Levels of Infinity

Peter Lynch
School of Mathematics & Statistics
University College Dublin

Berkeley Lecture

Maynooth, 29 September 2022

Outline

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"Levels of Infinity"

The above title is from an essay (in 1930) by Hermann Weyl. His opening statement is:

Mathematics is the science of the infinite.

The Enigma of Infinity

- The enigma of infinity has a long history.
- The stars in the sky seem beyond number.
- It has preoccupied and perplexed philosophers.
- It has mystified mathematicians for millennia.
- Children counting find no largest number.
- Given n, we have a larger number, $n + 1$.

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Preoccupation with the infinite was originally in the domains of philosophy and theology. Today, mathematics plays a central role.

Infinity in Ancient Greece

- Greeks from the time of Pythagoras.
- Anaximander used term $\alpha\pi\epsilon\iota\rho o\nu$
  (*unlimited*, *infinite*, *unbounded*.)
- Aristotle distinguished between potential and actual infinity:

The numbers 1, 2, 3, ... are potentially infinite, but we can never complete the count.

Euclid and Infinity

The word "infinite" does not appear anywhere in Euclid's *Elements.*

Euclid proved that there is no limit to the number of primes, but he never stated that there is an infinity of prime numbers.

"Given any prime number, there is a greater prime".

Parallel straight lines are straight lines which, being produced indefinitely, do not meet.

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“Given any prime number, there is a greater prime”.

Parallel straight lines are straight lines which, being produced indefinitely, do not meet.

This resonates with Aristotle’s potential infinity.

“You can keep going but you can never arrive.”

Ancient Indian Texts

Infinity treated in ancient texts from India and China.

Fascination with large numbers in Indian thought.

In the Jain mathematical manuscript **Surya Prajnapti** ($C4^{th}$ BC), there are three degrees of unboundedness:

Enumerable, Innumerable and Infinite.

This sounds like three "Levels of Infinity".

In Jaina cosmology, **time cycles eternally,** with a period of 10^{177} **years** (or 2^{588} years).

Infinity in the Early Church

Aristotle's ideas were taken up and developed by St. Augustine and St. Thomas Aquinas.

More paradoxes: God is omnipotent / omniscient. Can He set Himself a task that He cannot perform?

⋆ ⋆ ⋆

Potential or Actual?

Most philosophers and mathematicians
did not accept the concept of actual infinity.
Only potential infinity was accepted.

We mention a few key savants [there are many more]:

Galileo, Spinoza, Newton and Gauss all rejected it.

The first person to change things was Bernhard Bolzano. But Georg Cantor went much farther, constructing an entire hierarchy of infinities.

Infinity in Mathematics and Physics

Mathematics

- The sequence of natural numbers, 1, 2, 3, . . .
- The points on a continuous line.
- The continuum of real numbers.
- Sets of sets, and the class of all sets.

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- The points on a continuous line.
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- Sets of sets, and the class of all sets.

Physics

- Is there an infinitude of stars?
- Is space infinite in extent?
- Is space indefinitely divisible?
- Did time have a beginning?
- Does it endure forever?

The Zermelo-Fraenkel Axioms

1.1. *Axiom of Extensionality.* *If X and Y have the same elements, then $X = Y$.*

1.2. *Axiom of Pairing.* *For any a and b there exists a set $\{a, b\}$ that contains exactly a and b.*

1.3. *Axiom Schema of Separation.* *If P is a property (with parameter p), then for any X and p there exists a set $Y = \{u \in X : P(u, p)\}$ that contains all those $u \in X$ that have property P.*

1.4. *Axiom of Union.* *For any X there exists a set $Y = \bigcup X$, the union of all elements of X.*

1.5. *Axiom of Power Set.* *For any X there exists a set $Y = P(X)$, the set of all subsets of X.*

1.6. *Axiom of Infinity.* *There exists an infinite set.*

1.7. *Axiom Schema of Replacement.* *If a class F is a function, then for any X there exists a set $Y = F(X) = \{F(x) : x \in X\}$.*

1.8. *Axiom of Regularity.* *Every nonempty set has an $\in$-minimal element.*

1.9. *Axiom of Choice.* *Every family of nonempty sets has a choice function.*

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David Hilbert on "The Infinite!"

"No other question has ever moved so profoundly the spirit of man;

"no other idea has so fruitfully stimulated his intellect;

"yet no other concept stands in greater need of clarification than that of the infinite . . . ".

Philosophy. Metaphysics. Theology

Wovon man nicht sprechen kann, darüber muß man schweigen.

Ludwig Wittgenstein

Whereof one cannot speak, thereof one must be silent.

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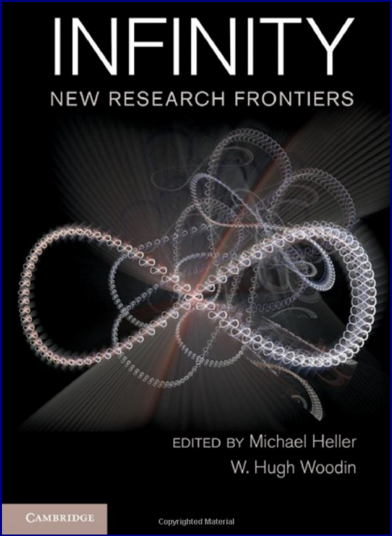


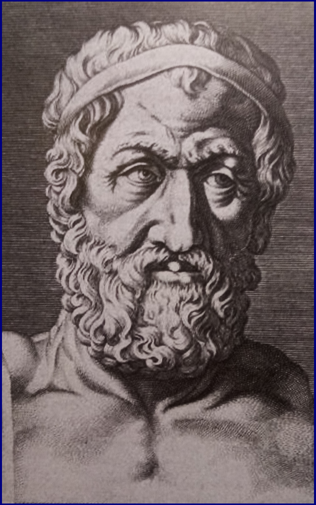
Philosophers who Contemplated Infinity

Bruno, Descartes, Kant, Voltaire

Spinoza, Schopenhauer, Nietzsche, Hegel.

Outline

Zeno of Elea



Zeno's Paradoxes

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To do so, he must first reach the half-way point. Before that, he must reach the quarter-point. ... He must arrive at an infinite number of midpoints.

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To do so, he must first reach the half-way point. Before that, he must reach the quarter-point. ... He must arrive at an infinite number of midpoints.

Zeno concludes that he cannot even begin to move:

Motion is impossible!

Infinite Sums

The Greeks could not imagine that the overall sum of an infinite number of distances could be finite.

James Gregory and, later, Gottfried Wilhelm Leibniz found an infinite series for π:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

This series was actually discovered much earlier by the mathematicians of the Kerala School in India.

But it was not until 1821 that Augustin Cauchy defined mathematical limits in a rigorous way.

Infinite Sums

Cauchy showed what it means to say that

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x} \qquad \text{for} \qquad |x| < 1\,.$$

In particular, when $x = \frac{1}{2}$, this gives the finite sum of the infinite number of stages that Achilles must run:

$$\frac{D}{2}\left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots\right) = D$$

Bertrand Russell described Zeno's ideas as "immeasurably subtle and profound".

Outline

Galileo's Number Paradox

Galileo discovered that the counting numbers can be matched, one for one, with the perfect squares.

$$n \longleftrightarrow n^2 \qquad \text{or} \qquad \begin{array}{ccccc} 1 & 2 & 3 & 4 & \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \cdots \\ 1 & 4 & 9 & 16 & \end{array}$$

This result is in direct conflict with Euclid:

The whole is greater than the part.

Galileo argued that concepts like "**equal to**" and "**greater than**" are meaningless for infinite sets.

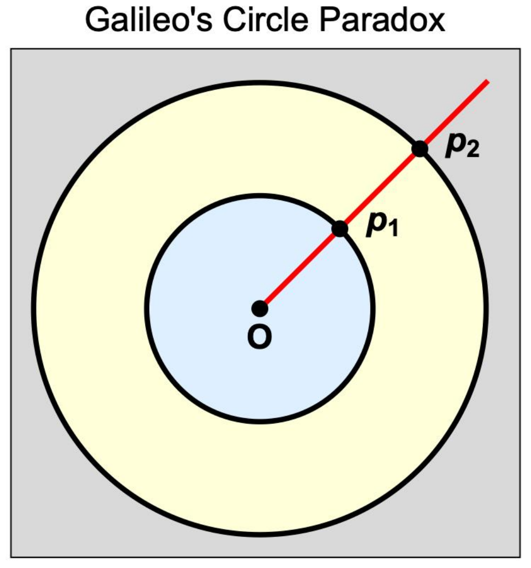
Galileo's Circle Paradox

Another paradoxical result of Galileo is that:

Two concentric circles of different radii contain the same number of points.

A simple diagram illustrates this paradox.

Galileo’s Circle Paradox



Outline

Georg Cantor

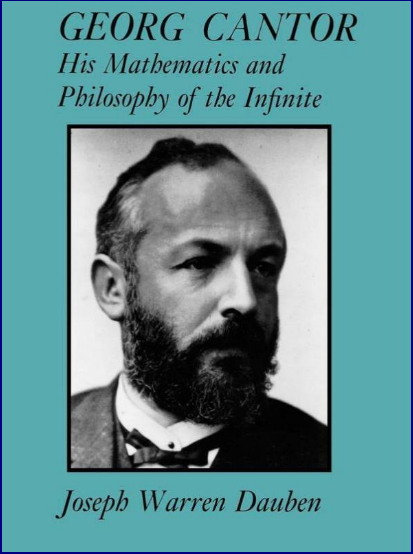
Inventor of Set Theory

Born in St. Petersburg, Russia in 1845.

Moved to Germany in 1856 at the age of 11.

His main career was at the University of Halle.

Dauben's Biography of Cantor



Georg Cantor (1845–1918)

- Invented Set Theory.
- One-to-one Correspondence.
- Infinite and Well-ordered Sets.
- Cardinal and Ordinal Numbers.
- Proved Rational Numbers are countable.
- Proved Real Numbers are uncountable.
- Infinite Hierarchy of Infinities.

A Passionate Mathematician

In 1874, Cantor married Vally Guttmann.
They had six children.

According to Wikipedia:

"During his honeymoon in the Harz mountains, Cantor spent much time in mathematical discussions with Richard Dedekind."

[Cantor had met the renowned mathematician Dedekind two years earlier while he was on holiday in Switzerland.]

Potential versus Actual Infinity

Cantor argued that a potentially infinite concept always arises from, and depends upon, a pre-existing actually infinite concept.

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Cantor's contributions were revolutionary. He showed that there is an unlimited hierarchy of infinite numbers.

Infinite sets do not all have the same size: Some infinities are more infinite than others.

Cantor's ideas have had an enormous impact, on both mathematics and philosophy.

Origins of Set Theory

Since the 5^{th} century BC, ancient Greek mathematicians and early Indian mathematicians wrestled with the concept of infinity.

In the early 19th century, Bernard Bolzano made some tentative advances:

- He wrote a book called Paradoxes of the Infinite
- He identified the characteristic property: an infinite set can be mapped $1-1$ onto a proper subset of itself.

This was similar to Galileo's discovery.
It was later used by Cantor as a definition.

Cantor's Theorem

Cantor's theorem states that, for any set A, the set of all subsets of A — $\mathcal{P}(A)$ — has a strictly greater cardinality than A itself.

Theorem (Cantor). Let f be a map from set A to its power set $\mathcal{P}(A)$. Then $f : A \to \mathcal{P}(A)$ is not surjective. As a consequence, $\operatorname{card}(A) < \operatorname{card}(\mathcal{P}(A))$ holds for any set A.

Proof: Consider the set $B = \{x \in A \mid x \notin f(x)\}$. Suppose to the contrary that f is surjective. Then there exists $\xi \in A$ such that $f(\xi) = B$. But by construction, $\xi \in B \iff \xi \notin f(\xi) = B$. This is a contradiction. Thus, f cannot be surjective. On the other hand, $g : A \to \mathcal{P}(A)$ defined by $x \mapsto \{x\}$ is an injective map. Consequently, we must have $\operatorname{card}(A) < \operatorname{card}(\mathcal{P}(A))$. ■

This implies that there is no "largest infinity".

Large Cardinals

Set theorists are producing ever-larger cardinals:
inaccessible, ethereal and remarkable cardinals.
There are even numbers called Berkeley cardinals.

A Berkeley cardinal is a cardinal κ in a model of ZF with the property that for every transitive set M that includes κ, there is a nontrivial elementary embedding of M into M with critical point below κ.

These were introduced by Hugh Woodin in a seminar at The University of California, Berkeley in 1992.

The Absolute

The Absolute Infinite (Ω) was proposed by Cantor.

It lies beyond all the ordinal numbers and is not itself an ordinal number.

> ***"I envisage the system of all ordinals and denote it Ω".***

Ω is bigger than any conceivable or inconceivable number, finite or transfinite.

Cantor, a deeply pious man, linked Ω with God.

Outline

Infinitely Large and Infinitely Small

There are two complementary aspects of infinity:

- Something may be infinite in terms of **extension.**
- Something may be infinite in terms of **divisibility.**

Can a time interval be sub-divided without limit?

Can a interval in space be so sub-divided?

The Calculus

Line segments of **infinitesimal length** are considered in calculus (**mathematical analysis**).

Velocity is the rate of change of position:

$$v(t) = \frac{x(t + \Delta t) - x(t)}{\Delta t}$$

when Δt is **indefinitely small:** $v(t) = \mathrm{d}x(t)/\mathrm{d}t$.

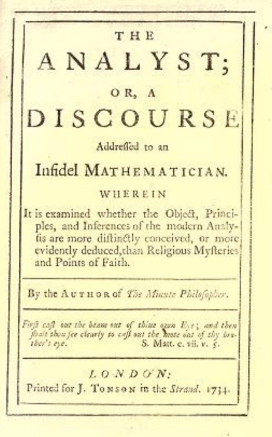
Quantities like "$\mathrm{d}t$" were not sharply defined.

Infinitesimals were treated as **non-zero** in some circumstances and as **zero** in others.

Controversy was rife and confusion abounded.

Bishop Berkeley (1685–1753)

Bishop Berkeley raised some pointed objections that were not satisfactorily answered for a century.



THE
ANALYST;
OR, A
DISCOURSE
Addreſſed to an
Infidel MATHEMATICIAN.
WHEREIN
It is examined whether the Object, Principles, and Inferences of the modern Analyſis are more diſtinctly conceived, or more evidently deduced, than Religious Myſteries and Points of Faith.

By the AUTHOR of *The Minute Philoſopher*.

Firſt caſt out the beam out of thine own Eye; and then ſhalt thou ſee clearly to caſt out the mote out of thy brother's eye. S. Matt. c. vii. v. 5.

LONDON:
Printed for J. TONSON in the *Strand*. 1734.

Bishop Berkeley

Berkeley, denied the existence of material substance: to exist is to be perceived.

> ***"All the choir of heaven and furniture of earth have no substance without a mind."***

$\star \quad \star \quad \star$

In 1734, Berkeley published "The Analyst", subtitled "A Discourse addressed to an Infidel Mathematician", a devastating critique of infinitesimals.

It was a direct attack on the foundations of calculus, specifically on the notion of infinitesimal change.

"Ghosts of Departed Quantities"

Berkeley had learned that Edmund Halley had dismissed Christian doctrine as inconceivable.

He argued that religious doctrines and beliefs were no more mysterious than the methods of analysis.

He devised the clever description of infinitesimals as "the ghosts of departed quantities".

These difficulties were not resolved for more than a century, when the epsilon-delta approach eliminated infinitesimals completely. [1]

[1] David Wilkins, Trinity College Dublin, maintains a website devoted to Berkeley.

Topological Calculus

The Abstract for this presentation mentioned a new formulation of Calculus, based on continuity.

I realised later that it was somewhat 'off-topic'.

Topological Calculus:
away with those nasty ε's and δ's[1]

A new approach to calculus has recently been developed by Peter Olver of the University of Minnesota. He calls it "Continuous Calculus" but indicates that the name "Topological Calculus" is also appropriate. He has provided an extensive set of notes, which are available online (Olver, 2022a).

An Article on Topological Calculus was posted today on my mathematical blog, thatsmaths.com.

Outline

The Shape of Space

The ultimate size of space is unknown:

We do not know whether space is limited in extent or whether it is unbounded.

We do not know whether time had a beginning or was “always there”.

Divisibility of Space and Time

It is possible that quantum mechanics may rule out an infinitely divisible space.

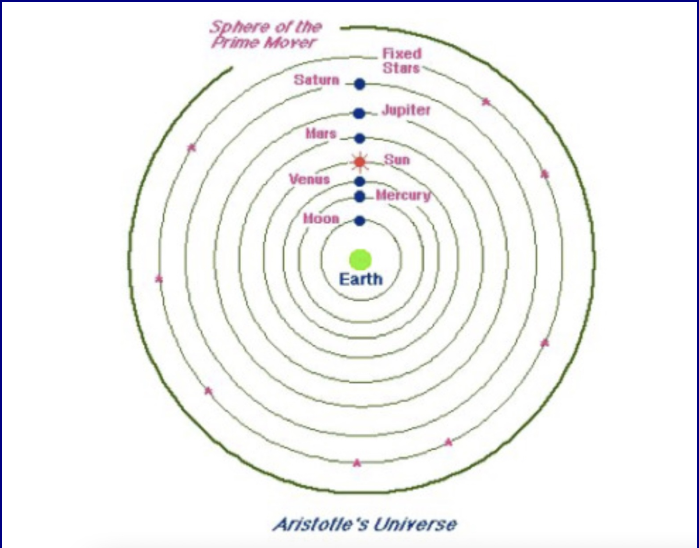
Planck's constant $\hbar$ may provide a measure of the granularity of space.

Using the physical constants, we can construct fundamental units of mass, length and time:

- Planck length: $l_\mathrm{P} = \sqrt{\hbar G/c^3} \approx 1.616 \times 10^{-35}$ m.
- Planck mass: $m_\mathrm{P} = \sqrt{\hbar c/G} \approx 2.176 \times 10^{-8}$ kg.
- Planck time: $t_\mathrm{P} = \sqrt{\hbar G/c^5} \approx 5.391 \times 10^{-44}$ s.

But could there be realms of physics below this level?

Aristotle's Universe of Celestial Spheres



Archytas (428–350 BC)

$A\rho\chi\upsilon\tau\alpha\varsigma$.

Born in Tarentum, son of Hestiaeus.

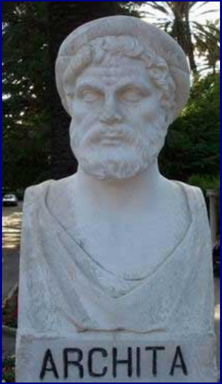
Mathematician and philosopher.

Pythagorean, student of Philolaus.

Formalised the Quadrivium

Provided a solution for the Delian problem of doubling the cube.

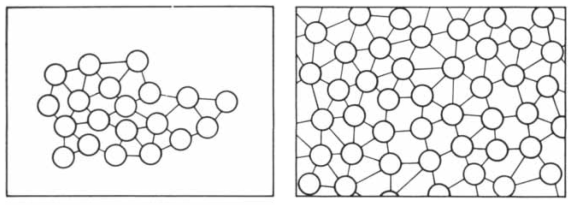
May have tutored Plato in mathematics.



Archytas and the size of the Universe

In a finite universe, what lies "beyond it"?

Newton. Leibniz. Kant



Newton argued that space could not be infinite:
"Only God could be infinite".

Leibniz argued that space could not be finite:
No Divine reason for it "here" rather than "there".

Kant argued that they were both correct:
We can never know whether it is one or the other.

Bounded and Unbounded Manifolds

A finite space can be unbounded:

- **The surface of a sphere.**
- **A solid 3-sphere.**

An infinite space can be bounded:

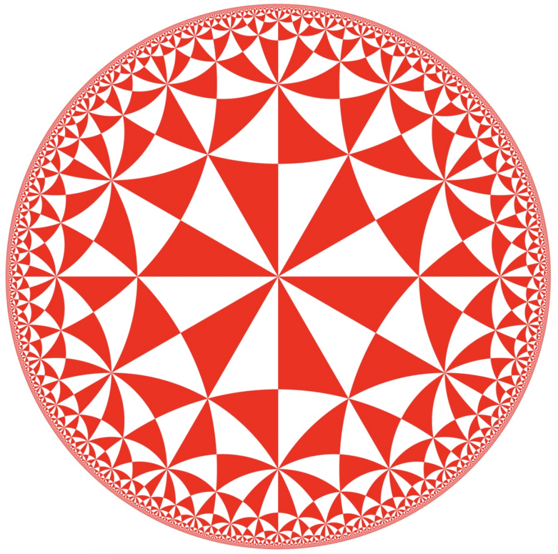
- **Poincaré Disk.**

Poincaré's Hyperbolic Disk

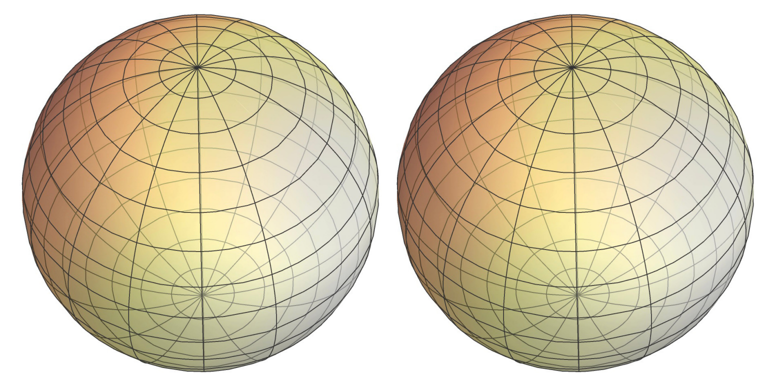
A bounded universe can be infinite in extent:

This is the **Poincaré Disk.**

The boundary is an infinite distance from the centre.

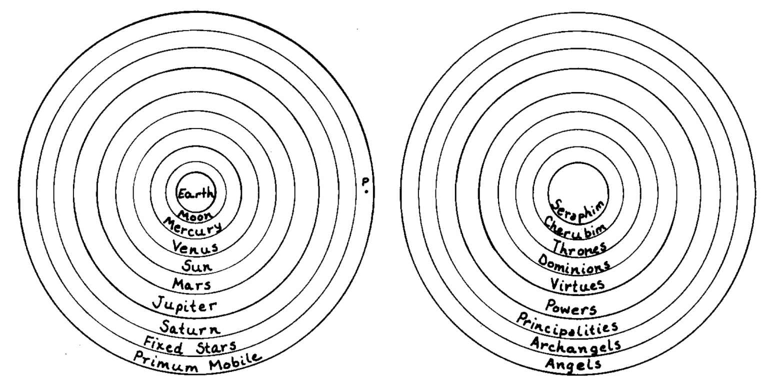


The Shape of Space: a 3-Sphere?



Imagine two solid spheres "glued together".
As we exit from one, we enter the other.

Dante's Universe: a 3-Sphere



The Universe as described in Dante's Divine Comedy.

The Empyrean

[Illustration by Gustave Doré]

Dante and Beatrice gazing into the Empyrean.

God
at the centre, surrounded by choirs of angels.

Jean Charles de Borda (1733–1799)

“Without mathematics, we do not get to the bottom of philosophy;

“without philosophy, we do not get to the bottom of mathematics;

“without both, we do not get to the bottom of anything.”

Thank you