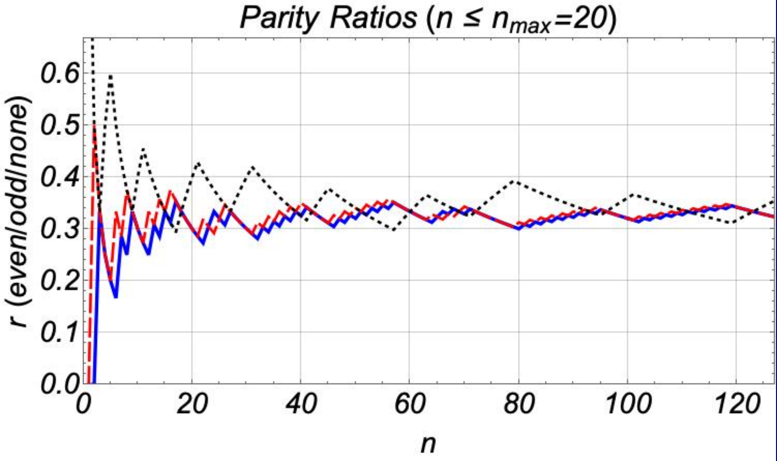
We may enquire about the relative sizes of the sets.

Numerical Evidence



Parity ratio r for denominator $\leqslant 20$.
Blue: Even. Red: Odd. Black: None.

Outline

Density of Subsets of $\mathbb{N}$

The set of even positive numbers is “the same size” as the set of all natural numbers:

Both are infinite countable sets.

However, cardinality is a blunt instrument:
Shouldn’t even numbers comprise 50% of $\mathbb{N}$?

Our intuition tells us that if B is a proper subset of A, it must be smaller than A.

The concept of *density* provides a measure of the relative sizes of sets that is more discriminating than cardinality.

Natural or Asymptotic Density

Assume a subset A of $\mathbb{N}$ is enumerated as $\{a_1, a_2, \ldots\}$.

We define the density of A in $\mathbb{N}$ as the limit, if it exists,

$$\rho_{\mathbb{N}}(A) = \lim_{n\to\infty} \frac{|\{a_k : a_k \leqslant n\}|}{n} .$$

If the fraction of elements of A among the first n natural numbers converges to a limit $\rho_{\mathbb{N}}(A)$ as $n \to \infty$, then A has density $\rho_{\mathbb{N}}(A)$.

For $A = \mathbb{N}_{\mathrm{E}}$ or $A = \mathbb{N}_{\mathrm{O}}$, we have $\rho(A) = \frac{1}{2}$, consistent with our intuitive notion.

Density Depends on Order

We can rearrange the natural numbers into a set F such that there are **twice as many even as odd numbers in F**.

We reorder $\mathbb{N}$ so that each odd number is followed by two even ones:

$$F = \{1,\ \ 2,4,\ \ 3,\ \ 6,8,\ \ 5,\ \ 10,12,\ \ldots\ ,2n-1,\ \ 4n-2,4n,\ldots\}\,.$$

It is easy to see that $\rho_F(\mathbb{N}_{\mathrm{E}}) = \frac{2}{3}$ and $\rho_F(\mathbb{N}_{\mathrm{O}}) = \frac{1}{3}$.

Proceeding further, we can construct a set H in which the n-th odd number is followed by n even numbers.

We find that $\rho_H(\mathbb{N}_{\mathrm{E}}) = 1$, so that "almost all the elements of H are even".

Partitioning the Rationals

Restricting attention to the even and odd rationals only — omitting those with no parity — we define

$$\mathbb{Q}_{\mathrm{P}} := \mathbb{Q}_{\mathrm{E}} \uplus \mathbb{Q}_{\mathrm{O}} \,.$$

This is the set of all rationals whose denominators are odd numbers in $\mathbb{Z}$.

$\mathbb{Q}_{\mathrm{P}}$ is closed under addition and multiplication and forms a commutative subring of $\mathbb{Q}$.

Moreover, since there are no divisors of zero, $\mathbb{Q}_{\mathrm{P}}$ is an integral domain.

Partitioning the Rationals

$\mathbb{Q}_{\mathrm{P}}$ is a (normal) subgroup of $\mathbb{Q}$.

We may enquire about its index $[\mathbb{Q} : \mathbb{Q}_{\mathrm{P}}]$ and its quotient group $\mathbb{Q}/\mathbb{Q}_{\mathrm{P}}$.

$\star \quad \star \quad \star$

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Somewhat out of context, we mention that all three parity classes, $\mathbb{Q}_{\mathrm{E}}$, $\mathbb{Q}_{\mathrm{O}}$ and $\mathbb{Q}_{\mathrm{N}}$, are (topologically) dense in the rationals.

Outline

2-Adic Valuation

"All multiples of 2 are even, but some are more even than others."

The *p*-adic valuation of an integer *n* is

$$\nu_p(n) = \begin{cases} \max\{k \in \mathbb{N} : p^k \mid n\} & \text{for } n \neq 0 \\ \infty & \text{for } n = 0 \end{cases}$$

This is extended to the rational numbers m/n:

$$\nu_p\left(\frac{m}{n}\right) = \nu_p(m) - \nu_p(n)\,.$$

We shall be concerned exclusively with the case $p = 2$. We note that

$$\mathbb{Q}_{\mathrm{P}} = \{q \in \mathbb{Q} : \nu_2(q) \geqslant 0\} \quad \text{and} \quad \mathbb{Q}_{\mathrm{E}} = \{q \in \mathbb{Q} : \nu_2(q) > 0\}$$

Partitioning $\mathbb{Q}$

For q rational with parity *even*, $\nu_2(q) > 0$,
For q rational with parity *odd*, $\nu_2(q) = 0$,
For q rational with parity *none*, $\nu_2(q) < 0$.

For all $k \in \mathbb{Z}$, we define

$$Q_k = \{q \in \mathbb{Q} : \nu_2(q) = k\} \qquad \text{and} \qquad Q_\infty = \{\, 0 \,\} .$$

The resulting partition yields a rich algebraic structure. The union of all the Q-sets comprises the rationals:

$$\mathbb{Q} = \{\, 0 \,\} \uplus \biguplus_{k=-\infty}^{\infty} Q_k .$$

Dyadic Rational Numbers

A dyadic rational is a fraction whose denominator is a power of two. We define

$$D_k = \{2^k(2\ell - 1) : \ell \in \mathbb{Z}\} \qquad \textbf{and} \qquad D_\infty = \{ 0 \} ,$$

and note that $\mathbb{D} = \biguplus_k D_k \uplus D_\infty$**.**
The dyadic rational numbers form a ring between the ring of integers and the field of rational numbers:

$$\mathbb{Z} \trianglelefteq \mathbb{D} \trianglelefteq \mathbb{Q} .$$

We construct a countable infinity of subgroups of $\mathbb{D}$**:**

$$\mathbb{D}_K := \{ 0 \} \uplus \biguplus_{k \geqslant K} D_k .$$

Particular cases include

$$\mathbb{D}_{-\infty} = \mathbb{D} , \quad \mathbb{D}_{-1} = \tfrac{1}{2}\mathbb{Z} , \quad \mathbb{D}_0 = \mathbb{Z} , \quad \mathbb{D}_1 = \mathbb{Z}_{\mathrm{E}} , \quad \mathbb{D}_\infty = \{ 0 \} .$$

Partitioning the Rationals

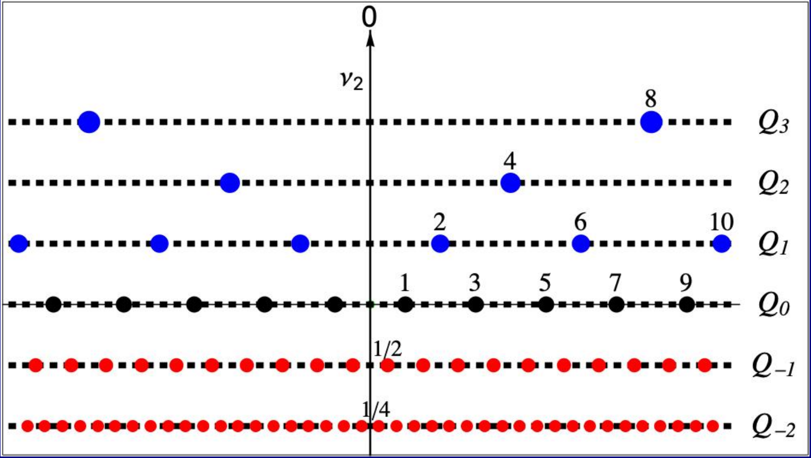


Figure: Partition of the rational numbers. The vertical axis is the 2-adic valuation ν_2. D_k indicated by marked points. Totality of these comprises dyadic rationals $\mathbb{D}$.

Partitioning the Rationals

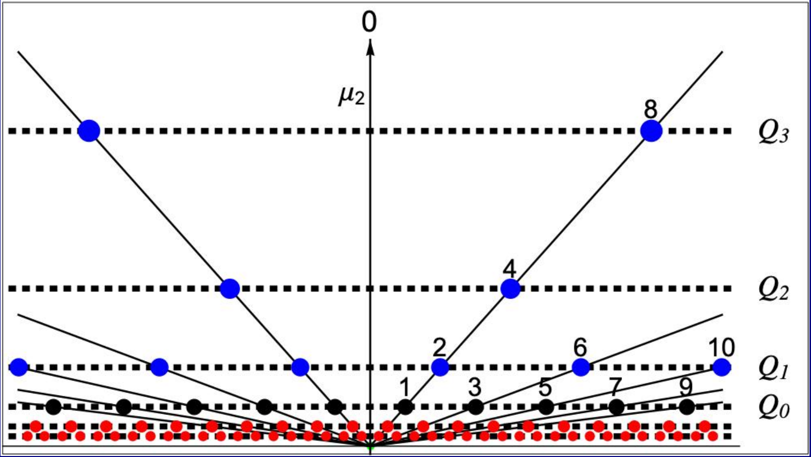


Figure: Partition of the rational numbers. The vertical axis is $\mu = 2^{\nu_2}$. D_k indicated by marked points. Totality of these comprises dyadic rationals $\mathbb{D}$.

Cosets of $\mathbb{Q}_{\mathrm{P}}$ in $\mathbb{Q}$.

For each $k > 0$, we define a set of values

$$q_k^\ell = 2^{-k}(2\ell - 1) \in Q_{-k} \quad \text{for} \quad \ell = 1, 2, 3, \ldots, 2^{k-1} .$$

These are the first 2^{k-1} positive values in D_{-k}.

We prove that these are representatives of 2^{k-1} cosets, which are all distinct and which provide a disjoint partition of Q_{-k}.

The cosets of $\mathbb{Q}_{\mathrm{P}}$ in $\mathbb{Q}$ are

$$q_k^\ell + \mathbb{Q}_{\mathrm{P}} , \qquad \ell = 1, 2, 3, \ldots 2^{k-1} , \quad k = 1, 2, \ldots .$$

Density of Q_k: Heuristic Discussion

The set $Q_{-1} = \frac{1}{2} + \mathbb{Q}_{\mathrm{P}}$ is a coset of $\mathbb{Q}_{\mathrm{P}}$. It can be visualized as a copy of $\mathbb{Q}_{\mathrm{P}}$ shifted by a distance $\frac{1}{2}$.

We argue heuristically that Q_{-1} is "as dense as $\mathbb{Q}_{\mathrm{P}}$".

More generally, for any k,

$$\frac{1}{2^k}\left(\frac{2m+1}{2n+1}\right) \in Q_k \longleftrightarrow \frac{1}{2^{k-1}}\left(\frac{2m+1}{2n+1}\right) \in Q_{k-1} \,.$$

Q_{k-1} is a compressed version of Q_k.

Since Q_{k-1} is "twice as dense as Q_k", we should have twice as many cosets in Q_{k-1} as in Q_k.

This has been proved rigorously.

Outline

The Calkin-Wilf Tree

The Calkin-Wilf tree is an enumeration of $\mathbb{Q}$.

The Calkin-Wilf tree is complete:

- It includes all the rationals;
- Each positive rational occurs just once.

Everything springs from the root $1/1$.

Each rational has two "children":
for the entry m/n, the children are

$$m/(m+n) \text{ and } (m+n)/n.$$

The Calkin-Wilf Tree

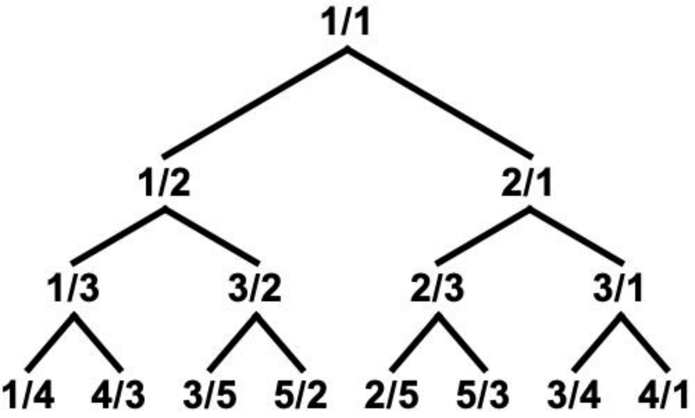
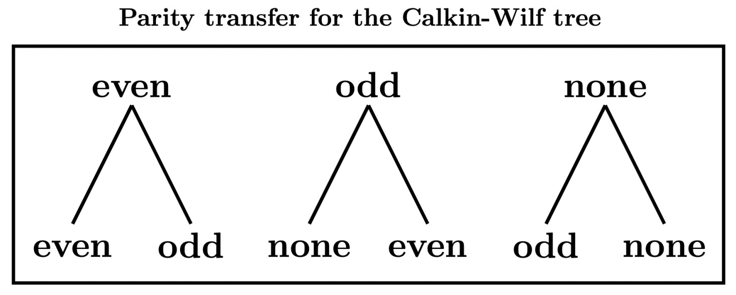


Figure: The initial rows of the Calkin-Wilf tree.

Mnemonic: $\frac{\text{Top}}{\text{Sum}}$ and $\frac{\text{Sum}}{\text{Bottom}}$.

Calkin-Wilf Parity Transfer



If each of the parity classes, even, odd and none, occurs with equal frequency at one generation, then this equality is passed on to the next generation:

$$\rho_{\mathbb{Q}}(\mathbb{Q}_{\mathrm{E}}) = \rho_{\mathbb{Q}}(\mathbb{Q}_{\mathrm{O}}) = \rho_{\mathbb{Q}}(\mathbb{Q}_{\mathrm{N}}) = \frac{1}{3} .$$

Stern-Brocot Tree

The Stern-Brocot tree is another ordering of $\mathbb{Q}$, very similar to the Calkin-Wilf tree.

The numbers at each level are formed from the **mediants** of adjacent pairs of numbers above.

The mediant of two rationals m_1/n_1 and m_2/n_2 is

$$M\left(\frac{m_1}{n_1}, \frac{m_2}{n_2}\right) := \frac{m_1 + m_2}{n_1 + n_2}$$

Stern-Brocot Tree

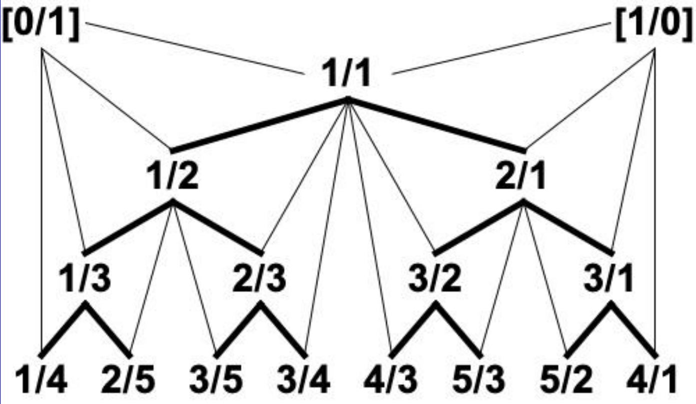


Figure: The initial rows of the Stern-Brocot tree.

Density for the Stern-Brocot Tree

The parity of the mediants of two numbers of different parity is the third parity:

$$M(e,o) = n\,, \qquad M(o,n) = e\,, \qquad M(n,e) = o\,.$$

From this, we can show for the Stern-Brocot tree:

$$\rho_{\mathbb{Q}}(\mathbb{Q}_{\mathrm{E}}) = \rho_{\mathbb{Q}}(\mathbb{Q}_{\mathrm{O}}) = \rho_{\mathbb{Q}}(\mathbb{Q}_{\mathrm{N}}) = \frac{1}{3}\,.$$

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The determination of the densities of parity classes for the ordering corresponding to the Farey sequences is left as a challenge!

Outline

Acknowledgments

We are grateful for helpful comments from:

Prof Tom Laffey (UCD)

Prof Tony O'Farrell (NUIM)

Summary

- We have extended parity from $\mathbb{Z}$ to $\mathbb{Q}$.
- Three parity classes were found.
- The even and odd rationals, $\mathbb{Q}_{\mathrm{E}}$ and $\mathbb{Q}_{\mathrm{O}}$, follow the usual rules of parity.
- The union of these, $\mathbb{Q}_{\mathrm{P}} = \mathbb{Q}_{\mathrm{E}} \uplus \mathbb{Q}_{\mathrm{O}}$, forms an additive subgroup of $\mathbb{Q}$.
- Using the 2-adic valuation, we partitioned $\mathbb{Q}$ into subsets and found a chain of subgroups.
- Using the natural density, we showed that the three parity classes are equally dense in the rationals for both the CW-Tree and the SB-Tree.