Met Éireann — The Irish Meteorological Service

Historical Note No. 3

Oscillations of an Elastic Pendulum as an Example of the Oscillations of Two Parametrically Coupled Linear Systems

A. Vitt and G. Gorelik

Journal of Technical Physics, Vol. 3 (2-3), 294–307 (1933)

This is a translation of a paper by A. Vitt and G. Gorelik

Kolebaniya uprugogo mayatnika kak primer kolebaniy dvukh parametricheski svyazannykh linejnykh sistem

originally published in 1933 in the Russian journal *Zhurnal Tekhnicheskoy Fiziki*, Vol. 3 (2-3), 294–307.

Translated from the Russian by Lisa Shields

with an Introduction by Peter Lynch

(A PostScript version of this paper is available on request from `Peter.Lynch@met.ie`)

Met Éireann, Glasnevin Hill, Dublin 9, Ireland

December, 1999

Introduction

by

Peter Lynch

The paper appearing here for the first time in an English translation is a detailed study of a low-order Hamiltonian system, the elastic pendulum. The original motivation for the study was the desire to find a simple classical description for the quantum mechanical phenomenon of the splitting of the spectral lines in the CO_2 molecule. The study was suggested to the authors by the Russian physicist L. I. Mandel'shtam. The 2:1 resonance of the pendulum provides a classical analogue to the resonance of the quantum system which has ionic oscillations with frequencies close to this ratio.

The simple system under study possesses a rich and varied range of dynamical behaviour. For large amplitudes the motion is chaotic. Breitenberger and Mueller (1981) remark that 'this simple system looks like a toy at best, but its behaviour is astonishingly complex, with many facets of more than academic lustre'. However, the concern here is the range of amplitudes where the motion is regular so that classical perturbation techniques yield meaningful results.

This work is the earliest comprehensive analysis of the elastic pendulum. Although the paper is frequently referenced by later authors, it is clear that, in some cases, they have not studied this work. Van der Burgh (1968) inaccurately describes the paper as 'a mainly qualitative description'; in fact, his own paper contains little that is not already contained in Vitt and Gorelik[1]. Breitenberger and Mueller (*loc. cit.*) note that this important paper has often been misquoted. Davidović, *et al.* (1996) give a brief but accurate synopsis of its contents, and state even more strongly that the paper has been 'more frequently quoted and misquoted than read by other authors'. I think this is a fair point; it is time the work was available in English.

⋆ ⋆ ⋆ ⋆ ⋆

The contents of the paper will now be summarised. Vitt and Gorelik (1933) consider the motion of an elastic pendulum confined to a plane, thus having two degrees of freedom. The authors set up the Lagrangian equations for the system, assuming the amplitude is sufficiently small that terms beyond cubic order can be ignored. They identify the linear vertical (springing) and horizontal (swinging) modes of the system. They concentrate on the special case where the vertical frequency is twice the horizontal frequency; in this case, each type of linear oscillation can induce the other through nonlinear interactions. Vertical oscillations can induce horizontal motion through parametric resonance, whereas horizontal or swinging motion can lead to vertical springing oscillations through direct resonant forcing.

In §2, periodic solutions are sought using the technique of secular perturbations. Two distinct solutions are found in which the trajectory of the bob is a parabola. For these particular solutions, the effect of the nonlinear interactions is to modify the frequency of the oscillations, but preserving the 2:1 ratio. The cup-like solutions, with

[1] Indeed, the incorrect reference given by Van der Burgh to the Vitt and Gorelik paper is identical to that in Minorsky (1962, p.506), suggesting that he took the reference from there and not from the original paper.

concave-upward parabola, have frequency slightly depressed, the cap-like ones with a concave-downward trajectory have a somewhat augmented frequency. There is no energy transfer between the springing and swinging motion. These solutions are easily demonstrated in the physical system.

In §3, solutions which transfer energy back and forth between the swinging and springing motion are considered. A perturbed Hamiltonian is constructed, action-angle variables are introduced, and the Hamiltonian is averaged with respect to the fast variations, so that the lowest-order solution is immediate. An equation (equation (20) in the paper) is derived for the slowly-varying amplitude of the horizontal oscillation. The integral curves of the equation are illustrated, and the patterns of the trajectories in phase-space are depicted, clearly illustrating both the generic behaviour and important limiting cases. Curiously, although Eq. (20) is easily solved in terms of Jacobian elliptic functions, the authors make no mention of this. A qualitative description of the energy transfer follows, and an explicit formula for the modulation period is derived (equation [21] in the translation, un-numbered in the original). Again, this may be expressed as a complete elliptic integral of the first kind, though the authors do not say this.

In §4, the authors describe a series of experiments, and show that the theoretically calculated results are in good agreement with the observed behaviour of the physical system. They make no reference to its three-dimensional motion. This is surprising because, in their experiments, they cannot have failed to have noticed the remarkable propensity of the bob to deviate from the original swing plane, either in a precessing elliptical orbit, or in successive horizontal excursions with different azimuthal directions. The three-dimensional motion is discussed in Lynch (1999). Probably, Vitt and Gorelik did notice the interesting behaviour, but found it not directly relevant to their goal of providing a classical analogue for quantum resonance.

In the concluding section, the nonlinear interaction of the elastic pendulum is compared and contrasted to modal interactions in linear systems. One of the crucial differences is the dependence of the non-linear interactions on the initial conditions. The authors then discuss the original motivation for the work, the phenomenon of Fermi resonance, seen in the line spectrum of CO_2 and in other molecules for which there is a frequency ratio close to 2:1. Although this is a quantum-mechanical effect, it is closely analogous to the classical phenomenon of nonlinear resonance seen in the swinging spring.

⋆ ⋆ ⋆ ⋆ ⋆

Current interest in the swinging spring arises from the rich variety of its solutions. For very small amplitudes, the motion is regular, and classical perturbation theory yields valid results. As the amplitude is increased, the regular motion breaks down into a chaotic regime which occupies more and more of phase space as the energy grows. However, for very large energies, a regular and predictable regime is re-established (Núñez-Yépez, *et al.*, 1990). This can easily be understood: for very high energies, the system rotates rapidly around the point of suspension and is no longer libratory.

Of course, the chaotic regime was not considered by Vitt and Gorelik, as the relevant concepts were unavailable to them. However, recent studies have examined this behaviour in some detail. A large list of references may be found in Lynch (2000).

That paper considers the elastic pendulum as a simple model for balance in the atmosphere. The concepts of filtering, initialization and the slow manifold, so important for atmospheric dynamics, can be introduced and lucidly illustrated in the context of the simple system. The swinging and springing oscillations act as analogues of the Rossby and gravity waves in the atmosphere.

Finally we may remark that Jin, *et al.*, (1994) have modelled the El Niño phenomenon using arguments based on transition to chaos through a series of frequency-locked steps induced by non-linear resonance with the Earth's annual cycle. Their model produces results consistent with currently available data. Thus, the non-linear resonance observed in our simple mechanical system may provide the basis for a paradigm of the most important interannual variation in the ocean-atmosphere climate system.

References

Breitenberger, E and R D Mueller, 1981: The elastic pendulum: a nonlinear paradigm. *J. Maths. Phys.*, **22**, 1196–1210.

Burgh, A van der, 1968: On the asymptotic solutions of the differential equations of the elastic pendulum. *J. Mécan*, **7**, 507–520.

Davidović, D , B A Aničin and V M Babović, 1996: The libration limits of the elastic pendulum. *Am. J. Phys.*, **64**, 338–342.

Jin, F-F, J.D. Neelin and M. Ghil, 1994: El Niño on the devil's staircase: annual subharmonic steps to chaos. *Science*, **264**, 70–72.

Lynch, Peter, 1996: The Elastic Pendulum: a Simple Mechanical Model of Atmospheric Balance. Tech. Note No. 54, Met Éireann, Dublin, Ireland.

Lynch, Peter, 1999: Resonant Motions of the Swinging Spring. Tech. Note No. 56, Met Éireann, Dublin, Ireland. (PostScript version available from Peter.Lynch@met.ie)

Lynch, Peter, 2000: The Swinging Spring: a Simple Model for Atmospheric Balance. To appear in *Proceedings of the Symposium on the Mathematics of Atmosphere-Ocean Dynamics.* Isaac Newton Institute, June–December, 1996. Cambridge University Press.

Minorsky, N, 1962: *Nonlinear Oscillations.* Van Nostrand, Princeton. p506.

Núñez-Yépez, H N, A L Salas-Brito, C A Vargas and L Vicente, 1990: Onset of chaos in an extensible pendulum. *Phys. Lett.*, **A145**, 101–105.

Vitt, A and G Gorelik, 1933: Oscillations of an Elastic Pendulum as an Example of the Oscillations of Two Parametrically Coupled Linear Systems. *Zh. Tekh. Fiz.* (*J. Tech. Phys.*) **3**(2-3), 294–307.

Oscillations of an Elastic Pendulum as an Example of the Oscillations of Two Parametrically Coupled Linear Systems

A. Vitt and G. Gorelik

Abstract*

This work investigates the oscillations of an elastic pendulum. Only planar oscillations are considered and therefore only two degrees of freedom investigated, namely the vertical and one of the horizontal dimensions. The investigation is based on the theory of secular pertubations. Of particular interest is the case where the frequency of the vertical motion is twice the frequency of the horizontal; this leads to a so-called parameteric resonance of the coupled system, which manifests itself as an energy transfer from one component to the other and vice versa. The speed and amplitude of the energy transfer depend essentially on the inital conditions. Other mechanical or electrical systems with two degrees of freedom can be treated in similar ways, e.g., two oscillating circuits coupled by a transformer with an iron core. The results of the theory are compared with experiment and are in complete agreement. Finally, a connection is indicated between the oscillations of an elastic pendulum and the model of the CO_2 molecule which was recently presented by Fermi to explain the splitting of the spectral lines for this compound.

* Translated from the German by Klara Finkele, Met Éireann.

Oscillations of an Elastic Pendulum as an Example of the Oscillations of Two Parametrically Coupled Linear Systems

A. Vitt and G. Gorelik

1. Introduction and Statement of the Problem

In this article a study is made of small oscillations around an equilibrium configuration of a conservative system with two degrees of freedom, which is profoundly different from the commonly studied and well-known linear[1] oscillatory systems with two degrees of freedom. The difference is shown in the fact that, however many small oscillations there are, the behaviour of the system in which we are interested here is essentially determined by nonlinear terms appearing in its differential equations and expressing the coupling between the two degrees of freedom. By way of a simple mechanical example of such a system, we will examine an elastic pendulum, that is, a weight hanging on a spring, the upper end of which is fixed in place. We shall assume that the movement takes place in one definite vertical plane. Let r denote the instantaneous value of the spring's length, ℓ_0 the length of the spring in the absence of a weight, φ the angle of deviation (we shall always assume it to be small), m the mass of the pendulum's bob, k the constant of elasticity of the spring, and g the acceleration of gravity. For the kinetic and potential energy of our system we have the following:

$$T = \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\varphi}^2)$$

$$V = \tfrac{1}{2}k(r - \ell_0)^2 - mgr(1 - \tfrac{1}{2}\varphi^2)$$

where differentiation in time is indicated by dots.

We shall replace r by the coordinate z, equal to the relative lengthening of the spring compared with its equilibrium length $\ell = \ell_0 + mg/k$, that is, we shall assume that

$$z = \frac{r - \ell}{\ell}\,.$$

As we are limited to the case of small oscillations, we shall consider that z is very small in comparison with unity. Ignoring terms of order higher than the third order in z and φ and their products we obtain, for the kinetic and potential energy, the new expressions

$$T = \frac{m\ell^2}{2}\left(\dot{z}^2 + \dot{\varphi}^2 + 2z\dot{\varphi}^2\right) \tag{1}$$

$$V = \frac{m\ell^2}{2}\left(\frac{k}{m}z^2 + \frac{g}{\ell}\varphi^2 + \frac{g}{\ell}z\varphi^2\right) \tag{2}$$

Using (1) and (2) we now formulate Lagrangian equations of motion:

$$\ddot{z} + \frac{k}{m}z + \left(\frac{1}{2}\frac{g}{\ell}\varphi^2 - \dot{\varphi}^2\right) = 0 \tag{3}$$

[1] That is, systems whose motion is represented by linear differential equations.

$$\ddot{\varphi} + \frac{g}{\ell}\varphi + \left(\frac{g}{\ell}z\varphi + 2\dot{z}\dot{\varphi} + 2z\ddot{\varphi}\right) = 0 \tag{4}$$

An ordinary linear system with two degrees of freedom may be regarded—especially when the coupling is weak—as a pair of two 'component' systems, each with one degree of freedom, linearly coupled to each other. For example, two pendulums joined by a weak spring are component systems doubly coupled one to the other, each of which possesses its own 'component frequency' and can be isolated from the full system by securing one of the pendulums, that is by depriving it of one of the degrees of freedom. In exactly the same way, as equations (3) and (4) demonstrate, our elastic pendulum can be regarded as a pair of two interconnected linear oscillators, each of which is isolated from the full system when one of its degrees of freedom is isolated: thus, preventing the weight from straying from the vertical (that is, assuming that $\varphi \equiv 0$) we obtain a vertical oscillator oscillating in accordance with the linear equation

$$\ddot{z} + \frac{k}{m}z = 0\,.$$

with an angular frequency $\alpha = \sqrt{k/m}$; in preventing the pendulum from changing its length, that is, by replacing the spring with a rigid shaft (in this case $z \equiv 0$), we obtain a horizontal oscillator oscillating in accordance with the linear equation

$$\ddot{\varphi} + \frac{g}{\ell}\varphi = 0$$

with an angular frequency $\beta = \sqrt{g/\ell}$. These vertical and horizontal oscillators are component systems with component frequencies α and β. The coupling between the component systems is nonlinear: this is shown by the nonlinear 'coupling terms' enclosed in brackets in equations (3) and (4).

We know that the behaviour of weakly coupled linear oscillatory systems depends essentially on the relationship between the component frequencies. If the latter differ strongly from each other, the oscillation as a whole differs little from those oscillations which would have been produced by component systems in the absence of a coupling; in the example of two pendulums connected by a weak spring, each pendulum would oscillate approximately in the same way as if it were free. But the situation is completely different when the component equations are equal to one another or, as we say, when resonance sets in between the component systems. In this case energy is periodically transferred from one pendulum to the other, and each pendulum thus performs a modulated oscillation which may be represented by the sum of two sinusoidal oscillations with frequencies one of which is greater and the other smaller than the component frequencies; it is as if, due to the coupling, the component frequency splits in two and we observe a beating of these two frequencies. The stronger the coupling the more the frequencies split up: that is, the faster the transfer of energy is produced.

In our case of nonlinearly coupled linear systems, when the oscillations are small the nonlinear terms consist of small values of a high order and, generally speaking, they are insignificant: the component systems have little influence on one another. We may be sure that in some particular relationship of component frequencies, namely when $\alpha = 2\beta$, a strong interaction between component frequencies should take place and we may expect resonance phenomena somewhat analogous to those which arise in the case of a linear coupling when component frequencies are equal.

In practice:

1) We move the weight vertically from a position of equilibrium and we release it, that is, we set up a vertical component oscillation

$$z = A\cos\alpha t\,. \tag{5}$$

As a result of this oscillation the length of our pendulum—its parameter—begins to change periodically; when we substitute (5) into (4) we obtain for φ a linear equation with periodic coefficients

$$(1 + 2A\cos\alpha t)\ddot{\varphi} - 2\alpha\sin\alpha t\,\dot{\varphi} + \beta^2(1 + A\cos\alpha t)\varphi = 0\,. \tag{6}$$

Since $\alpha = 2\beta$, the parameter of the system changes with a frequency double its own frequency. But it is known that in this case the phenomenon of parametric resonance begins. The pendulum becomes unstable. The smallest disturbance or deflection is enough for it to begin to undergo ever increasing horizontal swings. Thus, in the case of $\alpha = 2\beta$, the vertical oscillations cause the pendulum to swing in a horizontal direction.

2) First deflecting the pendulum, without stretching it, we set up a horizontal component oscillation

$$\varphi = B\cos\beta t\,. \tag{7}$$

The centrifugal force developing with this movement, reaching its maximum twice in each oscillation, will stretch the spring periodically. Substituting (7) in equation (3) we obtain

$$\ddot{z} + \alpha^2 z = \frac{\beta^2 B^2}{4}(1 - 3\cos 2\beta t)\,, \tag{8}$$

that is, the oscillator equation, under the influence of an external force with a sinusoidal component having the frequency 2β. But as $2\beta = \alpha$, this force will act in resonance on the vertical oscillator, and the latter will begin to perform oscillations of ever-increasing amplitude. Thus, in the case $2\beta = \alpha$, the horizontal oscillations cause the weight to oscillate in a vertical direction.

It is clear that the systems of equations (5), (6) and (7), (8) preserve engergy only at the start of the processes under consideration: each pair does not take into account the reciprocal action of the 'swinging' oscillator on the oscillator 'being swung'. But the nature of this action arises directly from the fact that our system is conservative: the energy of the 'swinging oscillator' can increase only at the expense of the weakening of the oscillator 'being swung'. Therefore in the case 1) the build-up of horizontal oscillations should be accompanied by a decrease in vertical oscillations, and in case 2) the build-up of vertical oscillations should take place at the expense of an attenuation of the horizontal ones. In exactly the same way, in the case of linearly coupled pendulums, the oscillations of the one build up at the same time as the oscillations of the other die down. This gives rise to the problem: not limiting ourselves to the initial stages of the movements, represented by equations (5), (6) and (7), (8), how to investigate the movements fully for any and every initial condition and to characterise them; may there not arise a periodic transfer of energy between the two degrees of freedom, analogous to the one taking place in linearly coupled systems? Ignoring, in our case of a nonlinear coupling, the reciprocal effect of the horizontal oscillation on the vertical one we obtain, instead of a system of nonlinear equations (3), (4) not explicitly containing time, the linear equation (6) which clearly contains time: this is an equation of a linear system with periodically changing parameters. Just as with a linear coupling, ignoring the action of one partial system on the other one, we replace a system of linear equations not explicitly containing time with a linear equation the right-hand side of which clearly contains time, that is, with an equation of forced oscillations.[2] And just as the theory of linearly coupled linear oscillatory systems is an extension of the theory of ordinary resonance to the case where one must not ignore the reciprocal effect of a resonator on the source of energy, the theory of our nonlinearly coupled systems may be viewed as an extension of the theory of parametric resonance to the case where one must not ignore the reciprocal effect of a parametrically created system on the source of energy which modulates its parameters. For that reason it is appropriate to say that we are dealing with parametrically coupled systems.

2. Periodic Solutions

Before we study the movements of our system in general we will satisfy ourselves that equations (3), (4) possess solutions where z and φ are periodic functions—in the first approximation, sinusoidal functions—of time and the initial conditions are such that there is no transfer of energy between the component systems.

As we shall employ the pertubation method, we shall—to limit the order of magnitude of the various values and in accordance with the assumption that z and φ are small—introduce the small parameter ϵ by means of the equations

$$z = \epsilon x\,,$$

$$\varphi = \epsilon y\,.$$

Introducing variables x, y in equations (3), (4), ignoring terms of order higher than the first in ϵ and assuming that $\sqrt{k/m} = 2\sqrt{g/\ell} = 2\beta$, we obtain

$$\ddot{x} + 4\beta^2 x = \epsilon\left(\dot{y}^2 - \tfrac{1}{2}\beta^2 y^2\right)\,, \tag{3$'$}$$

$$\ddot{y} + \beta^2 y = \epsilon\left(\beta^2 xy - 2\dot{x}\dot{y}\right)\,. \tag{4$'$}$$

When $\epsilon = 0$, these equations give us a periodic 'unperturbed' solution, in which x and y are sinusoidal functions of time, whose frequency has the ratio 2:1. We will assume that for $\epsilon \neq 0$ there is a periodic solution with ratio of frequencies 2:1, tending to this solution when $\epsilon = 0$, and we shall find this periodic solution. We shall let ω denote the frequency of the horizontal oscillation in this 'perturbed' solution and we shall assume that it is distinguished from the corresponding frequency of the unperturbed solution by a magnitude of the order ϵ. We have

$$\omega^2 = \beta^2 + \epsilon a$$

where a is a certain finite value. Introducing ω^2 in (3$'$), (4$'$) and once more rejecting terms of the order of ϵ^2, we obtain

$$\ddot{x} + 4\omega^2 x = \epsilon\left(\dot{y}^2 - \tfrac{1}{2}\omega^2 y^2 + 4ax\right)\,, \tag{3$''$}$$

[2] For example, in the case of two coupled electric circuits, if it is possible to ignore the action of the secondary circuit on the primary one, it is possible to consider that a sinusoidal emf (electro-motive force) is set up in the secondary circuit.

$$\ddot{y} + \omega^2 y = \epsilon\left(\omega^2 xy - 2\dot{x}\dot{y} + ay\right) . \tag{4''}$$

We shall seek a solution in the form of series in powers of ϵ:

$$x = x_0 + \epsilon x_1 + \dots$$

$$y = y_0 + \epsilon y_1 + \dots$$

Substituting these series in (3″), (4″) and equating the coefficients in equal powers of ϵ, we obtain

$$\ddot{x}_0 + 4\omega^2 x_0 = 0\,, \tag{3a}$$

$$\ddot{y}_0 + 4\omega^2 y_0 = 0\,, \tag{4a}$$

$$\ddot{x}_1 + 4\omega^2 x_1 = \dot{y}_0^2 - \tfrac{1}{2}\omega^2 y_0^2 + 4ax_0\,, \tag{3b}$$

$$\ddot{y}_1 + \omega^2 y_1 = \omega^2 x_0 y_0 - 2\dot{x}_0\dot{y}_0 + ay_0\,. \tag{4b}$$

We shall write the solution of equations (3*a*), (4*a*), selecting a determined origin of time, in the form:

$$x_0 = A\cos 2\omega t\,,$$

$$y_0 = B_1\cos\omega t + B_2\sin\omega t\,.$$

In order for x_0, y_0 to be an approximately periodic solution of equations (3″), (4″), it is necessary for the resonance terms to be reduced to zero when they are placed in the right-hand side of equations (3*b*), (4*b*); that is, in the right-hand side of (3*b*) the terms of frequency 2ω and in the right-hand side of (4*b*) the terms of frequency ω. These conditions give the following system of equations for determining the amplitudes A, B_1 and B_2 and the frequency ω:

$$B_1 B_2 = 0\,,$$

$$4aA - \tfrac{3}{4}\omega^2(B_1^2 - B_2^2) = 0\,,$$

$$\left(a - \tfrac{3}{2}\omega^2 A\right)B_1 = 0\,,$$

$$\left(a + \tfrac{3}{2}\omega^2 A\right)B_2 = 0\,.$$

This system permits a solution with three variants:

I) $$B_1 = \pm\sqrt{8}A\,,\quad B_2 = 0\,,\quad \omega = \omega_1 = \beta\left(1 + \tfrac{3}{4}\epsilon A\right)\,,$$

II) $$B_1 = 0\,,\quad B_2 = \pm\sqrt{8}A\,,\quad \omega = \omega_2 = \beta\left(1 - \tfrac{3}{4}\epsilon A\right)\,,$$

III) $$B_1 = B_2 = 0\,,\quad a = 0\,.$$

Case III is the vertical oscillation already considered in the introduction. As we know, it is unstable.

The Lissajous figure corresponding to oscillations of the types I and II are illustrated in Fig. 1 [Figs. 1 to 5 appear on page 15 below]. The ratio of frequencies of oscillation II (slower than the unperturbed) and oscillation I (faster than the unperturbed) is

$$\frac{\omega_2}{\omega_1} = 1 - \tfrac{3}{2}\epsilon A\,. \tag{9}$$

When the oscillations are of types I and II, the coupling between the component systems is effective only in that the frequency of their oscillation changes. No energy exchange takes place between component systems.

3. Energy Transfer

We shall now pass on to a more general investigation of the motion of our systems. Employing relations (1), (2), we shall introduce the momenta $p_1 = \partial T/\partial \dot{z}$, $p_2 = \partial T/\partial \dot{\varphi}$, and the conjucate coordinates z, φ, and we shall construct the Hamiltonian function:

$$H = \underbrace{\frac{1}{2m'}(p_1^2 + p_2^2) + \frac{m'}{2}(\alpha^2 z^2 + \beta^2 \varphi^2)}_{H_0} - \frac{1}{m'} z p_2^2 + \frac{m'}{2}\beta^2 z \varphi^2\,, \tag{10}$$

where $m' = m\ell^2$. (As before, we shall ignore the higher powers of z.) The terms designated together as H_0 correspond to the uncoupled component systems; the other terms correspond to the perturbation introduced by the coupling.

Using the standard [canonical] transform[3]

$$z = \sqrt{\frac{J_1}{\pi\alpha m'}}\sin 2\pi w_1\,, \qquad p_1 = \sqrt{\frac{\alpha m' J_1}{\pi}}\cos 2\pi w_1$$

$$\varphi = \sqrt{\frac{J_2}{\pi\beta m'}}\sin 2\pi w_2\,, \qquad p_2 = \sqrt{\frac{\beta m' J_2}{\pi}}\cos 2\pi w_2 \tag{11}$$

we shall transform to 'angle variables' w_1, w_2 and 'action variables' J_1, J_2 (*Winkel- und Wirkungsvariabeln*) of the unperturbed system, and the Hamiltonian function will be written thus:

$$H = \underbrace{\frac{1}{2\pi}(\alpha J_1 + \beta J_2)}_{H_0} + \frac{\beta}{2\pi\sqrt{\pi\alpha m'}} J_2 \sqrt{J_1}\sin 2\pi w_1 (\sin^2 2\pi w_2 - 2\cos^2 2\pi w_2)\,. \tag{10'}$$

We shall temporarily discard the supposition that $\alpha = 2\beta$; let α and β be arbitrary. We know from general theory that two different cases can occur:

a) α and β are not in a simple rational relationship; if the perturbation is small ($(H - H_0)/H \ll 1$) then the frequencies and amplitudes slowly change around their mean (unperturbed) values; the corresponding variations are of the same order as $(H - H_0)/H$.

b) α and β are in a simple rational relationship (degeneracy); in this case even a small perturbation can give rise to a large change in amplitude, that is, of the same order of magnitude as the unperturbed values. Since, in the problem being studied here, z and

[3] See, for example, M. Born, *Atommechanik*, p. 293.

φ are small values, the perturbation is small and consequently noticeable energy transfer from one component system to the other may be expected only where degeneracy exists. We shall proceed to its analysis, using the method of so-called 'secular perturbations'.[4]

Let $\alpha = n_1\omega$, $\beta = n_2\omega$, where n_1, n_2 are whole numbers. We have

$$H_0 = \frac{\omega}{2\pi}(n_1 J_1 + n_2 J_2)\,.$$

[**NOTE:** V&G write I instead of J here.] In our case of degeneracy we can, on the basis of general theory, introduce the new angle and action variables v_1, v_2, I_1, I_2, in such a way that the Hamiltonian function of the unperturbed problem depends on one only of the momenta, let us say I_1. We achieve this, for example, by means of a canonical transformation produced from the function

$$V = (n_1 J_1 + n_2 J_2)v_1 + J_2 v_2\,.$$

It gives the transformation equations

$$\begin{aligned}
I_1 &= \frac{\partial V}{\partial v_1} = n_1 J_1 + n_2 J_2 \\
I_2 &= \frac{\partial V}{\partial v_2} = J_2 \\
w_1 &= \frac{\partial V}{\partial J_1} = n_1 v_1 \\
w_2 &= \frac{\partial V}{\partial J_2} = n_2 v_1 + v_2
\end{aligned} \tag{12}$$

and we obtain

$$H = \underbrace{\frac{\omega I_1}{2\pi}}_{H_0} + \gamma I_2 \sqrt{\frac{I_1 - n_2 I_2}{n_1}}\,\sin 2\pi n_1 v_1 \left\{\sin^2 2\pi(n_2 v_1 + v_2) - 2\cos^2 2\pi(n_2 v_1 + v_2)\right\} \tag{10''}$$

where, for the sake of brevity, we set $\gamma = n_2\omega/(2\pi\sqrt{\pi n_1 \omega m'})$.

Hamilton's equations for the unperturbed motion ($H = H_0$) are

$$\begin{aligned}
\frac{dv_1}{dt} &= \frac{\partial H_0}{\partial I_1} = \frac{\omega}{2\pi}\,, \quad \text{whence} \quad v_1 = (\omega/2\pi)t \\
\frac{dv_2}{dt} &= \frac{\partial H_0}{\partial I_2} = 0\,, \quad \text{whence} \quad v_2 = \text{const.} \\
\frac{dI_1}{dt} &= -\frac{\partial H_0}{\partial v_1} = 0\,, \quad \text{whence} \quad I_1 = \text{const.} \\
\frac{dI_2}{dt} &= -\frac{\partial H_0}{\partial v_2} = 0\,, \quad \text{whence} \quad I_2 = \text{const.}
\end{aligned}$$

The following reasoning is the basis for the method of secular perturbations. It is assumed that, due to the perturbation, the value v_2 can become a function of time.

[4] See M. Born, *ibid*, p. 123.

But, as the speed of its change $dv_2/dt \to 0$ as $H \to H_0$, where values of $(H - H_0)/H$ are sufficiently small, the speed of variation of v_2 should be small in comparison with the speed of variation of v_1 ('secular' variation). In exactly the same way, when $(H - H_0)/H$ is small, I_1 and I_2, if they are functions of time, can change only slowly. Later it is assumed that in order to study the slow change of variables one can average the Hamiltonian function with respect to the swiftly changing variable v_1 and, using the averaged Hamiltonian function $\bar{H}$, construct new Hamilton's equations for the variables I_1, I_2 v_2.

In our problem

$$\bar{H} = \frac{\omega I_1}{2\pi} + \gamma I_2 \sqrt{\frac{I_1 - n_2 I_2}{n_1}} \bar{f} \tag{13}$$

where $\bar{f}$ is the mean with respect to v_1 of the function

$$\begin{aligned} f &= \sin 2\pi n_1 v_1 \{\sin^2 2\pi(n_2 v_1 + v_2) - 2\cos^2 2\pi(n_2 v_1 + v_2)\} \\ &= -\tfrac{1}{2} \sin 2\pi n_1 v_1 - \tfrac{3}{4} \cos 4\pi v_2 \{\sin 2\pi(n_1 - 2n_2) v_1 + \sin 2\pi(n_1 + 2n_2) v_1\} \\ &= \tfrac{3}{4} \sin 4\pi v_2 \{\cos 2\pi(n_1 - 2n_2) v_1 - \cos 2\pi(n_1 + 2n_2) v_1\}. \end{aligned}$$

If $n_1 = 2n_2$, and in this case only, $\bar{f}$ is different from zero and we have

$$\bar{f} = \tfrac{3}{4} \sin 4\pi v_2 . \tag{14}$$

We find, after simplifying, a secular perturbation of motion of our systems. (For other rational relations between frequencies, the perturbation becomes noticeable only when the oscillations are sufficiently strong for the terms higher than the third order, discarded by us in the Hamiltonian function, to have reached a significant size; this observation agrees with the fact that in ordinary parametric resonance the region of instability, corresponding to $\alpha = 2\beta$ is very much stronger—it has a different order of magnitude—than the other regions.)

And so we shall return to the case where $\alpha = 2\beta$. Assuming that $n_1 = 2n_2$, we obtain (assuming, without loss of generality, that $n_2 = 1$, $n_1 = 2$ (i.e., that $\omega = \beta$) it follows from (12) that

$$\begin{aligned} I_1 &= 2J_1 + J_2 \\ I_2 &= J_2 \\ w_1 &= 2v_1 \\ w_2 &= v_1 + v_2 \end{aligned} \tag{12'}$$

On the basis of (13), (14) we shall write the averaged Hamiltonian function

$$\bar{H} = \frac{\beta I_1}{2\pi} + \frac{3}{\sqrt{8}} \gamma I_2 \sqrt{I_1 - I_2} \sin 4\pi v_2 , \tag{15}$$

[leading to the canonical equations]

$$\frac{dI_1}{dt} = -\frac{\partial \bar{H}}{\partial v_1} = 0 , \quad \text{whence} \quad I_1 = a, \text{ an integration constant} \tag{16}$$

$$\frac{dI_2}{dt} = -\frac{\partial \bar{H}}{\partial v_2} = \frac{12}{\sqrt{8}} \gamma I_2 \sqrt{a - I_2} \cos 4\pi v_2 \tag{17}$$

$$\frac{dv_2}{dt} = \frac{\partial \bar{H}}{\partial I_2} = \frac{3}{\sqrt{8}} \gamma \left(\sqrt{a - I_2} - \tfrac{1}{2} I_2 \frac{1}{\sqrt{a - I_2}} \right) \sin 4\pi v_2 \tag{18}$$

[**NOTE:** Coefficient corrected in (18).] We shall introduce the designation

$$H' = \frac{\bar{H} - \dfrac{\omega I_1}{2\pi}}{\dfrac{3}{\sqrt{8}}\gamma}$$

[**NOTE:** correction in denominator.] Our system is conservative: $\bar{H}$ = const; and, because $I_1 = a$ is also constant we have, in accordance with (15), (16):

$$H' = I_2\sqrt{a - I_2}\,\sin 4\pi v_2 = \text{const.} \tag{19}$$

Denoting $I_2 = x$ for the sake of brevity, and eliminating v_2 from equations (17) and (19), we obtain the differential equation

$$\frac{dx}{dt} = \pm\sqrt{-x^3 + ax^2 - H'^2} \tag{20}$$

on which we shall base our discussion. [**NOTE:** the time variable has been re-scaled as $t' = (12\gamma/\sqrt{8})t$ and the prime dropped; V&G omit mention of this.] We shall explain beforehand the physical meaning of the variable x. On the basis of equataions (11) and (12) we have

$$z = \sqrt{\frac{a - x}{2\pi\alpha m'}}\,\sin 4\pi v_1$$

$$\varphi = \sqrt{\frac{x}{\pi\beta m'}}\,\sin 2\pi(v_1 + v_2) \tag{11'}$$

from which it follows that x is proportional to the square of the amplitude of the horizontal oscillation, and $(a - x)$ is proportional to the square of the amplitude of the vertical one. Equations (11′) show that when the amplitude of the horizontal oscillation increases, the amplitude of the vertical one decreases, and vice-versa.

The integral curves of the differential equation (20) are expressed by means of the equation

$$y = \pm\sqrt{\Phi(x)} \tag{20'}$$

where $y = dx/dt$ and $\Phi(x) = -x^3 + ax^2 - H'^2$. We shall introduce initial conditions: when $t = 0$, let $I_2 = b$ and $\sin 4\pi v_2 = c$; then

$$H' = bc\sqrt{a - b}$$

$$\Phi(x) = -x^3 + ax^2 - b^2c^2(a - b)\,.$$

Both z and φ are real valued; equations (11′) show that, thanks to this, a can assume only positive values, and b is confined to the range $0 \le b \le a$. With these conditions the function $\Phi(x)$ has the following properties:

1) $\Phi(x)$ has its minumum when $x = 0$:

$$\Phi(0) = \Phi(a) = -b^2c^2(a - b) \le 0\,;$$

2) $\Phi(x)$ becomes zero when $b = 0$ and when $b = a$;

3) $\Phi(x)$ has its maximum when $x = 2a/3$:

$$\Phi(2a/3) = \tfrac{4}{27}a^3 - b^2c^2(a-b) \geq 0\,;$$

4) $\Phi(2a/3)$ becomes zero when $b = 2a/3$, $c = 1$.

The form of the curves $\Phi(x)$ is shown in Fig. 2 for a constant value of a and for cases when $c = 1$, for various values of b. For $b = 0$ or $b = a$ we obtain curve I; for $b = 2a/3$ we get curve IV; and for the other values of b we get curves of the form of II, III.

Fig 3 shows the corresponding curves on the phase plane x, y. We have two singular points—the saddle point ($x = 0$, $y = 0$) and the centre ($x = 2a/3$, $y = 0$). All the integral curves not passing through these singular points have the form of closed cycles intersecting the x-axis at right angles (because

$$\frac{dy}{dx} = \frac{-3x^2 + 2ax}{2\sqrt{\Phi(x)}}$$

becomes infinite when $\Phi(x) = 0$ and $-3x^2 + 2ax \neq 0$).

The centre corresponds to those motions in which the amplitudes of the horizontal and vertical oscillations remain constant, i.e. periodic movements. It is not hard to be convinced that the centre corresponds precisely to the periodic motions which we calculated in §2. Furthermore, one or other of these oscillations occurs whichever sign we select before $\sqrt{\Phi(x)}$.

With b slightly different from $2a/3$, the representative point describes a small cycle round the centre, and there takes place a small periodic transfer of energy from the vertical oscillation to the horizontal and back: the amplitudes remain close to the values corresponding to periodic solutions. From this it follows that the latter are stable. The more accurately the initial conditions approximate $b = 2a/3$, $c = 1$, the more accurately can they, with experience, be realized.

The more strongly b differs from $2a/3$, the larger the cyclic changes of amplitude and the larger the energy transfer. With $b/a \ll 1$ or $(a-b)/a \ll 1$, we have an almost total energy transfer from an angular oscillation to the vertical and vice-versa (curve II). This transfer takes place periodically with a period of

$$\tau = \oint \frac{dx}{\sqrt{\Phi(x)}}\,, \qquad [21]$$

where the integral is taken along the corresponding closed cycle.

Finally, the saddle corresponds to a periodic movement in which $x = 0$, that is, there are no horizontal oscillations nor, consequently, any transfer of energy. We see once more that this movement is unstable; if there is the slightest change in initial conditions the representative point begins to move along one of the cycles. When the initial conditions correspond to curve I we have a critical solution: the representative point approaches the origin after infinite time. With initial conditions closely corresponding to curve I, there occurs a nearly total energy transfer from one oscillation to the next, and this process takes an extraordinarily long time.

We obtain the following general result: the speed and extent of the transfer of energy from one component system to another depend on the initial conditions. It is possible to have initial conditions under which energy transfer is completely absent (periodic solutions) and where component systems behave like uncoupled ones. It is possible also to have initial conditions under which energy transfer takes place fully and the 'coupling' of component systems is very great.[5] Finally, it is possible also to have all the intemediate degrees of 'coupling', depending on the initial conditions. These relationships are utterly alien to linearly coupled linear systems, where the extent and speed of the transfer—and hence also its 'coupling'—depend exclusively on the structure of the system itself (on the ratio of the component frequencies and on the coeffient of coupling). Translating this contrast into spectral language, we can say that, when we have a linear coupling, the freqencies and relative intensities of the coupled system do not depend on initial conditions; where the coupling is nonlinear, both the freqencies and the relative intensities of the components are essentially determined by initial conditions.

A completely analogous treatment holds too in the case where α is not exactly equal to 2β. Introducing a new small parameter, the 'frequency difference' η, we have

$$\alpha = 2\beta + \eta\,.$$

In this case the Hamiltonian function can be presented in the form

$$H = \frac{\beta}{2\pi}(2J_1 + J_2) + \frac{\eta J_1}{2\pi} + \cdots$$

Here the 'unperturbed' Hamiltonian function is once more degenerate, and we can employ transformation (12′) and the method of secular perturbations.

There is one more remark to be made. For the application of the method of secular pertubations to be valid, it is necessary for v_2 to be a slowly changing function of time over the entire course of the motion. On the basis of equations (18) and (19) we obtain

$$\frac{dv_2}{dt} = H' \frac{a - \frac{3}{2}x}{x\sqrt{a-x}}\,,$$

from which it follows that this requirement breaks down around the points $x = 0$ and $x = a$, and it is possible therefore that a doubt may arise as to the correctness of our judgments. But this doubt is easy to eliminate: in fact, the points $x = 0$ and $x = a$ correspond, on the basis of (11′), to the conditions where there are respectively only a vertical and only a horizontal oscillation. Around these points, therfore, the course of the process can be traced perfectly strictly with the help of the linear equations (6) and (8), which corroborate the results obtained above.

4. Experiments

All the results obtained here can be very easily verified and demonstrated in an experiment. We used a good (weakly damped) steel spring, from which it was possible to

[5] Here we employ, following L.I.Mandel'stam, the term 'coupling' for the characteristic of interaction of partial systems, while the term 'coefficient of coupling' describes only a mechanism by means of which component systems can interact with each other. In linear terms, in particular, 'coupling' depends not only on the degree of interconnection but also on the ratio of the component frequencies.

suspend various weights. The spring was mounted on a stand. The periods of oscillation were counted on a stop watch. With the use of an electric light, a shadow of the weight and the spring was projected onto a screen. A grid of the polar coordinates was traced on the screen, allowing us to assign the necessary values to z and φ. (If the oscillations of the spring take place in a plane parallel to the screen and if the vertical plane passing through the lamp and the point where the spring is suspended is perpendicular to the screen, then z and φ can be measured directly from the position of the shadow on the coordinate grid.)

By varying the mass of the weight (by trial and error) the relation $\alpha = 2\beta$ was achieved, in which, by drawing out the spring by any small amount we chose, it was possible to observe the phenomenon of parametric resonance and the resulting transfer of energy. In accordance with theory, the complete energy transfer began to take place after very small initial sideways displacements. Again, in accordance with the theory of parametric resonance, when there was a frequency difference, energy transfer was observed only where the initial stretching exceeded a certain minimum value. These minimum values increased as the frequency difference increased.

We demonstrated, for the case $\alpha = 2\beta$, motions corresponding to the periodic solutions found earlier. A random initial angular deviation was taken and the initial values of z were calculated for the two corresponding periodic solutions (Fig. 1). The weight was displaced in a plane parallel to the screen so that its shadow fell on the point calculated, and was released without any initial velocity. It was possible to observe a $\bigcup$-shaped and a $\bigcap$-shaped oscillation, depending on whether the initial z was positive or negative. The stop-watch detected the difference in frequency of these oscillations, and this could be compared with the one calculated from equation (9).

In the table below the calculated and observed values of the ratio ω_2/ω_1 are compared.

Table

φ_0 (Degrees)	ω_2/ω_1 Observed	ω_2/ω_1 Calculated
3.75	0.99	0.97
7.5	0.95	0.93
15	0.85	0.86

A study of the transfer of energy was carried out in the following fashion: a series of initial declinations φ_0, z_0 was assigned without any initial velocity, corresponding to various values of b with the same value of integration constant a (see §3). The size of the initial declination and the initial tension are coupled, with a and b constant and in the absence of initial velocity, by the equation,

$$\varphi_0^2 + 4z_0^2 = \frac{a}{\pi\omega m'},$$

obtained from (11), (12′) and (16).

In practice, complete energy transfer was observed, in accordance with theory, with very small values of b, and also where b is close to a. With intermediate values of

b the energy transfer was not total, and when $b = 2a/3$, there was a complete absence of energy transfer, that is, we encountered once more periodic movements.

The curve in Fig. 4 represents the measured dependence of the duration τ of the cycle of energy transfer on the size of b when a is unchanged. Where $\varphi < 1.8°$, we did not obtain definite results, as the accuracy of the assignment of initial conditions was then less than the size of chance deviations.

5. Conclusion

To conclude, we shall make a short comparison between the oscillations of linearly coupled linear systems and the case analysed here of the oscillations of nonlinearly coupled linear systems, and then we shall indicate the link between the the problems considered here and other problems of physics.

	In linearly coupled systems:	**In our example:**
1)	We have a generalization of the theory of normal resonance, where reciprocal action should be taken into account.	We have an analogous generalization of the theory of parametric resonance.
2)	A strong reciprocal action of the component systems occurs when their frequencies are close to one another.	A strong reciprocal action of the component systems is possible when one of them has a frequency about twice that of the other one.
3)	The rate of the energy transfer does not depend on the initial conditions	The rate of the energy transfer depends on the initial conditions.

The present work may be of interest for the explanation and calculation of phenomena taking place in more complex mechanical systems, and also in electromagnetic circuits with a nonlinear coupling (a magnetic one through a transformer with iron, or an electrical one, through a condensor with Seignette [or Rochelle] salt). It has already arisen in connection with an examination of a model of the molecule CO_2, the quantum theory which Fermi recently provided.[6] This theory explains the structure of the lines of Raman scattering in carbonic acid.

Optical and electrical data lead to the model of the molecule CO_2, illustrated in Fig. 5a. Spectral measurements and theoretical considerations lead to the conclusion which, in terms of classical mechanics, can be formulated thus: in the molecule it is possible for there to be ionic oscillations, whose form is shown in Fig. 5b, 5c, and the frequency of the first oscillation is approximately equal to twice that of the second one.[7] This model of the CO_2 molecule is analogous to our elastic pendulum: the role of the vertical oscillation is played by the oscillation in Fig. 5b, and the role of the horizontal oscillation is played by the one in Fig. 5c. Transferring the results obtained by us to the molecule CO_2, we see that there is, according to classical mechanics, an energy transfer from one oscillation to the other which likewise gives rise to a splitting of the lines of

[6] E. Fermi, *Z. für Physik*, **71**, p. 250, 1931.

[7] More accurately, these frequencies are respectively equal to 3.90×10^{-13} sec. and 2.02×10^{-13} sec. There is also a third oscillation which has no role in the phenomena of interest to us. [**Note:** V&G write 10^{13}, not 10^{-13}].

Raman scattering.[8] We obtain a sophisticated result, partially agreeing with the one given by quantum mechanics.

Naturally, the only theory adequate for the phenomena taking place within atoms and molecules is quantum mechanics. Nevertheless, in the area of those comparitively slow ionic oscillations which generate infrared radiation and Raman scattering, classical mechanics can still give a certain good quality approximate representation of the true relationships—a representation which has the advantage of being clear. From the point of view of classical mechanics, the oscillation of ions in the molecule should be viewed as an oscillation of linear oscillators coupled either linearly or nonlinearly. For this reason, when the optics of molecules are given a classical interpretation, cases can occur which are not only appropriate to the usual model of linearly coupled systems, but also analogous to the case which is being studied here.

The subject of the present work has been initiated and formulated by L. I. Mandel'shtam. We are truly grateful to him for his valuable comments.

Moscow,
Scientific Research Institute of Physics,
Moscow State University, Oscillations Laboratory.

Received by the editors
8 October 1932.

Figure Captions

Figs. 1 to 5 (opposite) are reproduced directly from the original Russian version.

Fig. 1: Lissajous figure of periodic motions.

Fig. 2: Family of curves $\Phi(x)$ for various values of b.

Fig. 3: Family of curves in a phase plane.

Fig. 4: Dependence of the period of energy transfer on the initial declination for constand values of a.

Fig. 5: Molecule of CO_2 and its component oscillations.

---

[8] The scattered light wave will not be modelled periodically, but by a near-periodic ionic oscillation.

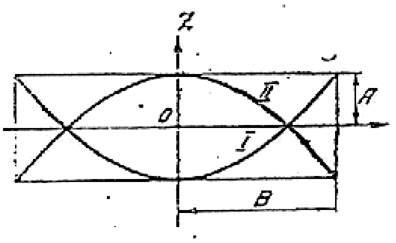

Рис. 1. Фигура Лиссажу периодических движений.

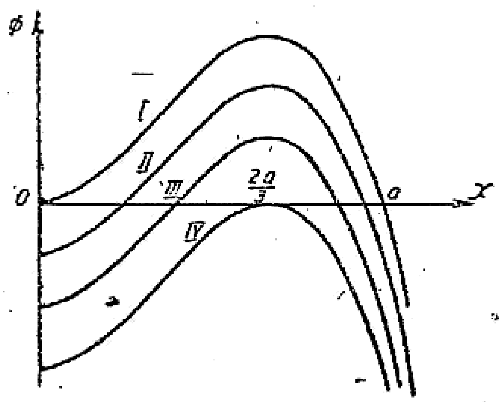

Рис. 2. Семейство кривых Φ (x) при различных b.

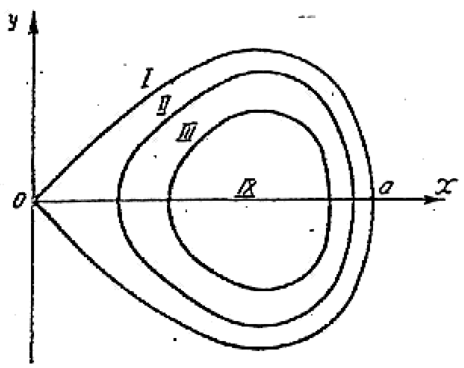

Рис. 3. Семейство кривых на фазовой плоскости.

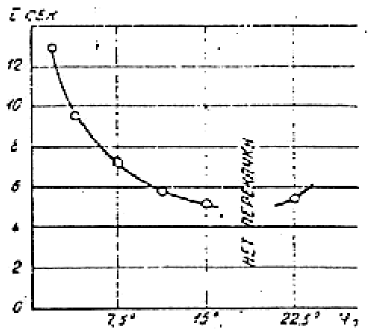

Рис. 4. Зависимость периода перекачки энергии от начального отклонения при постоянном значении a.

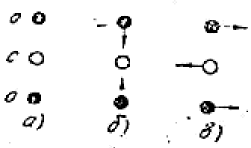

Рис. 5. Молекула CO^2 и ее парциальные колебания.