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Resonant Dynamics near the Triangular Lagrange Points

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Abstract

We investigate dynamics near the stable (triangular) Lagrange points in the planar circular restricted three-body problem. There are two eigenmodes of the linearized equations, one with low frequency and one with high frequency. There is resonant interaction between the modes when the eigenfrequencies are commensurate. The Lagrangian is approximated to a given order in the amplitudes and the averaged Lagrangian technique is applied. The resulting Euler-Lagrange equations, the modulation equations, describe the dynamics of the slowly-varying amplitudes and phases.

In the case of 2:1 resonance, the modulation equations are a special case of the 'explosive interaction' three-wave equations. They are integrable, with solutions becoming infinite in finite time. On the basis of general dynamical systems theory, the 3:1 resonance is also expected to be unstable, but all higher-order resonances should be stable. We present numerical evidence of a stable 5:1 resonance, whose solution exhibits a slow periodic exchange of energy between the two types of oscillation.

The three body problem

"The most celebrated of all dynamical problems"

Whittaker, 1937

The history of the three body problem is reviewed in
June Barrow-Green, 1997:
Poincaré and the Three Body Problem.
American Mathematical Society, 272pp.

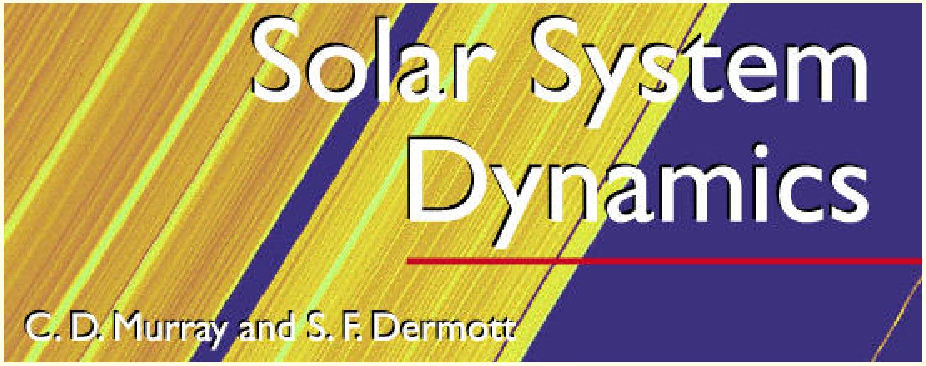
The theory of the three body problem is reviewed in
Christian Marchal, 1990:
The Three-Body Problem.
Elsevier Science Publishing, 576pp.

The restricted three-body problem

We consider the planar, circular, restricted 3-body problem.

- Two bodies of masses m_1 and m_2 $(m_1 > m_2)$.
- Rotating at constant angular speed n about their CoM.
- Assume unit mean motion $n = 1$.
- Choose units such that $\mu = \mathcal{G}(m_1 + m_2) = 1$.
- Define: $\mu_1 = \mathcal{G}m_1$ and $\mu_2 = \mathcal{G}m_2 = m_2/(m_1 + m_2)$.
- Assume $m_2 \ll m_1$. Then $\mu_1 \approx 1$ and $\mu_2 \ll 1$.

We (generally) follow the notation in



Murray, C. D. and S. F. Dermott, 1999:
Solar System Dynamics.
Cambridge Univ. Press, 592pp.

Let (x, y) be coordinates of the particle in a reference frame with origin at the CoM and x-axis pointing towards m_2. The distances from the particle to the masses are

$$r_1 = \sqrt{(x+\mu_2)^2 + y^2}\,, \qquad r_2 = \sqrt{(x-\mu_1)^2 + y^2}\,.$$

The equations of motion of the particle are

$$\ddot{x} - 2n\dot{y} - n^2 x = -\left[\mu_1 \frac{x+\mu_2}{r_1^3} + \mu_2 \frac{x-\mu_1}{r_2^3}\right]$$

$$\ddot{y} + 2n\dot{x} - n^2 y = -\left[\mu_1 \frac{y}{r_1^3} + \mu_2 \frac{y}{r_2^3}\right]$$

We define an ***effective potential***

$$V(x,y) = -\tfrac{1}{2}n^2(x^2+y^2) - \frac{\mu_1}{r_1} - \frac{\mu_2}{r_2}$$

Then the equations of motion become

$$\ddot{x} - 2n\dot{y} = -\frac{\partial V}{\partial x}$$

$$\ddot{y} + 2n\dot{x} = -\frac{\partial V}{\partial y}$$

Note that the potential V is ***negative definite.***

It is conventional in celestial mechanics to define the pseudo-potential $U(x,y) = -V(x,y)$. U is positive definite.

$$\ddot{x} - 2n\dot{y} = \frac{\partial U}{\partial x} \qquad (1)$$

$$\ddot{y} + 2n\dot{x} = \frac{\partial U}{\partial y} \qquad (2)$$

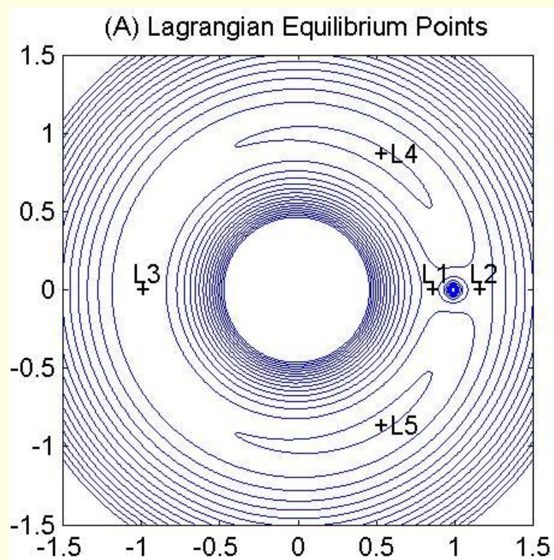
The system (1)–(2) is not integrable. However, there is a constant of motion, the Jacobi integral

$$C_{\mathrm{J}} = -(\dot{x}^2 + \dot{y}^2) - 2V(x,y)\,.$$

We note that $C_{\mathrm{J}} = 2U - |\mathbf{v}|^2$ where $\mathbf{v}$ is the velocity in the rotating frame.

There are five equilibrium points of (1)–(2) in the xy-plane. These are the well-known Lagrangian equilibrium points.

The Lagrange Equilibrium Points



The five Lagrange equilibrium points for $\mu_2 = 0.01$.

The 'triangular points' L_4 and L_5 are linearly stable equilibria provided

$$\mu_2 \leq \mu_0 \equiv \frac{27 - \sqrt{621}}{54} \approx 0.0385$$

However, this is not sufficient for stability in the nonlinear case.

We shall find that, in certain resonant cases, the motion near L_4 is nonlinearly unstable.

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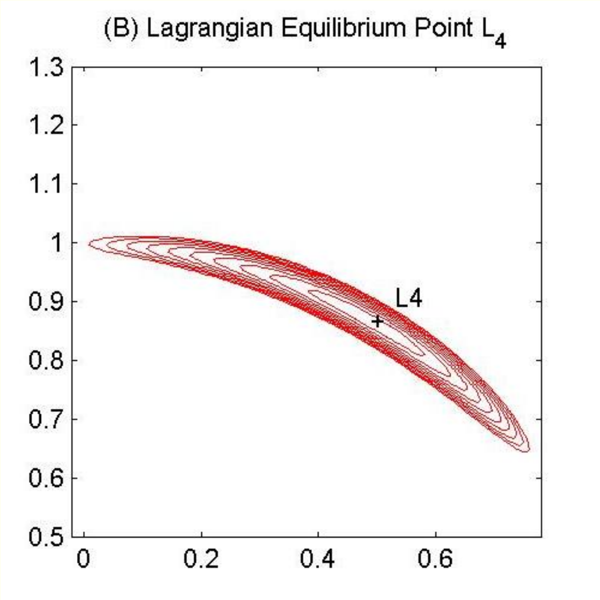
The following figure shows a zoom in on the potential lines near the triangular point L_4.

The L_4 triangular equilibrium point



Zoom on the L_4 equilibrium point for $\mu_2 = 0.01$.

Atmospheric Analogy

- L_4 is a maximum of the potential function $V(x, y)$.
- In a non-rotating frame, this would imply instability.
- However, the rotation has a stabilizing effect. Small-amplitude motions near this point may persist.
- Equations (1)–(2) may be written

$$\dot{\mathbf{v}} + 2\mathbf{n} \times \mathbf{v} + \nabla V = \mathbf{0}$$

  where $\mathbf{v} = (u, v) = (\dot{x}, \dot{y})$ and $\mathbf{n} = n\hat{\mathbf{z}}$.i The term $2\mathbf{n} \times \mathbf{v}$ is the Coriolis acceleration.
- This equation is formally identical to the equations of motion governing atmospheric and oceanic flows on an f-plane.

Two special cases

Inertial flow

If the potential gradient term is negligible, the acceleration and Coriolis terms must balance, leading to the solution

$$u = A \sin[2n(t - t_0)] , \qquad v = A \cos[2n(t - t_0)] .$$

The *trajectories are circular* and are followed in a clockwise direction with frequency $f = 2n$, independent of the amplitude A. This is called *inertial flow.*

Geostrophic Flow

If the acceleration terms are small, there is balance between the Coriolis and potential gradient terms, resulting in *geostrophic flow*

$$\mathbf{v} = \frac{1}{2n} \hat{\mathbf{z}} \times \nabla V .$$

The flow is along lines of equal potential.

In geophysical flows, the potential function V represents the field of pressure, that is, the mass distribution. The motion alters this, so that V changes in time.

In the astronomical context, V is independent of time. We may think of the equi-potential lines as *isobars on a frozen weather map*.

The pattern near $\mathbf{L}_4$ is like an elongated ridge of high pressure — a banana-shaped anticyclone.

If the gradient of V is negligible, the flow is inertial and the trajectories are clockwise circles.

If the acceleration is negligible, the flow is again clockwise, but parallel to the lines of equal potential.

As we will see, the eigensolutions of the celestial problem fall between the purely inertial and purely geostrophic trajectories.

Linear solutions near $\mathrm{L}_4=(x_0, y_0)$

We introduce coordinates centered at the equilibrium point:

$$X = x - x_0 \,, \qquad Y = y - y_0 \,.$$

Noting that there is a maximum of $V(X,Y)$ at $\mathbf{L}_4$,

the potential surface is given to quadratic order by

$$V(X,Y) = V_0 + \tfrac{1}{2}\left[V_{XX}X^2 + 2V_{XY}XY + V_{YY}Y^2\right] \,.$$

Here, subscripts 0 indicate evaluation at $(X,Y)=(0,0)$ and

$$V_{XX} = \left(\frac{\partial^2 V}{\partial X^2}\right)_0 , \quad V_{XY} = \left(\frac{\partial^2 V}{\partial X \partial Y}\right)_0 , \quad V_{YY} = \left(\frac{\partial^2 V}{\partial Y^2}\right)_0 ,$$

Then the equations may be written

$$\ddot{X} - 2n\dot{Y} + V_{XX}X + V_{XY}Y = 0$$
$$\ddot{Y} + 2n\dot{X} + V_{XY}X + V_{YY}Y = 0$$

We can write the system as a set of four first-order equations

$$\frac{d}{dt}\begin{pmatrix} X \\ Y \\ \dot{X} \\ \dot{Y} \end{pmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -V_{XX} & -V_{XY} & 0 & 2n \\ -V_{XY} & -V_{YY} & -2n & 0 \end{bmatrix} \begin{pmatrix} X \\ Y \\ \dot{X} \\ \dot{Y} \end{pmatrix}$$

The solution is determined by the initial conditions.

The eigensolutions of this system are easily found. We assume $X = X_0 \exp(\mathrm{i}\sigma t)$ and $Y = Y_0 \exp(\mathrm{i}\sigma t)$ and find that

$$\sigma^4 - \left(4n^2 + V_{XX} + V_{YY}\right)\sigma^2 + (V_{XX}V_{YY} - V_{XY}) = 0 \,.$$

The partial derivatives of V at $\mathbf{L}_4$ are

$$V_{XX} = -\frac{3}{4}, \qquad V_{YY} = -\frac{9}{4}, \qquad V_{XY} = -\frac{3\sqrt{3}}{4}(1-2\mu_2) \,.$$

so the eigenvalue equation reduces to the biquadratic:

$$\sigma^4 - \sigma^2 + \frac{27}{4}\mu_2(1-\mu_2) = 0 \,.$$

The criterion for *linear stability* is that *all roots are real.*

The reality of the roots is assured if:

$$\mu_2 \le \mu_0 \equiv \frac{27 - \sqrt{621}}{54} \approx 0.0385$$

The real and imaginary parts of the roots of the biquadratic are plotted in the figure to follow.

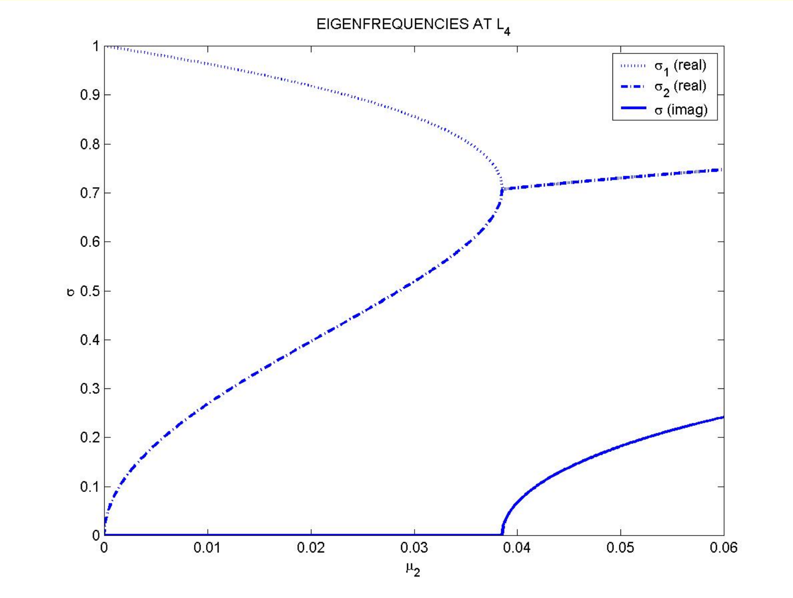
For small μ_2 it is easily shown that

$$\sigma_1 \approx \frac{3}{2}\sqrt{3\mu_2} \,, \qquad \sigma_2 \approx 1 - \frac{27}{8}\mu_2 \,.$$

Since for $\mu_2 \ll \mu_0$ we have $|\sigma_1| \ll |\sigma_2|$, we may describe the two modes as ***slow*** and ***fast*** respectively.

The ratio $|\sigma_2/\sigma_1|$ increases from 1 to ∞ as μ_2 decreases from μ_0 to 0. Thus, given any rational number $n_1/n_2 \ge 1$, there is a value of μ_2 in the interval $[0, \mu_0]$ for which $(n_1\sigma_1 - n_2\sigma_2) = 0$.

Such conditions give rise to *resonant solutions*.



Absolute values of the real and imaginary parts of the roots σ_1 and σ_2 as functions of the mass μ_2. For $\mu_2 \le \mu_0 \approx 0.0385$ all the roots are real.

Eigenvectors

To compute the eigenvectors, eliminate the cross-derivative term in

$$V(X,Y) = V_0 + \tfrac{1}{2}\left[V_{XX}X^2 + 2V_{XY}XY + V_{YY}Y^2\right]$$

by a rotation to new coordinates

$$\begin{pmatrix}\xi\\ \eta\end{pmatrix} = \begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}X\\ Y\end{pmatrix}$$

The angle θ is given by

$$\tan 2\theta = \frac{2V_{XY}}{V_{XX}-V_{YY}} = -\sqrt{3}(1-2\mu_2)\,.$$

Since μ_2 is small, this implies $\theta \approx -\pi/6$, a clockwise rotation through 30°.

Retaining terms to quadratic order in ξ and η, we find that the potential becomes

$$V(\xi,\eta) = -\left[\left(\frac{3}{2}-\frac{1}{2}\mu_2\right) + \frac{9}{8}\mu_2\xi^2 + \left(\frac{3}{2}-\frac{9}{8}\mu_2\right)\eta^2\right]\,.$$

The linear equations in the new coordinates may be written

$$\ddot{\xi} - 2n\dot{\eta} - \frac{9}{4}\mu_2\xi = 0$$

$$\ddot{\eta} + 2n\dot{\xi} - 3(1-\frac{3}{4}\mu_2)\eta = 0$$

We seek eigenvector solutions of the form

$$\begin{pmatrix}\xi\\ \eta\end{pmatrix} = \begin{pmatrix}A\sin(\sigma t-\psi)\\ \rho A\cos(\sigma t-\psi)\end{pmatrix}$$

It follows that

$$\varrho = \frac{2\sigma}{3-\frac{9}{4}\mu_2+\sigma^2}\,.$$

Since μ_2 is small we may write

$$\varrho \approx \frac{2\sigma}{3+\sigma^2}\,.$$

Slow Mode.

For $\sigma = \sigma_1 \approx \frac{3}{2}\sqrt{3\mu_2}$, we find that $\varrho = \varrho_1 \approx \sqrt{3\mu_2}$, so that

$$\begin{pmatrix}\xi_1\\ \eta_1\end{pmatrix} = A_1\begin{pmatrix}\sin(\sigma_1 t-\psi_1)\\ \sqrt{3\mu_2}\cos(\sigma_1 t-\psi_1)\end{pmatrix}$$

where A_1 and ψ_1 are arbitrary. This is a clockwise rotation round an ellipse elongated in the ξ-direction, with aspect ratio $\sqrt{3\mu_2}$. Note that this ratio is twice that of the zero velocity curves. The frequency of the oscillation is small, approximately proportional to $\mu_2^{1/2}$.

Fast Mode.

For $\sigma = \sigma_2 \approx (1-\frac{27}{8}\mu_2)$, we find that $\varrho = \varrho_2 \approx \frac{1}{2}$, so that

$$\begin{pmatrix}\xi_2\\ \eta_2\end{pmatrix} = A_2\begin{pmatrix}\sin(\sigma_2 t-\psi_2)\\ \frac{1}{2}\cos(\sigma_2 t-\psi_2)\end{pmatrix} \qquad (3)$$

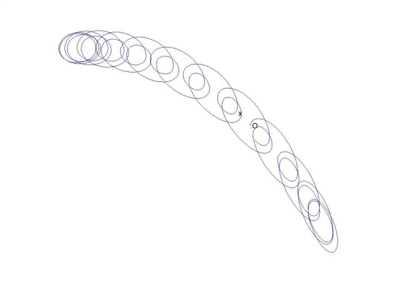
where A_2 and ψ_2 are arbitrary. This is a clockwise rotation round an ellipse with aspect ratio $2:1$. Its frequency is close to unity.

The <u>general solution</u> of the linear problem is

$$\begin{aligned} \xi &= \quad A_1 \sin(\sigma_1 t - \psi_1) + \quad A_2 \sin(\sigma_2 t - \psi_2) \\ \eta &= \varrho_1 A_1 \cos(\sigma_1 t - \psi_1) + \varrho_2 A_2 \cos(\sigma_2 t - \psi_2) \end{aligned}$$

where $\varrho_1 \approx \sqrt{3\mu_2}$ and $\varrho_2 \approx \frac{1}{2}$. The four arbitrary constants are determined from the initial conditions.

This solution comprises a fast cyclic motion of period $2\pi/\sigma_2$ superimposed on the slower cycling of a guiding centre rotating around an elongated ellipse with period $2\pi/\sigma_1$.



Energy of Eigenmodes

Assume that $V = 0$ at $\mathbf{L}_4$ and that

$$\xi = A \sin(\sigma t - \psi)\,, \qquad \eta = \varrho A \cos(\sigma t - \psi)\,.$$

The kinetic and potential energy are given by

$$\begin{aligned} T &= \tfrac{1}{2}\sigma^2 (c^2 + \varrho^2 s^2) A^2 \\ V &= -\left[\frac{9}{8}\mu_2 s^2 + (\frac{3}{2} - \frac{9}{8}\mu_2)\varrho^2 c^2\right] A^2 \end{aligned}$$

where $c = \cos(\sigma t - \psi)$ and $s = \sin(\sigma t - \psi)$.

For the first eigenmode

$$E = E_1 = -\frac{9}{8}\mu_2 A^2 < 0\,.$$

For the second eigenmode

$$E = E_2 = \frac{1}{8} A^2 > 0\,.$$

There cannot be complete pulsation, where the state changes back and forth between the two modes.

We recall that the relative energy (or Jacobi integral) is constant, but the physical energy of the particle is *not* conserved for the restricted three-body problem: the particle can exchange energy and angular momentum with the massive bodies. That is why it can move from $\mathbf{L}_4$ to more-or-less anywhere under certain conditions.

There is a possibility that <u>both eigenmodes grow and decay</u> together, pulsating on a long-period basis.

For stability, both modes may oscillate together, between small and large amplitudes.

For equi-partition of energy between the two modes there is a relatively small amplitude for the faster mode.

Resonance

Resonant interaction between the eigenmodes occurs if the eigenfrequencies are <u>commensurate</u>, that is, if there are integers n_1 and n_2 such that

$$n_1 \sigma_1 - n_2 \sigma_2 = 0\,.$$

While $\mu_2 \le \mu_0$ is necessary for linear stability, *it is not sufficient in the nonlinear case*. There are two exceptional values of μ_2, corresponding to the lowest-order resonances $2:1$ and $3:1$ for which stability is not assured:

$$\begin{aligned} n_1 : n_2 &= 2:1 \qquad & \mu_2 &= \frac{45 - \sqrt{1833}}{90} \approx 0.0243 \\ n_1 : n_2 &= 3:1 \qquad & \mu_2 &= \frac{15 - \sqrt{213}}{30} \approx 0.0135 \end{aligned}$$

The region of instability is a neighbourhood surrounding each exceptional point, the extent of which diminishes as the amplitude of the motion decreases.

Absolute values of the eigenfrequencies σ_1 and σ_2 as functions of μ_2. Some resonant values are indicated by vertical lines.

The 2 : 1 Resonance

The strongest interaction between the modes occurs for $\sigma_2 = 2\sigma_1$, corresponding to $\mu_2 = 0.0243$.

$$L = \tfrac{1}{2}(\dot{x}^2 + \dot{y}^2) + n(x\dot{y} - \dot{x}y) + \tfrac{1}{2}n^2(x^2 + y^2) + \frac{\mu_1}{r_1} + \frac{\mu_2}{r_2}\,.$$

As before, $X = x - x_0$ and $Y = y - y_0$. To cubic order, the Lagrangian becomes

$$\begin{aligned} L = \; & \tfrac{1}{2}(\dot{X}^2 + \dot{Y}^2) + n(X\dot{Y} - \dot{X}Y) + \tfrac{3}{8}X^2 + \tfrac{3\sqrt{3}x_0}{2}XY + \tfrac{9}{8}Y^2 \\ & + \frac{1}{8}\left[7x_0X^3 - \frac{3\sqrt{3}}{2}X^2Y - 33x_0XY^2 - \frac{3\sqrt{3}}{2}Y^3\right] \end{aligned}$$

The quadratic cross-term is eliminated by rotating the axes through an angle $\theta \approx -30°$. Then

$$L = \tfrac{1}{2}(\dot{\xi}^2 + \dot{\eta}^2) + n(\xi\dot{\eta} - \dot{\xi}\eta) + \left[\tfrac{9}{8}\mu_2\xi^2 + (\tfrac{3}{2} - \tfrac{9}{8}\mu_2)\eta^2\right] + \tfrac{3}{2}\xi^2\eta - \eta^3\,.$$

We now apply the *averaged Lagrangian technique*.

$$\begin{aligned} \xi &= \Im\{\quad \alpha_1(t)\exp(i\sigma_1 t) + \quad \alpha_2(t)\exp(i\sigma_2 t)\} \\ \eta &= \Re\{\varrho_1\alpha_1(t)\exp(i\sigma_1 t) + \varrho_2\alpha_2(t)\exp(i\sigma_2 t)\} \end{aligned}$$

where α_1 and α_2 vary slowly with time. After 'considerable algebraic manipulation' the averaged Lagrangian is found:

$$\langle L \rangle = \Im\left\{\kappa_{20}\dot{\alpha}_1\alpha_1^* + \kappa_{02}\dot{\alpha}_2\alpha_2^*\right\} + \Re\left\{\kappa_{21}\alpha_1^2\alpha_2^*\right\}.$$

The parameters κ_{20}, κ_{02} and κ_{21} are defined to be

$$\begin{aligned} \kappa_{20} &= \tfrac{1}{2}[(1 + \varrho_1^2)\sigma_1 - 2n\varrho_1] \\ \kappa_{02} &= \tfrac{1}{2}[(1 + \varrho_2^2)\sigma_2 - 2n\varrho_2] \\ \kappa_{21} &= -\tfrac{3}{8}[\varrho_2 - 2\varrho_1 + 2\varrho_1^2\varrho_2] \end{aligned}$$

Substituting the values for ϱ_1, ϱ_2, σ_1 and σ_2 we find that

$$\kappa_{20} = -0.0421 < 0\,, \quad \kappa_{02} = +0.0715 > 0\,, \quad \kappa_{21} = +0.0048 > 0\,.$$

The Euler-Lagrange equations may now be derived, the *modulation equations* governing the slowly-varying envelope of the motion:

$$\begin{aligned} i\kappa_{20}\dot{\alpha}_1 &= \kappa_{21}\alpha_1^*\alpha_2 \\ i\kappa_{02}\dot{\alpha}_2 &= \tfrac{1}{2}\kappa_{21}\alpha_1^2 \end{aligned}$$

We can show that the quantity

$$H_{12} = \Re\{\alpha_1^2\alpha_2^*\}$$

is a constant of the motion. The following quantity is also conserved:

$$N_{21} = \kappa_{20}|\alpha_1|^2 + \kappa_{02}|\alpha_2|^2\,.$$

It is highly significant that κ_{20} and κ_{02} are of opposite sign, because $|\alpha_1|^2$ and $|\alpha_2|^2$ are not bounded.

Introducing new variables A and C defined by

$$A = \sqrt{\frac{\kappa_{21}}{-\kappa_{20}\kappa_{02}}}\alpha_1 , \qquad C = -\frac{\kappa_{21}}{\kappa_{20}}\alpha_2 ,$$

the modulation equations become

$$\begin{aligned} i\dot{A} &= -A^*C \\ i\dot{C} &= +A^2 \end{aligned}$$

These are a special case of the *three wave equations* — the explosive case.

These equations are ***integrable***, with two invariants

$$\begin{aligned} N &= 2(|C|^2 - |A|^2) \\ H &= \Re\{A^2C^*\} \end{aligned}$$

A single equation can be derived for the squared amplitude

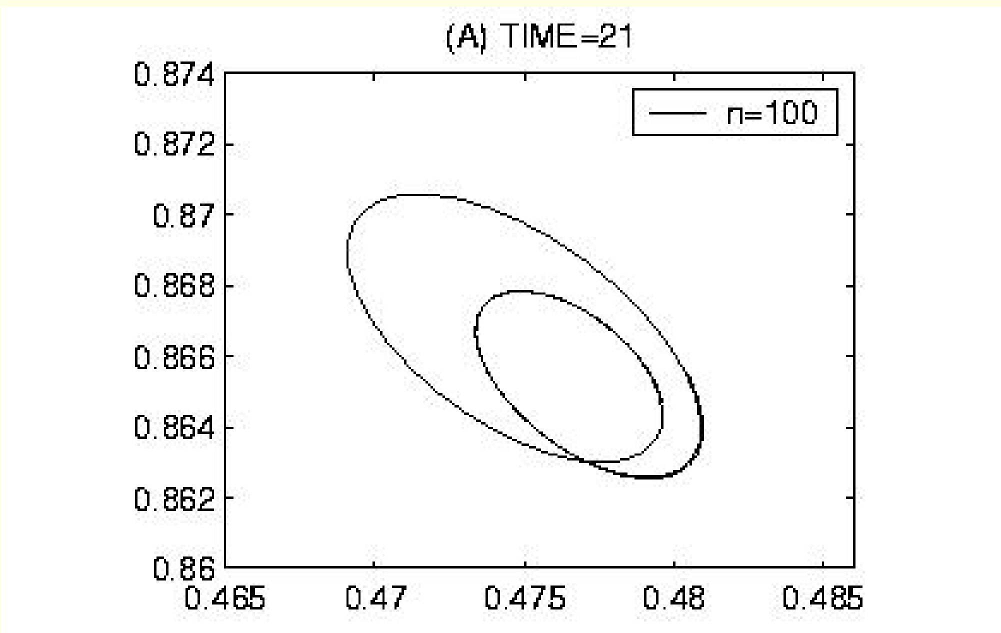
$$\left(\frac{dw}{d\tau}\right)^2 = 4\left[(w-\tfrac{1}{2})^2 w - \frac{H^2}{N^3}\right] \equiv \Phi_{21}(w) ,$$

with $w = |C|^2/N$ and $\tau = \sqrt{N}t$. The range of $|C|^2$ is $[\frac{1}{2}N, \infty)$.

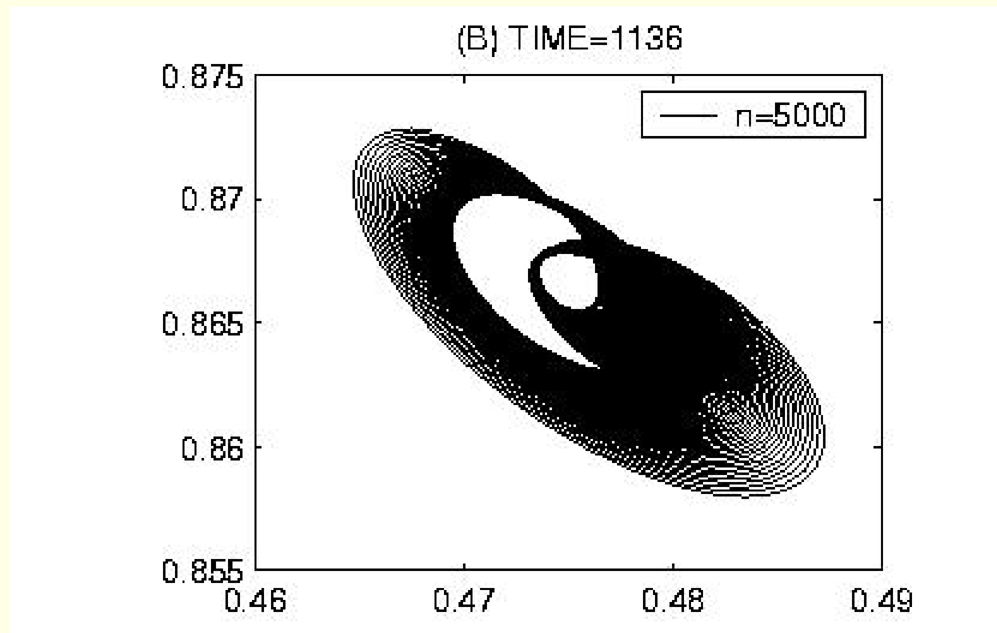
The solution is

$$|C|^2 = Nw = N\left[\frac{1}{3} + \wp\left(\sqrt{N}(t-t_0)\right)\right]$$

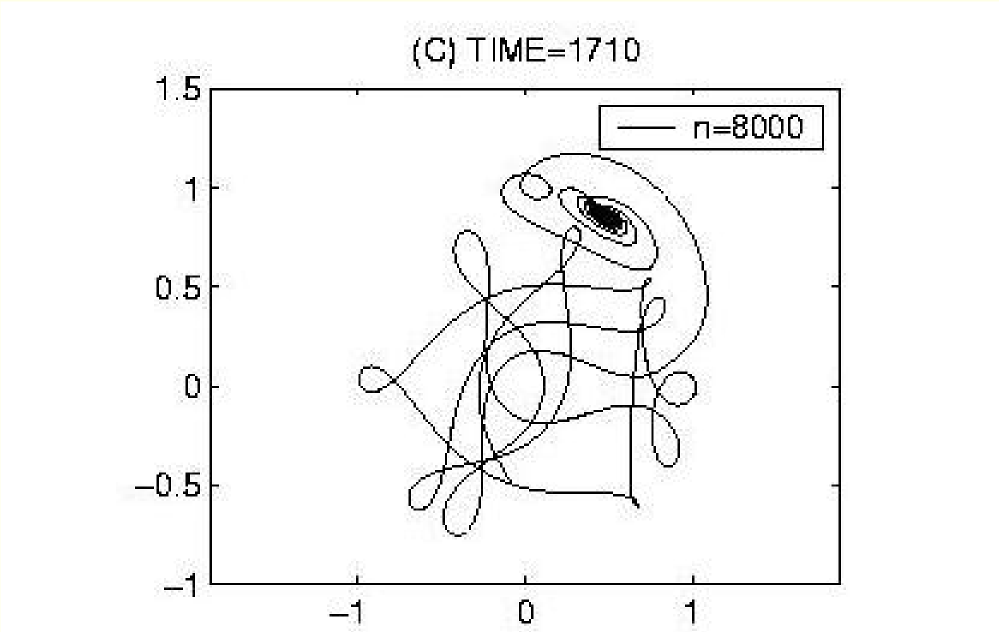
where $\wp(z)$ is the Weierstrass elliptic function and t_0 is an arbitrary ***real*** number.
Since $\wp(z)$ has poles periodically on the real axis, the solution becomes singular in finite time.
Thus, the $2:1$ resonance is nonlinearly unstable.



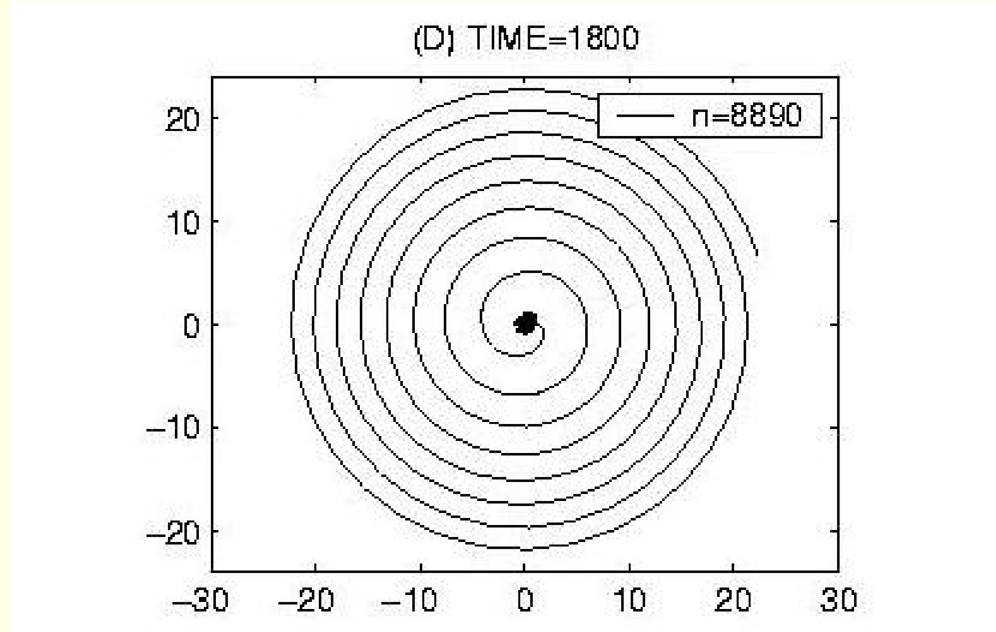
Trajectory for the unstable frequency ratio $2:1$. $t = 21$



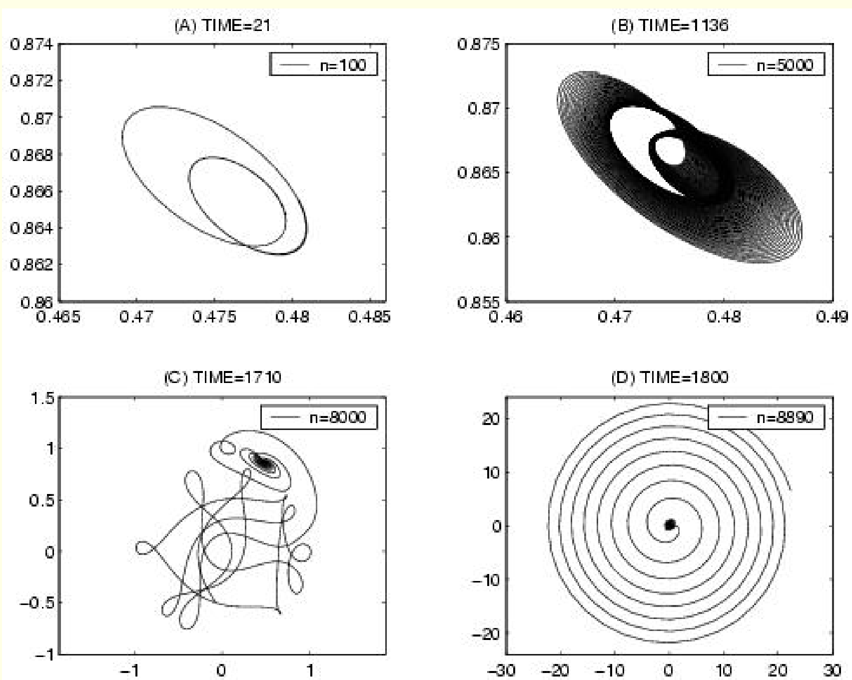
Trajectory for the unstable frequency ratio $2:1$. $t = 1136$



Trajectory for the unstable frequency ratio $2:1$. $t = 1710$



Trajectory for the unstable frequency ratio $2:1$. $t = 1800$



Trajectory for the unstable frequency ratio $2:1$.

The $\mathbf{3:1}$ Resonance

We next consider the $3:1$ resonance, with $\sigma_2 = 3\sigma_1$, which occurs for $\mu_2 = 0.0135$.
The *considerable algebraic manipulation* required in the case of $2:1$ resonance becomes even more 'considerable' for higher order resonances. We have therefore resorted to the use of a symbolic algebra package, MAPLE.

As a result of the complexity, the analysis has not yet been completed, but is still in progress.

$\star \quad \star \quad \star$

However, we have numerical evidence of resonant behaviour which will now be presented.

Numerical Evidence of Resonance

We present some *prima facie* numerical evidence of resonance between the modes near L_4 <u>for $5:1$ resonance</u>.

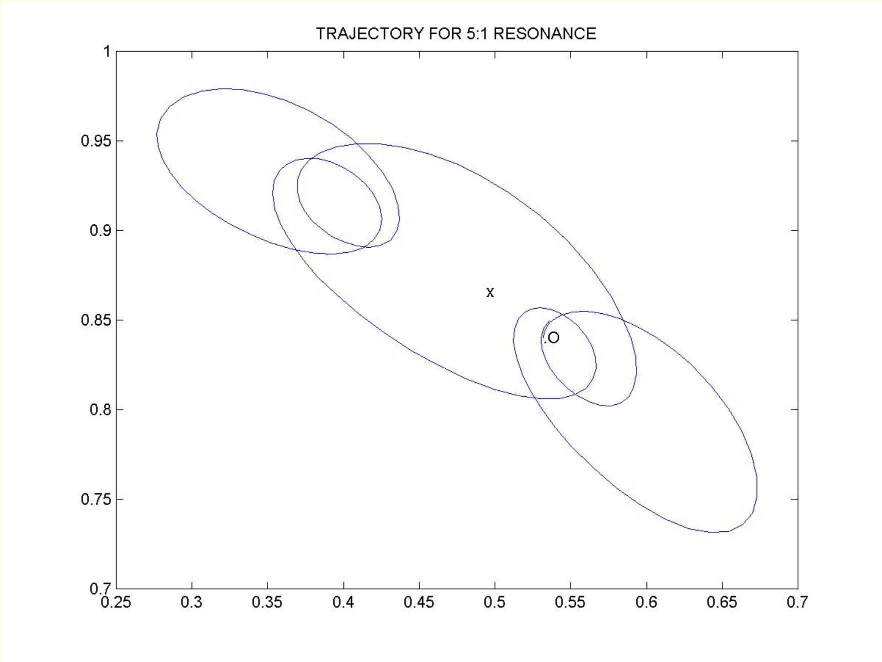
We will show the evolution of several quantities:

- x **versus** y
- x **and** y **against time**
- ξ **and** η **against time**
- A_1 **and** A_2 **against time**

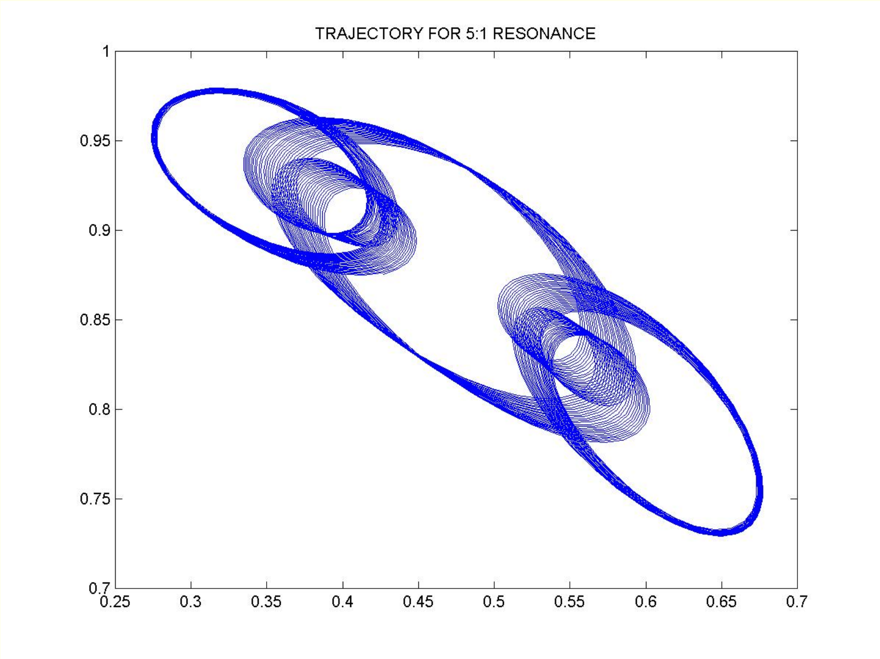
The envelope amplitudes A_1 and A_2 will indicate that resonance is occuring.

This numerical evidence <u>needs to be supported</u> by analytical results.

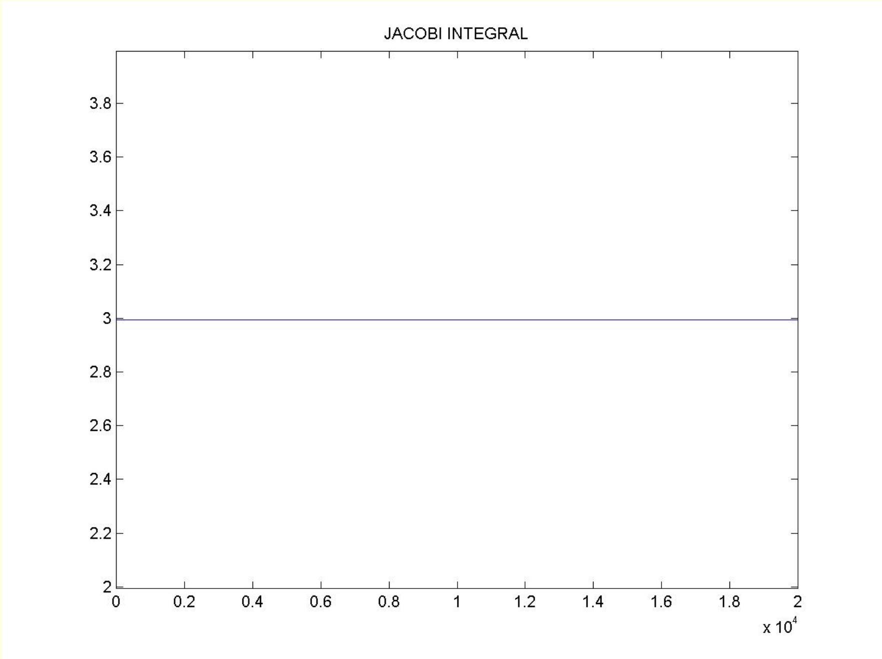
So far, we have not obtained such results, but are currently seeking them.



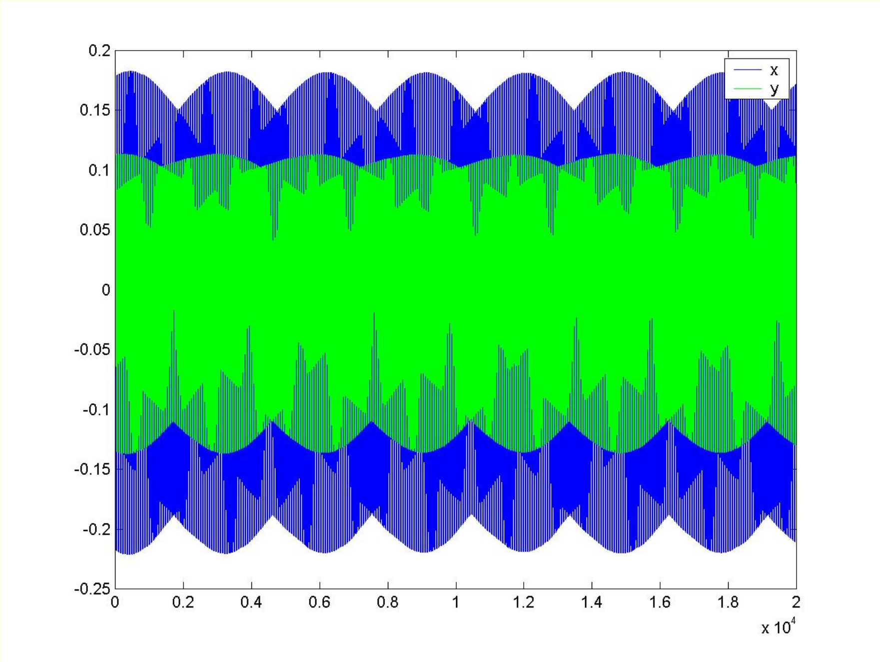
$5:1$ **Resonance. Plot of y versus x for about one cycle.**



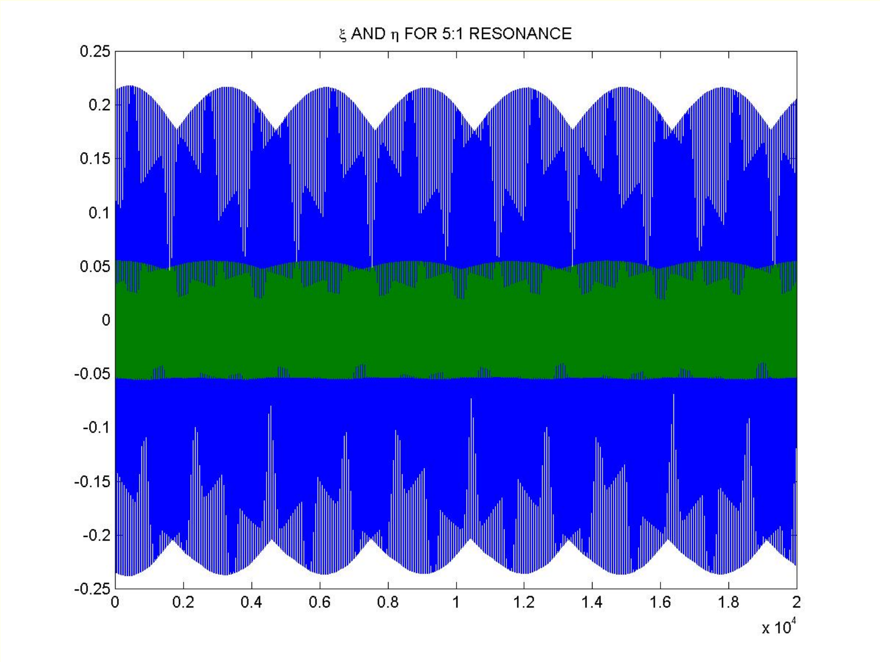
$5:1$ **Resonance. Plot of y versus x for about 25 cycles.**



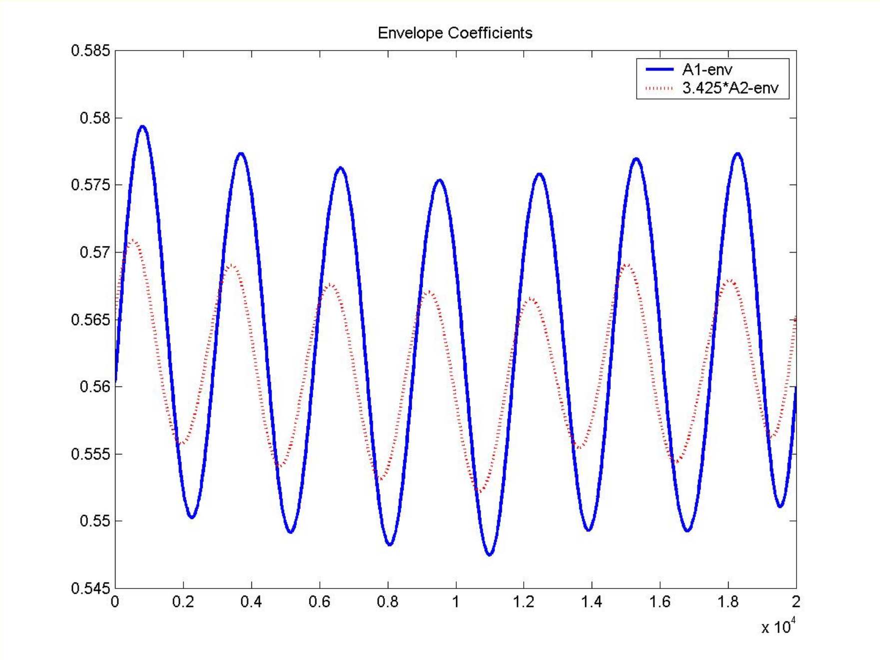
The Jacobi Integral for the $5:1$ case. $T = 20,000$ years.



Plot of x and y for a 20,000 year integration



Plot of ξ and η for a 20,000 year integration



Plot of A_1 and A_2 for a 20,000 year integration

The Earth-Moon System

For the Earth/Moon system, $\mu_2 = 0.012153$.

The eigenfrequencies are

$$\sigma_1 = 0.29824\,, \qquad \sigma_2 = 0.95449\,,$$

giving a frequency ratio:

$$\frac{\sigma_2}{\sigma_1} = 3.2004 \approx 3.2000 \equiv \frac{16}{5}\,.$$

It is possible that an object close to the L_4-point of the Earth/Moon system may be found to be in $16:5$ resonance.

Of course, this is speculative. Near-resonances are ubiquitous in the solar system, and many are without dynamical consequence. We shall see!

Thank you for listening.

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