Deducing the Wind from Vorticity and Divergence

PETER LYNCH

Irish Meteorological Service, Glasnevin Hill, Dublin

(Manuscript received 9 March 1987, in final form 2 June 1987)

ABSTRACT

The horizontal wind field may be deduced from the vorticity and divergence by solving Poisson equations for the velocity potential and streamfunction or, more directly, by the solution of a single Poisson equation for one of the velocity components. Both methods are examined here, and both are shown to be equally accurate.

If the domain is of limited extent, boundary conditions must be specified. It is sufficient to prescribe a single component of the boundary velocity. Methods which use both components overdetermine the solution and may not converge in general.

1. Introduction

The deduction of the wind from the vorticity and divergence fields is frequently necessary in numerical weather prediction. For example, in a numerical model formulated in terms of the differentiated equations of motion, winds must be derived at each time step. If the model covers a limited area, appropriate boundary conditions for this inversion must be specified. Although it is sufficient to specify a single wind component on the boundary, a number of methods have been proposed which use both components. The resulting mathematical problem has, in general, no solution: it is impossible to derive winds which satisfy these boundary conditions and yield vorticity and divergence fields in agreement with those given.

To clarify this difficulty, we must make a sharp distinction between two situations in which a solution is sought. In the *special case* we assume a wind field given, and derive vorticity and divergence fields from it. We then try to deduce from these a wind field which agrees with that originally given. This problem has been used for evaluating various methods, since the precise answer is known in advance. However, because of the manner in which the data are produced, a correct solution may be found even by methods which use an overspecified mathematical formulation. Furthermore, integral constraints necessary for a solution are automatically satisfied in this case.

In the *general case* we assume the vorticity and divergence fields given, together with wind information on the boundary. It will be shown that a single wind component suffices to determine a solution. The other component can be derived, and if prespecified will overdetermine the solution. Methods which use both winds on the boundary are not well posed and are, therefore, unsatisfactory in general.

The calculation of the streamfunction and velocity potential for a given wind field is an example of the special case. However, for a limited area these fields are of doubtful utility: they are not uniquely defined by the given data. As shown in the Appendix, either the streamfunction or the velocity potential can be modified to assume completely arbitrary values on the boundary, and provided the other field is modified accordingly, the implied wind is unchanged. Thus, the streamfunction and velocity potential have physical significance only insofar as *together* they define a wind—neither field is meaningful in itself.

Sangster (1960) proposed a solution method which has been widely used. The normal wind is given on the boundary and the velocity potential is assumed to vanish there. This may seem arbitrary but it leads to a satisfactory solution: the streamfunction and velocity potential are not uniquely determined by the limited area data, and partitioning of the wind into irrotational and nondivergent parts is not unique (see the Appendix). This allows us a degree of freedom; Sangster's choice of boundary condition minimizes the divergent kinetic energy of the solution (Pedersen, 1971).

A number of investigators who have used Sangster's method (for the special case of a prescribed wind field) have reported poor results. Hawkins and Rosenthal (1965) studied a number of formulations; their best results, using centered differences at the boundaries, involved errors of about 0.5 m s^{-1} in the reconstructed winds. They used a regular grid but concluded that the staggered grid suggested by Sangster would probably prove more accurate.

Shukla and Saha (1974) also used a regular (unstaggered) grid in implementing Sangster's method. Furthermore, they used a linear extrapolation in applying

Corresponding author address: Dr. Peter Lynch, Meteorological Service, Dept. of Communications, Glasnevin Hill, Dublin 9, Ireland.

the boundary conditions. Their errors in the reconstructed winds were greater than 1 m s^{-1}. They proposed an iterative procedure which gave a marginal improvement; however, this procedure requires the use of both the normal and tangential boundary winds, and is therefore unsuitable in the general case. A similar iterative technique was outlined by Haltiner and Williams (1980, p. 257) who pointed out that there is no guarantee of convergence for such a scheme.

The indifferent results reported for Sangster's scheme led Bijlsma et al. (1986, denoted hereafter as BHL) to seek an alternative solution method. They developed a method of solving the two Poisson equations for the streamfunction and velocity potential simultaneously. A staggered grid was used and all finite differences, including those at the boundaries, were centered. In the special case tested, the method worked very well and the prescribed winds were reconstructed to high accuracy. However, both boundary velocities are used, which overdetermines the solution in the general case. The method is reexamined here and found not to converge in general.

The poor results reported for Sangster's scheme are not due to any inherent deficiency of the scheme, but rather to the manner in which it has been implemented; in particular, the failure to use a staggered grid precludes the use of centered normal differences at the boundary. In section 4 it is shown that the method gives results of high accuracy when a staggered grid and centered boundary differences are used. Reconstructed winds agree (to machine accuracy) with those originally prescribed. For the general case, satisfactory convergence of the scheme is indicated by the diminishing residues calculated in the successive overrelaxation (SOR) scheme used to solve the Poisson equations. In this case the scheme of BHL fails to converge.

The method of Sangster involves the solution of two Poisson equations, for the streamfunction and velocity potential. An alternative method, requiring solution of only one Poisson equation, is discussed in this paper. The velocity potential and streamfunction for a limited area have meaning only insofar as they imply a wind field (Miyakoda, 1960). Therefore, it is proposed to disregard them and to solve directly a Poisson equation for a component of the wind. Specification of the normal boundary velocity determines the solution. The second wind component is then obtained from the definitions of vorticity and divergence. The formulation of the method makes it clear why only one component of the boundary velocity is required. This method will be referred to here as the *direct method*.

Results obtained using the direct method are presented, and compared to those using Sangster's method. Both methods give completely satisfactory results: the errors in the reconstructed winds are negligible, and convergence is satisfactory in the general case. The advantage of the direct method is that only one Poisson equation need be solved.

2. Background: Sangster's method

According to the Helmholtz theorem, the horizontal wind field **V** may be defined in terms of a streamfunction ψ and velocity potential χ by

$$\mathbf{V} = \nabla\chi + \mathbf{k}\times\nabla\psi \qquad (1)$$

We define the vorticity ζ and divergence δ by

$$\zeta = \mathbf{k}\cdot\nabla\times\mathbf{V} \quad \delta = \nabla\cdot\mathbf{V}. \qquad (2)$$

Then the divergence of (1) leads to a Poisson equation

$$\nabla^2\chi = \delta \qquad (3)$$

for χ. Similarly, we may obtain a Poisson equation for ψ

$$\nabla^2\psi = \zeta. \qquad (4)$$

Two problems are distinguished in BHL. The *partitioning problem* is that of separating a given wind field into rotational and divergent components. The problem is solved as follows: the divergence is derived, (3) is solved for χ with zero boundary conditions, $\mathbf{V}_\chi = \nabla\chi$ is the divergent wind, and the residual $(\mathbf{V} - \mathbf{V}_\chi)$ is the rotational wind. This solution is not unique, but it minimizes the divergent kinetic energy. The nonuniqueness of the solution is discussed in the Appendix; the partitioning problem will not be considered further here.

The *reconstruction problem* concerns the deduction of the winds from the vorticity and divergence fields. If (3) and (4) are to be solved in a limited domain, boundary conditions are required. From (1) it follows that

$$V_s \equiv \mathbf{s}\cdot\mathbf{V} = \frac{\partial\psi}{\partial n} + \frac{\partial\chi}{\partial s} \qquad (5)$$

$$V_n \equiv \mathbf{n}\cdot\mathbf{V} = -\frac{\partial\psi}{\partial s} + \frac{\partial\chi}{\partial n} \qquad (6)$$

where **s** and **n** are tangent and normal unit vectors and s and n are distances along and normal to the boundary. These relations couple the variables χ and ψ so that either (3) and (4) must be solved simultaneously or they must be decoupled by further assumptions.

The method of BHL solves both equations simultaneously, but it requires knowledge of both V_s and V_n on the boundary. In general there is no solution, as this problem is overdetermined. Thus, the method is unsatisfactory; the difficulty will be illustrated by an example in section 4.

The method of Sangster (1960) decouples the variables by assuming that the velocity potential χ vanishes on the boundary. The solution involves the following sequence of steps (subscript B denotes boundary values).

Step 1: Assume χ_B vanishes, and solve (3) for χ.
Step 2: Integrate (6) stepwise around the boundary,

using specified values of the normal velocity V_n, to obtain ψ_B.

Step 3: Solve (4), with these boundary values, for ψ.

Step 4: Derive the wind field by using (1).

Note that only the normal component of the boundary velocity is used.

In order that step 2 should yield a single-valued streamfunction, it is necessary that the input data satisfy an integral constraint. If we integrate the definitions of vorticity and divergence (2) over the domain Ω, bounded by $\partial\Omega$, and use Gauss and Stokes' theorems, it follows that

$$\int\int_{\Omega} \delta da = \oint_{\partial\Omega} V_n ds \tag{7}$$

$$\int\int_{\Omega} \zeta da = \oint_{\partial\Omega} V_s ds. \tag{8}$$

These conditions must be satisfied by the prescribed data. It is assumed that the input information is suitably modified so that (7) and (8) hold (in the special case where the winds are prescribed throughout Ω, these conditions are automatically fulfilled).

An alternative method, also discussed by Sangster, is to set ψ to zero on the boundary. Then (4) is solved for ψ, and (6) provides Neumann boundary conditions for the solution of (3). The data must still satisfy the compatibility condition (8). Although the ψ and χ fields will be different from before, the wind field is unchanged. We will not consider this alternative formulation further here.

In the Appendix we discuss the nonuniqueness of the velocity potential and streamfunction, and show that χ may be chosen so as to take *arbitrary* boundary values. This discussion illustrates the lack of physical significance of the χ and ψ fields for flows specified on a limited domain, and justifies Sangster's choice of zero χ boundary values.

In section 4 Sangster's method will be used to solve for the winds in a special and a general case, and will be seen to give completely satisfactory results.

3. The direct method

Considering the Laplacian of the horizontal wind, and using well-known vector identities (e.g., see Morse and Feshbach, 1953) we can write

$$\nabla^2\mathbf{V} = \nabla\delta + \mathbf{k}\times\nabla\zeta. \tag{9}$$

Given δ and ζ, this vector Poisson equation can be solved for $\mathbf{V}$ if the boundary velocity $\mathbf{V}_B$ is specified. However, the vorticity ζ_0 and divergence δ_0 derived from this solution will not generally agree with those prescribed. This is most easily seen as follows: if ζ, δ and $\mathbf{V}_B$ are arbitrary, they cannot be expected to satisfy the integral constraints (7) and (8); but these constraints automatically hold for ζ_0 and δ_0 [simply integrate the definitions (2)]. Therefore, the calculated vorticity and divergence cannot be the same as those prescribed. The problem is deeper than this, however: even if the data are modified so that (7) and (8) hold, the solution may still fail to generate the specified ζ and δ; if we add an *arbitrary* harmonic function to δ and its harmonic conjugate to ζ, the right-hand side of (9) remains unchanged (see the Appendix); the difficulty arises through the differentiations in deriving (9), which introduce solutions other than those of the original problem.

In order to deduce winds which imply the specified vorticity and divergence, we propose the following procedure. A single component of (9) is solved for one component of the wind. This solution requires knowledge of one wind component on the boundary $\partial\Omega$. The definitions (2) of ζ and δ are then integrated stepwise, starting from specified boundary values, to derive the other wind component. The use of (2) guarantees that the solution generates the original vorticity and divergence. Note that only one boundary wind is prescribed.

a. Implementation

To implement the method, we consider a limited domain Ω on the sphere, specified by the coordinates, longitude λ and latitude ϕ, and bounded west and east at λ_W and λ_E, and south and north at ϕ_S and ϕ_N. The vector Laplacian in (9) may be separated into components (Morse and Feshbach, p. 116) but it is easier to proceed directly as follows. The definitions (2) may be written

$$\zeta = \frac{1}{a\sigma}\left[\frac{\partial v}{\partial\lambda} - \frac{\partial(\sigma u)}{\partial\phi}\right] \tag{10}$$

$$\delta = \frac{1}{a\sigma}\left[\frac{\partial u}{\partial\lambda} + \frac{\partial(\sigma v)}{\partial\phi}\right] \tag{11}$$

where u and v are eastward and northward winds, $\sigma = \cos\phi$ and a is the earth's radius. It is straightforward to eliminate v between these equations, obtaining an equation for $\hat{u} = \sigma u$:

$$\nabla^2\hat{u} = \frac{1}{a}\left[\frac{\partial\delta}{\partial\lambda} - \frac{1}{\sigma}\frac{\partial(\sigma^2\zeta)}{\partial\phi}\right] \equiv F_u. \tag{12}$$

This Poisson equation for $\hat{u}$ corresponds to a component of (9), and is the equation that we will solve. The other component, for $\hat{v} = \sigma v$, will not be required.

To solve (12) we need boundary conditions. We may specify u everywhere on $\partial\Omega$ [Dirichlet conditions] or v everywhere on $\partial\Omega$ [Neumann conditions, through (10) and (11)]. Usually, we prefer to specify the normal or tangential velocity on the boundary. We consider the former case, with u given at λ_W and λ_E and v given at ϕ_S and ϕ_N. Using (10) we deduce $\partial\hat{u}/\partial\phi$ on the south and north boundaries. Thus, we have the following problem for $\hat{u}$:

$$\left.\begin{array}{l}\nabla^2\hat{u} = F_u, \quad \text{in } \Omega \\ \hat{u} \text{ given at } \lambda_W \text{ and } \lambda_E \\ \partial\hat{u}/\partial\phi \text{ given at } \phi_S \text{ and } \phi_N\end{array}\right\}. \tag{13}$$

This elliptic equation with mixed boundary conditions has a unique solution for $\hat{u}$. Uniqueness may easily be seen by considering the homogeneous system corresponding to (13). Let W be a solution of the resulting Laplace equation. If this equation is multiplied by W, integrated over Ω and Gauss' theorem used, we get

$$\iint_{\Omega} |\text{grad } W|^2 da = \oint_{\partial\Omega} W\frac{\partial W}{\partial n} ds. \tag{14}$$

Since the boundary conditions ensure that the right-hand side vanishes, the solution W must be equal to zero everywhere. Therefore, any two solutions of the inhomogeneous system (13) must be identical. Existence of the solution may be established by means of the general theory of integral equations (e.g., see Pogorzelski, 1966, §XII.2) or, more simply, by separation of variables and expansion in eigenfunctions (Morse and Feshbach, 1953).

Having obtained the solution of (13), we may integrate (11) northwards, using the known value of v at ϕ_S, to obtain v throughout the domain. It is straightforward to show that the eastward gradient of v at the northern boundary ϕ_N calculated from the solution is equal to that prespecified. Thus, a necessary and sufficient condition that the specified and calculated velocities v agree there is that the integral constraint (7) be satisfied. We should modify the input data (either δ or V_n) so that this is so.

Consider now the case where both u and v are given on $\partial\Omega$. Using (10) and (11) we can calculate the normal derivative of u. Thus, we have Cauchy conditions for (12), which overspecify the solution. It is now clear why the methods of Shukla and Saha and of BHL are unsatisfactory in general: "For an elliptic equation inside a closed boundary Cauchy conditions on any portion of the boundary are too many conditions" (Morse and Feshbach, 1953, p. 702).

b. Direct solution

The *direct* solution method is summarized here for convenience.

Step 1: ζ and δ are given in Ω, and V_n on $\partial\Omega$. Modify V_n and/or δ to satisfy (7).

Step 2: Calculate $\partial\hat{u}/\partial\phi$ at ϕ_S and ϕ_N using (10) and known values of v and ζ.

Step 3: Solve the system (13) for $\hat{u}$.

Step 4: Integrate (11) northward, starting from specified values of $\hat{v}$ at ϕ_S, to obtain $\hat{v}$.

The boundary conditions described here will be denoted by BC = 1. For comparison, we also considered the case where the zonal velocity u was specified everywhere on $\partial\Omega$; these Dirichlet boundary conditions will be denoted by BC = 2. To obtain v, we specify $v_{SW} = v(\lambda_W, \phi_S)$, integrate (10) eastward to get v on the southern boundary, and then integrate (11) northward to obtain v everywhere. The recursion in step 4 proved to be stable with respect to error propagation. In section 4 the direct method will be compared with Sangster's method and shown to give equally satisfactory results.

4. Numerical examples

To demonstrate the efficacy of the method of Sangster and the direct method, we employ them in a limited area model to derive winds from the vorticity and divergence. The method of Bijlsma, Hafkenscheid and Lynch is also used and the difficulties encountered with it illustrate the problems arising through overspecification of the solution.

Initially, the wind field is given and ζ and δ are calculated—this is the *special case*—and all three methods give excellent results. We then make a one hour forecast, using the equation of conservation of potential vorticity, and derive the divergence through a diagnostic relationship. The boundary winds u_B, v_B are not changed during this process. We now derive the winds from the new vorticity and divergence, together with wind information on the boundary—this is the *general case,* since the automatic compatibility between (u_B, v_B) and (ζ, δ) no longer holds. The divergence is modified by the addition of a small constant so that (7) holds. An alternative method of satisfying (7) by modification of the normal boundary velocities led to virtually indistinguishable results. For the BHL method ζ is also modified to satisfy (8). Despite this, the method fails to converge. Other means of satisfying the integral constraints, through modification of the boundary velocities, did not improve the situation. The method is trying to find winds which imply the given (ζ, δ) and also agree with (u_B, v_B) on the boundary—but in general no such winds exist, and the method cannot converge. Sangster's method and the direct method both converge satisfactorily in the general case.

The forecast model uses a rotated latitude/longitude grid (whose north pole is at the geographical position 30°N, 150°E), bounded by (λ_W, λ_E) = (−40°, +38°) and (ϕ_S, ϕ_N) = (−25°, +25°). The grid spacing is $\Delta\lambda = \Delta\phi = 2.0°$ and 40 × 26 points cover the area shown in Fig. 1. (Note that terms like northward component, eastern boundary, etc., refer to the transformed grid.)

The staggered grid used for the finite difference formulation of Sangster's method is shown in Fig. 2. All differences are centered. If (1) and (2) are discretized and the velocities eliminated, five-point Laplacians arise in the discrete analogues of (3) and (4). The boundary velocities which are specified are encircled.

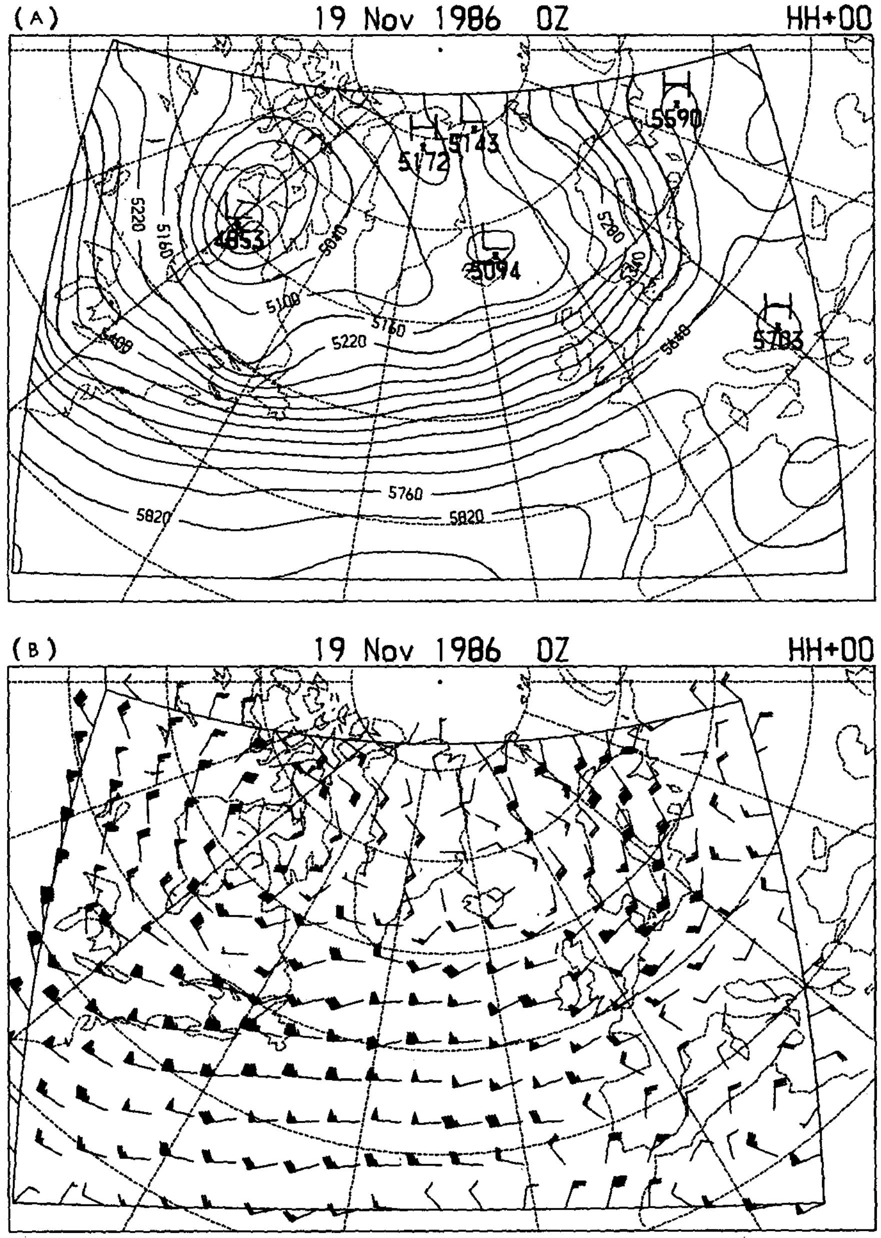

FIG. 1. (a) Initial 500 mb height field; (b) corresponding wind field (only values in every alternate row and column are plotted).

The χ values which are set to zero are denoted by 0 subscripts and the ψ values deduced in step 2 by integration of (6) are denoted by subscript b. The grid for the direct method is shown in Fig. 3. The boundary velocities (for BC = 1) are encircled as before. Note that the vorticity at ϕ_S and ϕ_N is needed; in consequence, u on the north and south boundaries is derived. The finite difference equation is derived by elimination of v from the discrete forms of (10) and (11), yielding again a five-point Laplacian analogue of (12). The Poisson equations are solved by a successive overrelaxation (SOR) technique, which is iterated until there is no further significant decrease in the residue.

Initial data are the 500 mb height and wind fields for 0000 UTC 19 November 1986. These are depicted in Fig. 1.

In Table 1 we show the results, using the three methods, in the *special case* where the initial winds are prescribed. The methods are indicated as follows: Sangster's method (SANSTR); the direct method (DIRECT); and that of Bijlsma et al. (BIHALY). For the direct method, results are shown for both the mixed boundary

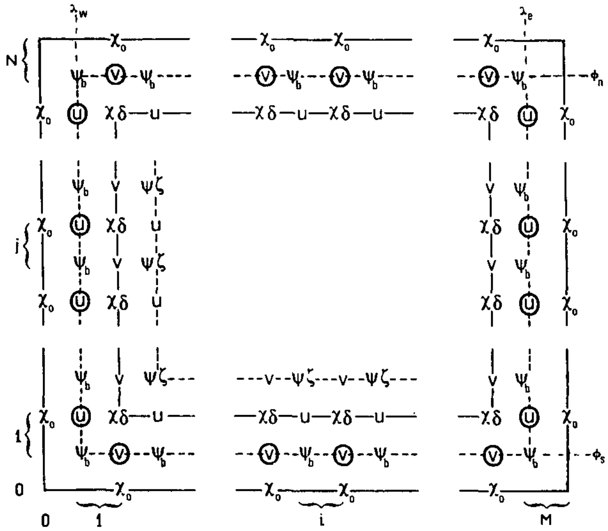

FIG. 2. Discrete grid used for Sangster's method. The values denoted by χ_0 are set to zero, and the ψ-values obtained from (6) are indicated by subscript b. Specified boundary velocities are encircled.

conditions (BC = 1) and the Dirichlet conditions (BC = 2). The rms vector error in the reconstructed wind is defined by

$$E=\left[\frac{1}{N}\sum\{(u_o-u_i)^2+(v_o-v_i)^2\}\right]^{1/2} \quad (15)$$

where the subscripts i and o indicate input and output values and the sum is over all the N gridpoints. We also show the rms and maximum errors for each component of the winds. These results show clearly that, in the special case, all three methods of retrieving the winds are highly accurate; the reconstructed winds differ from those originally prescribed by amounts close to the minimum value that can be represented on the computer. Use of double precision reduces these differences by ten orders of magnitude. Thus, all methods give satisfactory results.

The success of Sangster's method is in stark contrast to the results reported by Hawkins and Rosenthal (1965) and Shukla and Saha (1974), who obtained errors of about 1 m s^{-1} in the reconstructed winds using this method. It seems that their use of uncentered boundary differences caused these large errors. The results presented here show that the iterative technique proposed by Shukla and Saha, and reviewed by Haltiner and Williams, is neither computationally necessary nor mathematically desirable.

In the general case the correct winds are not known exactly, and we cannot use the vector error of the wind as a measure of the accuracy of the methods. Instead, we calculate the normalized maximum residual differences between the calculated and original values of the vorticity and divergence. For the SANSTR and BIHALY methods these are defined by

$$R_\psi=\max[\nabla^2\psi_o-\zeta_i]/\bar{\zeta}$$

$$R_\chi=\max[\nabla^2\chi_o-\delta_i]/\bar{\delta}$$

and for the DIRECT method we define

$$R_\zeta=\max[\mathbf{k}\cdot\nabla\times\mathbf{V}_o-\zeta_i]/\bar{\zeta}$$

$$R_\delta=\max[\nabla\cdot\mathbf{V}_o-\delta_i]/\bar{\delta}$$

where subscripts o and i denote calculated and prescribed quantities, $\bar{\zeta}$ and $\bar{\delta}$ are mean absolute vorticity and divergence values used for normalization, and the maxima are taken over all gridpoints.

In Table 2 the values of these quantities for the three methods are given for the special case. All the residuals are very small, indicating adequate convergence and confirming the results in Table 1. The values obtained using double precision demonstrate that the accuracy of the results is limited only by machine precision.

In sharp contrast, the figures in Table 3 (for the general case) show that, whereas the Sangster and direct methods converge satisfactorily, the method of Bijlsma, Hafkenscheid and Lynch is unable to converge. The results for this method in double precision are no different from those in single precision. The method cannot converge since it is trying to find a solution which agrees with incompatible data on the boundaries and in the interior—this is the price paid for overspecifying the problem.

Both the method of Sangster and the direct method give excellent results in this general case. The tiny residuals indicate that the methods have converged. This

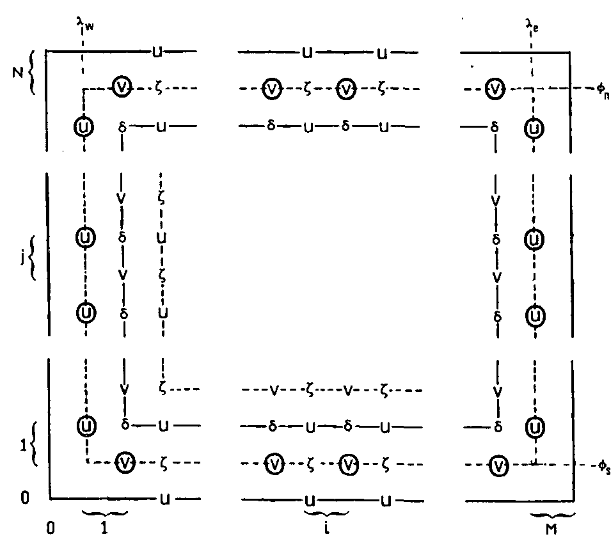

FIG. 3. Discrete grid used for the direct method. The specified boundary velocities are encircled.

TABLE 1. Root-mean-square vector error, and rms and maximum errors in the components of the reconstructed winds, using various methods: Bijlsma et al. (1986) (BIHALY), Sangster (1960) (SANSTR), and the direct method (DIRECT). The direct method is run with both mixed boundary conditions (BC = 1) and Dirichlet conditions (BC = 2). Results are given for both single and double precision code.

Method	Rms errors	Single precision		Double precision	
	rms vector error	2.572×10^{-06}		7.998×10^{-17}	
BIHALY	rms errors in u and v	2.308×10^{-06}	1.136×10^{-06}	7.235×10^{-17}	3.408×10^{-17}
	max errors in u and v	6.914×10^{-06}	4.113×10^{-06}	1.874×10^{-16}	1.214×10^{-16}
	rms vector error	8.167×10^{-06}		2.515×10^{-16}	
SANSTR	rms errors in u and v	6.755×10^{-06}	4.590×10^{-06}	2.080×10^{-16}	1.413×10^{-16}
	max errors in u and v	2.551×10^{-05}	1.788×10^{-05}	7.355×10^{-16}	4.233×10^{-16}
	rms vector error	1.182×10^{-05}		3.406×10^{-16}	
DIRECT	rms errors in u and v	7.847×10^{-06}	8.844×10^{-06}	2.078×10^{-16}	2.699×10^{-16}
(BC = 1)	max errors in u and v	2.861×10^{-05}	3.311×10^{-05}	3.886×10^{-16}	8.422×10^{-16}
	rms vector error	1.442×10^{-05}		5.161×10^{-16}	
DIRECT	rms errors in u and v	1.958×10^{-06}	1.428×10^{-05}	7.148×10^{-17}	5.111×10^{-16}
(BC = 2)	max errors in u and v	5.007×10^{-06}	2.754×10^{-05}	1.804×10^{-16}	8.496×10^{-16}

is the acid test: the residuals measure the extent to which the calculated solutions satisfy the equations.

As a final test, we made two parallel 24 hour forecasts, using the SANSTR and DIRECT methods to retrieve the winds at each time step. The same normal boundary winds at the same gridpoints were used in each case. The direct method uses extra vorticities and, correspondingly, produces extra values of u on the south and north boundaries. To attain maximum similarity between the parallel runs, these were reset each timestep to their initial values, and the vorticity was rederived. The rms difference between the forecast wind fields after 24 hours was only 0.13 m s^{-1}; the height fields differed by less than one meter. For all practical purposes, the forecasts were identical.

5. Conclusions

We have examined some methods of deducing winds from the vorticity and divergence fields. Both Sangster's method and the direct method described in this paper have been shown to give correct results in both the special case of prescribed winds and the more general case where arbitrary vorticity and divergence fields together with boundary wind data are given. Therefore, if the streamfunction and velocity potential are not required, the direct method may be preferable, since only one Poisson equation needs to be solved using this technique. (For a limited domain, the streamfunction and velocity potential have no physical significance per se, since they are not uniquely defined.) Methods which use both boundary wind components are overdetermined and do not converge in general. Therefore, they should not be used.

The SOR method used here is useful in providing a clear indication of the extent to which the computational solutions are converging. However, it is computationally expensive. Highly efficient direct methods (*fast solvers*) are available (e.g., see Swartztrauber and Sweet, 1975) for solving Poisson equations with boundary conditions of the type occurring here, and can obviously be used to advantage.

TABLE 2. Residues (see text) of the calculated solutions in the *special case* of prescribed winds. The various methods are indicated as in Table 1. These residues provide a useful indication of the degree of convergence of the methods.

Method	Residue	Single precision	Double precision
BIHALY	R_ψ	1.805×10^{-6}	5.880×10^{-17}
	R_χ	2.928×10^{-6}	1.091×10^{-16}
SANSTR	R_ψ	5.983×10^{-6}	2.001×10^{-16}
	R_χ	3.001×10^{-7}	1.023×10^{-17}
DIRECT (BC = 1)	R_ζ	9.719×10^{-7}	2.920×10^{-17}
	R_δ	2.928×10^{-7}	6.924×10^{-18}
DIRECT (BC = 2)	R_ζ	1.059×10^{-6}	3.832×10^{-17}
	R_δ	2.562×10^{-7}	8.522×10^{-18}

TABLE 3. Residues (see text) of the calculated solutions in the *general case*. The vorticity and divergence fields are those resulting from a 1-hour forecast; the boundary velocities are unchanged from their initial values. The methods are indicated as in Table 1.

Method	Residue	Single precision	Double precision
BIHALY	R_ψ	2.055	2.055
	R_χ	6.166	6.166
SANSTR	R_ψ	5.444×10^{-6}	1.832×10^{-16}
	R_χ	5.124×10^{-8}	1.325×10^{-18}
DIRECT (BC = 1)	R_ζ	1.354×10^{-6}	2.578×10^{-17}
	R_δ	2.462×10^{-7}	1.800×10^{-16}
DIRECT (BC = 2)	R_ζ	1.189×10^{-6}	3.835×10^{-17}
	R_δ	2.480×10^{-7}	9.737×10^{-17}

Finally, although we have discussed the problem in the context of a limited-area domain, the direct solution method can also be used for global gridpoint models, with consequent computational savings. The only conditions on the solution of the Poisson equation (13) are that it be periodic in longitude and regular at the poles.

Acknowledgments. As always, my colleagues Ray Bates, Jim Hamilton and Aidan McDonald have been generous with help and encouragement. Thanks also to Joop Bijlsma of KNMI for helpful comments.

APPENDIX

Nonuniqueness of Partitioning in a Limited Domain

The streamfunction and velocity potential for a flow *in a limited domain* Ω have no physical significance in themselves, but only insofar as they imply a wind field (Miyakoda, 1960). We show here that an arbitrary harmonic function may be added to χ, provided ψ is also modified accordingly, without changing the winds. This means that χ (or alternatively ψ) may be made to assume completely arbitrary values on the boundary.

For a limited domain the kinetic energy does not partition into divergent and rotational components, but also contains a cross-product term.

Let the horizontal wind be partitioned into irrotational and nondivergent components as follows

$$\mathbf{V} = \mathbf{V}_\chi + \mathbf{V}_\psi = \nabla\chi + \mathbf{k}\times\nabla\psi. \quad \text{(A1)}$$

We can represent a point on the sphere by a complex variable z (e.g., $z = \lambda + i\phi$). Now let $\xi(\lambda, \phi)$ be *any* harmonic function, i.e., any function satisfying Laplace's equation. There is a corresponding harmonic conjugate function $\eta(\lambda, \phi)$, unique up to an additive constant, such that $f(z) = \xi + i\eta$ is an analytic function (Carrier et al., 1966). The Cauchy-Riemann equations then imply that

$$\nabla\xi + \mathbf{k}\times\nabla\eta = 0. \quad \text{(A2)}$$

The modified velocity potential and streamfunction (χ', ψ') defined by

$$\chi' = \chi + \xi; \quad \psi' = \psi + \eta \quad \text{(A3)}$$

represent the same wind field as (χ, ψ) (but this field partitions into irrotational and nondivergent components in a different way). Thus, χ, for example, may be altered by the addition of an *arbitrary* harmonic function provided that ψ is altered accordingly. In this way χ may be adjusted so as to satisfy *arbitrary* boundary values. The irrotational and nondivergent components $\mathbf{V}_\chi$ and $\mathbf{V}_\psi$ are not uniquely defined.

[In the same way, the divergence and vorticity may be altered by the addition of a conjugate pair of harmonic functions without altering the right-hand side of equation (9)].

If the domain considered is the entire sphere, the only nonsingular harmonic functions are constants, χ and ψ are essentially unique, and the partitioning of the wind is well defined.

We define the kinetic energy (assuming density $\rho = 1$) by

$$K \equiv \frac{1}{2}\int\!\!\int_\Omega \mathbf{V}\cdot\mathbf{V}da, \quad \text{(A4)}$$

and its divergent and rotational components by

$$K_\chi \equiv \frac{1}{2}\int\!\!\int_\Omega \mathbf{V}_\chi\cdot\mathbf{V}_\chi da; \quad K_\psi \equiv \frac{1}{2}\int\!\!\int_\Omega \mathbf{V}_\psi\cdot\mathbf{V}_\psi da.$$

The kinetic energy may be expanded, using (A1) above:

$$K = K_\chi + K_\psi + \int\!\!\int_\Omega \mathbf{V}_\chi\cdot\mathbf{V}_\psi da. \quad \text{(A5)}$$

If the domain Ω is limited, the cross-product term will generally not vanish. Using (A1) again, we may write

$$\int\!\!\int_\Omega \mathbf{V}_\chi\cdot\mathbf{V}_\psi da = \oint_{\partial\Omega} \psi\frac{\partial\chi}{\partial s}ds = -\oint_{\partial\Omega} \chi\frac{\partial\psi}{\partial s}ds. \quad \text{(A6)}$$

This term may be positive or negative, and the kinetic energy does not partition into divergent and rotational components. Sangster's choice of boundary condition ($\chi = 0$) ensures that the cross term vanishes, and the partitioning is well defined. Furthermore, this choice minimizes the divergent component of the energy (Pedersen, 1971).

If the domain is the entire sphere, the boundary integral does not arise, the cross term vanishes, and the partitioning of kinetic energy is unique.

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