RESONANT ROSSBY WAVE TRIADS AND THE SWINGING SPRING

BY PETER LYNCH

A mathematical equivalence with a simple mechanical system sheds light on the dynamics of resonant Rossby waves in the atmosphere.

Rossby waves are solutions of simplified forms of the equations governing the dynamics of the atmosphere and oceans. They serve as archetypes for the sinuous large-scale motions of the midlatitude troposphere. They are horizontal transverse waves with large values of vorticity and with divergence that is negligible by comparison. Their most characteristic feature is that they move westward relative to the zonal atmospheric flow. This strange lopsidedness, or chirality, is a result of the earth's rotation, which breaks the symmetry of east–west reflection. The Rossby wave was the topic chosen by Professor George Platzman for his Symons Memorial Lecture to the Royal Meteorological Society, and an expository review has appeared (Platzman 1968). Several interesting articles on Rossby have appeared recently in *BAMS:* in particular, see Phillips (1998) and Lewis (1992). The dynamics of Rossby waves are discussed in considerable depth in Pedlosky (1987).[1]

The full equations governing atmospheric dynamics are overly complicated and include, in addition to the meteorologically significant motions, physical phenomena that have little import on the weather. Thus, the full spectrum of sound waves is embraced within the set of solutions. Gravity waves are another class of solutions of the full system, which, for many purposes, can be regarded as a noisy nuisance. One of the key advances enabling the application of quantitative methods to weather forecasting was the development of simplified systems of equations, from which irrelevant or unimportant solutions were eliminated or filtered out. And one of the outstanding contributions to this development was the seminal paper of Charney (1948).

Charney (Fig. 1) introduced scale analysis to examine and compare the relative sizes of the various terms in the equations of motion. He recognized that the dominant motion is approximately hydrostatic, geostrophic, adiabatic, and horizonal; that the grav-

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[1] A more detailed historical discussion is contained in the Web supplement to this paper (http://dx.doi.org/10.1175/BAMS-84-5-Lynch).

ity waves are of secondary importance; and that only the vortical waves—the rotational waves with large vorticity and small divergence—are of importance for modeling and prediction of large-scale weather phenomena. By elimination of divergence and systematic use of the geostrophic relationship, he reduced the system to a single equation for a single variable, the *potential vorticity*. The conservation of quasigeostrophic potential vorticity is the fundamental principle governing large-scale atmospheric dynamics. The historical development of quasigeostrophic theory has been described by Phillips (1990). For biographical information on Charney, and his most important publications, see Lindzen et al. (1990).

The simplest context for the study of Rossby waves is a shallow layer of incompressible fluid on a rotating earth. The geometry is greatly simplified by ignoring the effect of sphericity except for one crucial respect: we allow for the change in the vertical component of the earth's rotation with latitude. This is called the beta-plane approximation: it was introduced by Rossby et al. in their ground-breaking paper of 1939 in which the Rossby wave formula first appeared. Rossby had the intuitive genius to isolate the factors that were essential for the existence of these waves, and to elucidate their dynamics by studying them in a model of maximum simplicity. A fuller discussion of the development of Rossby wave theory is presented in Platzman's (1968) Symons Lecture, (loc. cit.), and also in a recent scientific history of tides (Cartwright 1999).

FIG. 1. Jule Charney (1917–81), from the cover of *Eos,* 57, Aug 1976 (copyright Nora Rosenbaum).

CONSERVATION OF POTENTIAL VORTICITY. In this expository paper, mathematical details are omitted. However, the key equations are included here to facilitate and clarify the discussion. The notation is generally consistent with meteorological convention. A more complete mathematical presentation may be found in the expanded version of the paper (electronic supplement available online at http://dx.doi.org/10.1175/BAMS-84-5-Lynch; hereafter ES).

We shall consider a shallow layer of incompressible fluid on a rotating planet. We can remove the complications of spherical geometry by means of the beta-plane approximation mentioned above. Under the assumptions of quasigeostrophic theory, the dynamics reduce to an equation expressing the conservation of potential vorticity. It is a single partial differential equation for the streamfunction, ψ:

$$\frac{\partial}{\partial t}\left(\nabla^2\psi - F\psi\right) + \left(\frac{\partial\psi}{\partial x}\frac{\partial\nabla^2\psi}{\partial y} - \frac{\partial\psi}{\partial y}\frac{\partial\nabla^2\psi}{\partial x}\right) + \beta\frac{\partial\psi}{\partial x} = 0. \qquad (1)$$

The variation with latitude of the planetary vorticity due to the earth's rotation, the β term, is the crucial factor for the existence of Rossby wave solutions. The parameter F is equal to $1/L_R^2$, where L_R is the Rossby radius of deformation.

It is easy to find linear wave-like solutions of (1). Such solutions exist provided that the frequency σ satisfies a simple dispersion relationship:

$$\sigma = -\frac{k\beta}{k^2 + \ell^2 + F}. \qquad (2)$$

This is the celebrated Rossby wave formula. In fact, Rossby assumed conservation of *absolute* vorticity and obtained a simpler form of the dispersion relationship. He also allowed for a mean zonal velocity that Doppler shifts the wave solution. A detailed discussion of these solutions is presented in Pedlosky (1987). We note only that for these waves the zonal phase speed is always negative, which means that they always travel toward the west. As long as we neglect the nonlinear terms, we can superimpose a number of Rossby wave solutions with differing wavenumbers and frequencies. Each component will travel at a different rate, and each will evolve independently of its fellow travelers.

Nonlinear effects vanish for a single Rossby wave, because the isolines of vorticity are parallel to the streamlines, so that the gradient of vorticity is perpendicular to the velocity and the advection, the term in (1) enclosed in braces, vanishes. However, if more than one component is present, the velocity of one component advects the vorticity of another, and the components are no longer independent but interact through the nonlinear term in (1). Suppose, we start with just two components. They will interact with each other to produce a third component whose wavenumber and frequency are the sums of their wavenumbers and frequencies. The third component will in turn interact with the first two, producing further components. In this sense, a pure Rossby wave is an unstable solution; inevitable small perturbations will have projections onto other components, and these will interact nonlinearly with the primary wave to produce still further components. Eventually, the solution will be transformed out of all recognition.

It is remarkable that Rossby waves have been "rediscovered" in a completely different physical context—that of instabilities in a magnetically confined plasma. Hasegawa and Mima (1977) investigated wave motions of an inhomogeneous plasma and derived an equation, which is mathematically identical to (1). Their wave solutions, called *drift waves,* are dynamically equivalent to Rossby waves. The quantity corresponding to the variation of the ambient vorticity in a fluid (the beta parameter) is the variation of the background plasma density. The correspondence between Rossby waves in the atmosphere and drift waves in plasmas has been thoroughly explored by Horton and Hasegawa (1994).

When considering the meteorological origins of (1), plasma physicists have adopted the name *Charney's equation* (e.g., Horton and Hasegawa 1994) or the *Charney–Obukhov equation* (e.g., Nezlin and Snezhkin 1993). In the plasma context it is called the *Hasegawa–Mima equation* (Hasegawa and Mima 1977). Charney (1948) was first to present a systematic derivation based on scale analysis and to clarify the precise conditions for its validity. Obukhov (1949) derived an equation of essentially the same form [he omitted the beta term in his analysis, but was aware of its importance for planetary-scale motions; see Phillips et al. (1960)]. However, the equation was known before the publications of Charney and Obukhov and was used by other workers, most notably by Rossby. Thus, we feel it is inappropriate to follow the practice in plasma physics, so we will continue to refer to (1) by its "dynamical" title, the quasigeostrophic barotropic potential vorticity equation.

A more appropriate equation to bear Charney's name is the three-dimensional quasigeostrophic quasi-potential vorticity equation. This was first derived by Charney (1948), and was presented later in a more elegant formulation by Charney and Stern (1962). It may be written as

$$\left(\frac{\partial}{\partial t}+\frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}-\frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}\right)\times \left[\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{1}{\rho_S}\frac{\partial}{\partial z}\left(\frac{\rho_S}{S}\frac{\partial\psi}{\partial z}\right)+\beta y\right]=0. \quad (3)$$

[See Pedlosky (1987) for a derivation, discussion, and applications of this equation.] Charney contributed substantially to our understanding of atmospheric dynamics by showing that for synoptic-scale three-dimensional motions, the vertical velocity, which enables the stretching of planetary vorticity filaments, can be eliminated by means of the thermodynamic equation, which leads to Eq. (3), a single equation for a single unknown, the quasigeostrophic potential vorticity.

RESONANT ROSSBY WAVE TRIADS. There is a case of special interest in which two wave components produce a third wave, such that its interaction with each of them generates the other. In this case the nonlinear interaction is essentially confined to three components that exchange energy but do not produce any further waves. These three waves are called a *resonant triad.* But not just any three waves will do; they must satisfy restrictive conditions on their wavenumbers and frequencies. The wavenumbers in each spatial direction must sum to zero, so must the frequencies of the three components. This guarantees that each pair of components interacts nonlinearly to produce a total phase corresponding to that of the third component, ensuring that strong interaction between the waves occurs.

To study the dynamics of Rossby wave triads, Pedlosky (1987) used a two-timing perturbation approach: he assumed that each component comprised a sinusoidal oscillation with an amplitude, the envelope amplitude, which varied slowly compared to variations due to the movement of the wave. We refer the interested reader to Pedlosky's excellent text for a full exposition. The Rossby wave combination satisfies the complicated *partial* differential equation (1). The three amplitudes of a resonant triad satisfy a drastically simplified system of three *ordinary* differential equations. We assume that the wave components are ordered so that the *third* component has an *intermediate* horizontal scale. This is consistent with arrangement in order of increasing frequency. After some mathematical transformations (for details, see ES) we obtain three equations governing the amplitudes A_n of the components:

$$
\begin{aligned}
\dot{A}_1 &= -A_2^* A_3, \\
\dot{A}_2 &= -A_3 A_1^*, \\
\dot{A}_3 &= +A_1 A_2,
\end{aligned}
\tag{4}
$$

where dots denote time derivatives and asterisks denote complex conjugates. These are the modulation equations, that is, the equations for the envelope amplitudes. They are in the canonical form of the system known as the *three-wave equations* (see, e.g., Holm and Lynch 2002).

With periodic boundary conditions, the solutions of the barotropic potential vorticity equation (1) conserve not only the total energy, but also the potential enstrophy, the square of the relative potential vorticity. The conservation of these two quantities has a profound influence on the character of the flow. For a wave triad, the conservation of energy and enstrophy has special significance. If the energy of the wave with intermediate scale (the third component) grows, both the smaller and larger waves must lose energy. Similarly, the largest and smallest components grow together at the expense of the intermediate one. Thus, it is impossible for energy to be transferred only to larger or smaller scales. These conditions put strong constraints on the distribution of energy in the atmosphere and are the primary reason why there is a preponderance of energy at large scales.

The triad solutions are based on an assumption of small amplitudes, so that the effects of nonlinearity act like perturbations of a predominantly linear wave evolution. For larger amplitudes, further components are generated by nonlinear interactions, which enables a flux of energy to the largest scales. The very nature of the flow changes completely as the energy increases: mathematically, the equations are no longer integrable and physically the motion is no longer regular but becomes chaotic.

The conservation of energy and potential enstrophy are properties not only of resonant triads, but of the solutions of the complete equation (1) and, indeed, of its three-dimensional generalization, Charney's equation (3). The consequences for the energy spectrum of these constraints were investigated by Fjørtoft (1953) in the context of a nonrotating, nondivergent barotropic fluid. He showed that if a fraction of the energy flows into smaller scales, then a greater fraction must flow into larger scales. Platzman (1962) investigated the analytical dynamics of the spectral vorticity equation for nondivergent motions on the sphere. He showed that, with three components, concurrent energy changes in the components of smallest and largest scale are of the same sign, and opposite in sign to that of the component of intermediate scale. He pointed out that this "spectral blocking" is a direct consequence of the existence of two spectral invariants. Following the work of Kraichnan (1967), Fjørtoft's results were greatly extended by Charney (1971), in a paper titled *Geostrophic Turbulence.* Charney showed that an energy cascade to small scales was also precluded for three-dimensional quasigeostrophic flow, and he deduced a *minus three* power law for the energy spectrum.

Let us consider an initial distribution of energy concentrated near a particular wavenumber. As the flow evolves, the energy spectrum of the motion will broaden. But the enstrophy constraint requires that the mean wavenumber of the spectrum and the mean frequency must decrease. In other words, the spatial and temporal scales of the motion must *increase* with time. At the same time, the enstropy is transferred to smaller scales. This is in marked contrast to the character of fully three-dimensional turbulence, where the energy cascades to smaller scales until frictional mechanisms begin to act.

NUMERICAL EXAMPLE OF TRIAD RESONANCE. In this section, numerical solutions of the barotropic potential vorticity equation (1) will be presented. The initial conditions correspond to a superposition of three Rossby wave components satisfying the conditions for resonance. We shall see that the solutions display the characteristics of a resonant triad, with a periodic interchange of energy between the modes. We first note an important property of the three-wave equations (4): if the amplitudes are scaled by a constant value and the time is contracted by a similar factor, the form of the equations is unchanged. Thus, the period of the modulation envelop will vary inversely with its amplitude. However, this scaling property will be inherited by the full equation (1) only as long as the modulation equations faithfully reflect the envelope dynamics of the full solution. This is the case as long as the perturbation procedure is valid, and this in turn requires that the nonlinear term in (1) is relatively small compared to the other terms. So, the scale invariance should be observed for small-amplitude waves, but may be expected to break down for larger-sized waves.

The parameters chosen for the numerical experiments are given in ES. The means of defining the wavenumbers and frequencies so that the conditions for resonance are obtained are discussed in Pedlosky (1987). The wavenumbers of the three components are given in Table 1of ES. The three-wave components are plotted in Fig. 2, together with the initial streamfunction, which is a linear combination of them. For illustration, the waves are scaled to have unit amplitude. We refer to wave 3 as the *primary component,* because it predominates at the initial time, and to the other components as secondary waves.

Equation (1) is solved by an elementary numerical technique: the quantity whose time derivative occurs in the equation is stepped forward using a leapfrog scheme, and the streamfunction is deduced by solving a Helmholtz equation with periodic boundary conditions at each time step. The numerical scheme is designed to conserve both energy and enstrophy.

To illustrate resonance in its pure form, we choose the amplitude to be very small. According to the discussion above, this implies a very long interaction time. We show in Fig. 3 the evolution of the coefficients of the three components, obtained by calculating the Fourier transform of the streamfunction at each time step. The periodic exchange of energy between the components is clear. Waves 1 and 2 grow and decay together, in antiphase with the third, or primary, wave. The modulation period is about 800 days,[2] with six cycles over the 4800-day duration of the integration.

In Fig. 4, we see the streamfunction valid at three different times. At the initial time the streamfunction is dominated by the primary component, wave 3. At the final time, the solution looks very similar to that at the initial time (allowing for phase shifts due to the wave motion). The system has gone through six full cycles at this time. One-quarter of the way through the in-

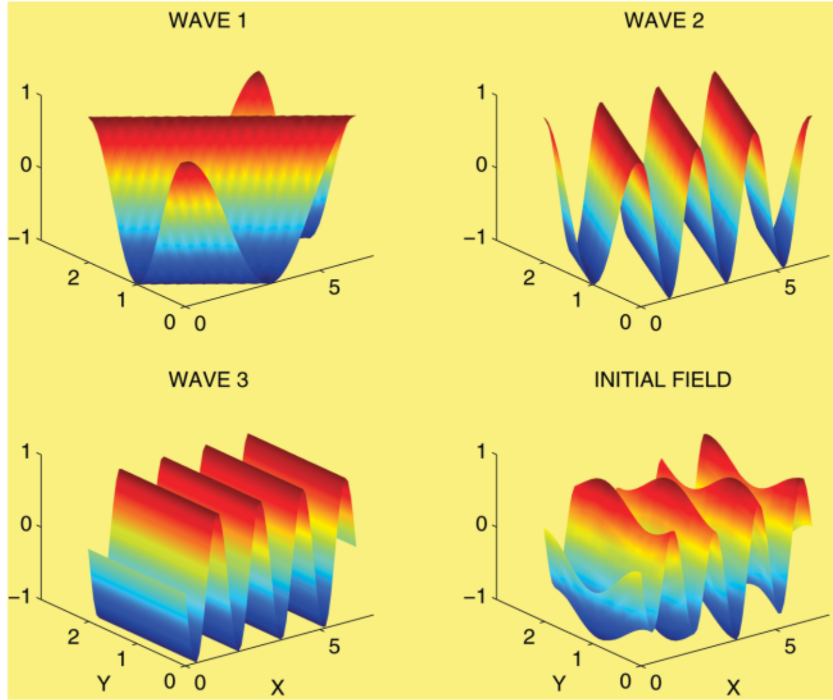

Fig. 2. Components of a resonant Rossby wave triad and the initial field constructed from them. All fields are scaled to have unit amplitude.

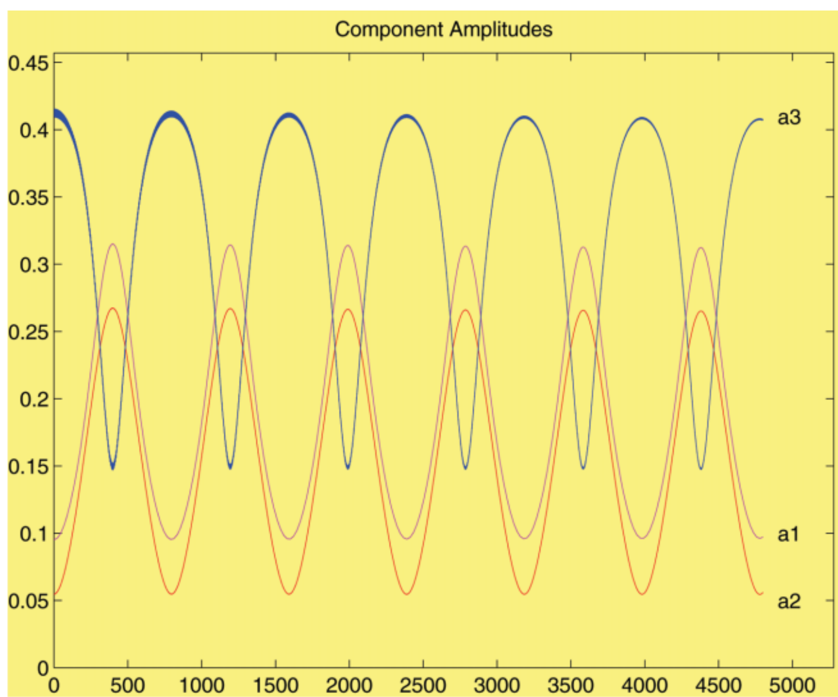

Fig. 3. Variation with time (days) of the amplitudes of the three components of the streamfunction.

[2] Choosing a more realistic amplitude drastically reduces the modulation period, as seen in the section titled "Precession and Predictability of Triads."

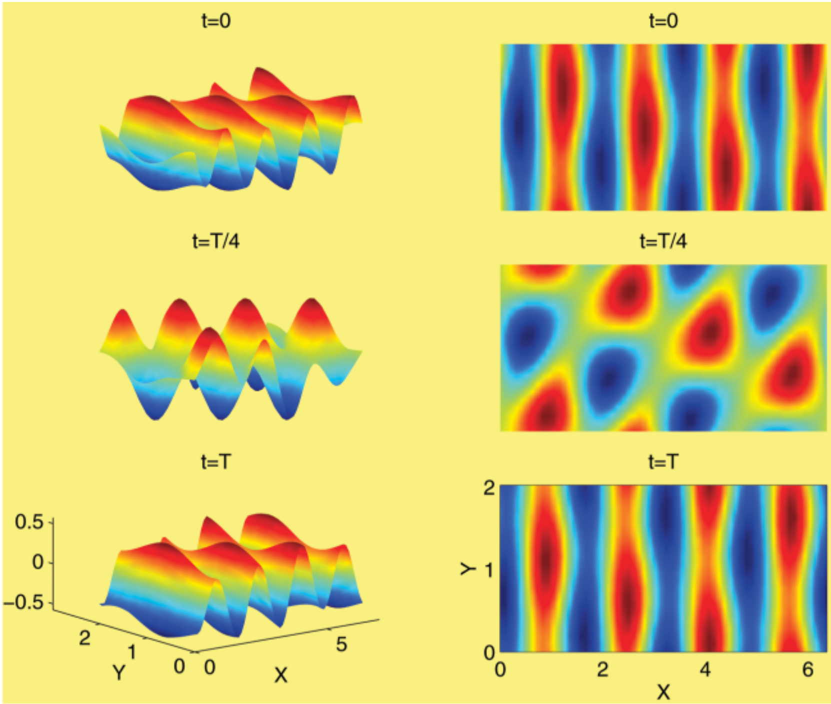

Fig. 4. Streamfunction at 3 times during an integration of duration 4800 days. Left-hand panels show a perspective view and right-hand panels show a plan view.

tegration (center panels) there is clearly a substantial contribution from the other waves, consistent with the values of the coefficients at this time (see Fig. 3). We remark here that while the amplitudes are clearly periodic, the phases need not return to their original values at the end of each modulation cycle.

THE SWINGING SPRING. The elastic pendulum or swinging spring is a simple mechanical system with highly complex dynamics. It comprises a heavy mass suspended from a fixed point by a light spring, which can stretch but not bend, moving under gravity. The equations of motion are easy to write down but, in general, impossible to solve analytically. For finite amplitudes, the motion of the system exhibits *chaos,* and predictability is severely limited. For small amplitudes, perturbation techniques are valid, the system is integrable, and approximate analytical solutions can be found.

The linear normal modes of the system are of two distinct types, a vertical or springing oscillation, in which elasticity is the restoring force, and quasi-horizontal swinging oscillations, in which the system acts like a pendulum. When the frequencies of the springing and swinging modes are in the ratio 2:1, an interesting nonlinear resonance phenomenon occurs in which energy is transferred periodically back and forth between the springing and swinging motions. The resonance phenomenon was first examined by Vitt and Gorelik (1933), who were inspired by the analogy between this system and the Fermi resonance of a carbon dioxide molecule.

Lynch (2002b) considered the swinging spring as a simple model of balance in the atmosphere, assuming the frequency of the elastic oscillations to be much greater than that of the pendular motions. He drew an analogy between the elastic and pendular modes of the spring and the gravity and Rossby modes of oscillation in the atmosphere. He showed that the dynamics of several phenomena could be illustrated by the system, for example, the nonlinear interplay between low-frequency Rossby waves and high-frequency gravity waves, the normal mode initialization of data to prevent spurious oscillations, the filtering of the equations to eliminate the high-frequency solutions, the existence and structure of a slow manifold, and the onset of chaos. There are numerous references in Lynch (2002b) to earlier work on the elastic pendulum.

In Lynch (2002a), the resonance of the spring was studied. Asymptotic solutions were obtained, and an expression was derived for the precession of the swing plane. This was later generalized by Holm and Lynch (2002) who used Hamiltonian reduction and pattern evocation techniques to derive a formula for the stepwise precession of the azimuthal angle. Holm and Lynch discovered that the perturbation equations describing the motion of the swinging spring could be reduced to the three-wave equations; this is the *key result* leading to the present work. The relevance of the three-wave equations in a broad range of physical contexts was discussed by these authors.

EQUATIONS OF THE SPRING. The mechanical system is illustrated schematically in Fig. 5. The equations of motion are formulated in terms of Cartesian coordinates *x, y,* and *z,* centered at the point of equilibrium. Expressions for the kinetic and potential energy are easily derived; the difference between them is the Lagrangian. The Lagrange equations of motion may then be written in the usual way (see, e.g., Synge and Griffith 1959); they provide three equations for the three variables, *x, y,* and *z.* If we assume that the amplitude of the motion is small, the solutions may written in the form of pure sinusoidal oscillations. In this linear limit, there is no interaction between the oscillations in each direction. In general, there are two constants of the motion, the energy and the angular momentum about the vertical. Because the system has 3 degrees of freedom and only two invariants, it is not integrable. We must employ perturbation techniques to obtain an approximate solution.

We confine attention to the resonant case where the frequency of the vertical or elastic oscillations is twice that of the horizontal or pendular motion. The averaged Lagrangian technique is applied (see Holm and Lynch 2002, for details); the solution is assumed to be a rapidly varying sinusoidal oscillation with a slowly varying amplitude (we made the same assumption for the Rossby wave triad). If the Lagrangian is averaged over the period of the rapid oscillations, approximate equations for the slowly varying envelop amplitudes can be derived. After some transformations (the details of which may be found in ES), the equations take the following form:

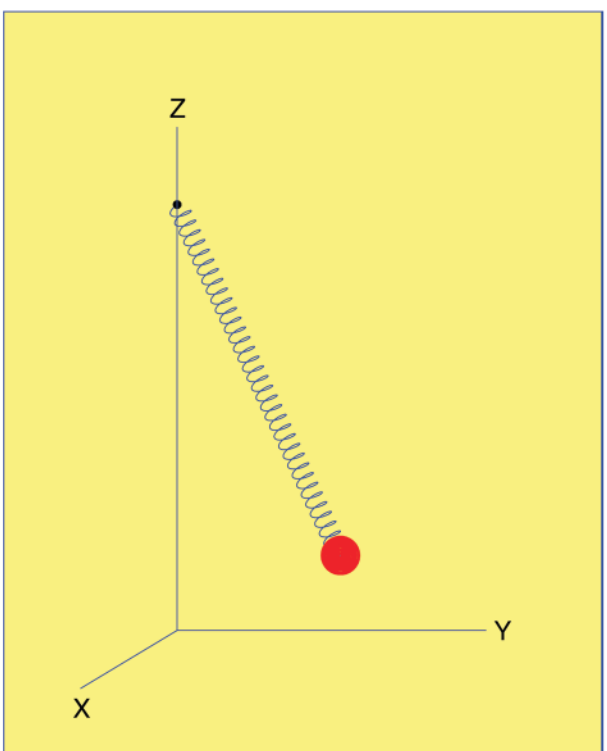

FIG. 5. The swinging spring: Cartesian coordinates are used, with the origin at the point of stable equilibrium of the bob.

$$\begin{aligned} \dot{A}_1 &= -A_2^* A_3, \\ \dot{A}_2 &= -A_3 A_1^*, \\ \dot{A}_3 &= +A_1 A_2. \end{aligned} \tag{5}$$

But these are exactly the same as (4). Thus, the modulation equations for the swinging spring are transformed into the *three-wave equations,* mathematically identical to the equations obtained above for resonant Rossby triads. The three-wave equations conserve three quantities: *H,* the Hamiltonian of the system; *N,* the energy of the oscillations; and *J,* the angular momentum (see Holm and Lynch 2002). There are thus three independent constants of the motion.

The three-wave equations model the nonlinear dynamics of the amplitudes of three waves in fluids or plasmas. Resonant wave–triad interactions play an essential role in the generation of turbulence and in determining the statistics of the power spectrum. We have seen that energy and enstrophy are conserved for a Rossby wave triad. Equation (5) is also equivalent to the Maxwell–Schrödinger envelope equations for the interaction between radiation and a two-level resonant medium in a microwave cavity. The three-wave system describes the dynamics of the envelopes of light waves interacting quadratically in nonlinear material, and of triplets of phonons, vibrations in crystal lattices. Using a geometrical approach, the reduced dynamics for the wave intensities may be represented as motion on a closed surface in three dimensions—the three-wave surface (see Holm and Lynch 2002, for a fuller discussion of the three-wave system, and for further references).

For the special case where the Hamiltonian takes the value zero, the system (5) reduces to three *real* equations equivalent to Euler's equations for the rotation of a free rigid body rotating about its center of gravity (Synge and Griffith 1959). Thus, the simple spring pendulum, which was first studied to provide a classical analog to the quantum phenomenon of Fermi resonance, provides a concrete mechanical system, which simulates a wide range of physical phenomena, in particular, the phenomenon of interest here, the resonance of Rossby wave triads.

PRECESSION OF THE SWING PLANE. There is a particular feature of the behavior of the physical spring, which is fascinating to watch. When started with almost vertical springing motion, the movement gradually develops into an essentially horizontal swinging motion. This does not persist, but is soon replaced by springy oscillations similar to the initial motion. Again a horizontal swing develops, but now in a different direction. This variation between springy and swingy motion continues indefinitely. The change in direction of the swing plane from one horizontal excursion to the next is difficult to predict; the plane of swing precesses in a manner that is quite sensitive to the initial conditions.[3]

NUMERICAL EXAMPLE OF SPRING PRECESSION. To illustrate the nature of the modulated motion, we present the results of some numerical integrations of the spring equations. The parameter values are chosen so that the linear swinging mode has a period of 2 s, and the springing mode has half this period. The parameters and initial conditions may be found in ES. In Fig. 6 we plot the solutions *x, y,* and *z* obtained by integrating the equations numerically. Also plotted (lower-right-hand panel) are the components of energy, showing the periodic exchange between the horizontal and vertical components. During the integration time of 150 s there are six horizontal excursions, so the modulation period is about 25 s.

[3] A Java Applet illustrating the precession of the swinging spring may be found online at www.maths.tcd.ie/~plynch/SwingingSpring/SS_Home_Page.html.

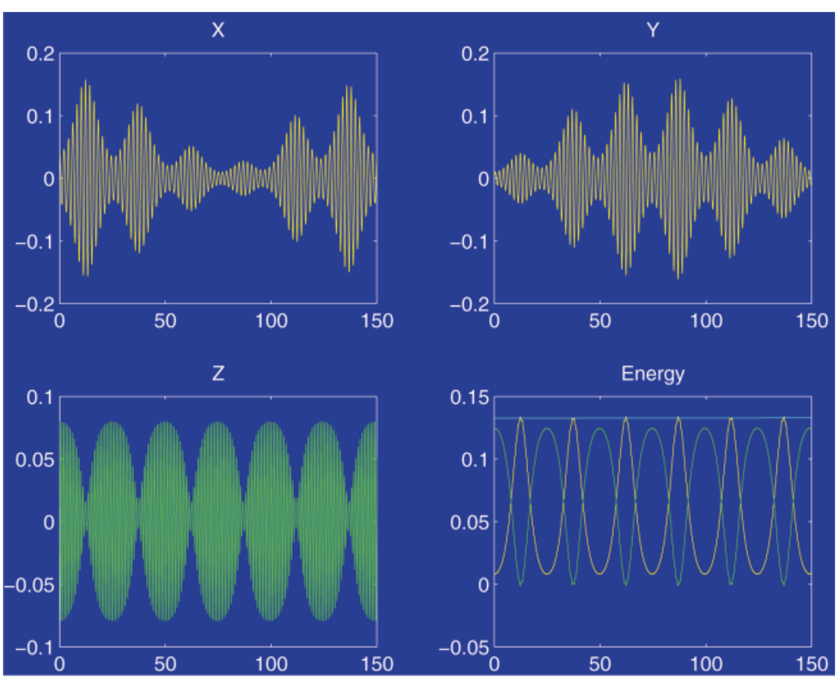

Fig. 6. Solutions *x, y,* and *z* obtained by integrating the equations of motion of the spring. Also plotted (lower-right-hand panel) are the components of energy (yellow: horizontal; green: vertical; cyan: total).

The stepwise precession of the swinging spring will now be illustrated. In Holm and Lynch (2002) the concept of an *instantaneous ellipse* was introduced: at any time, the spring trajectory can be approximated by a central ellipse, and the rotation of its major axis represents the precession of the amplitude envelope. In Fig. 7 (upper panel) we plot the azimuthal angle (in degrees) and magnitude of the major axis of the ellipse (scaled to have a maximum value of 180). The stepwise precession is clearly seen. In the lower panel, we plot the horizontal projection of the position of the bob and obtain a star-shaped pattern. The precession angle between horizontal excursions is 30° (the initial values were tweaked to tune the precession angle to an even fraction of 360°). Thus, the major axis passes through 180° in 150 s.

PRECESSION AND PREDICTABILITY OF TRIADS. Analogies between physical systems are a powerful means of gaining understanding of abstruse and complex phenomena from more familiar and simple systems. When the equations describing the systems are identical, more concrete conclusions can be reached. Because the same equations apply to both the spring and triad systems, the stepwise precession of the spring must have a counterpart for triad interactions. Expressions for the axes and azimuth of the instantaneous ellipse in terms of the amplitudes of the spring were given in Holm and Lynch (2002). In terms of the variables of the three-wave equations, they are even simpler. The semiaxis major A_{maj} is the sum of the amplitudes of the two lowest-frequency triad components, and the azimuthal angle θ is half of the difference of their phases (see ES for details). The initial conditions for the spring, which were used to generate the solution shown in Fig. 7, were transformed to obtain corresponding initial conditions for (1). The initial field was then scaled to ensure that the small-amplitude approximation was accurate (the amplitude of the primary component was set to 0.4 m).

The coefficients of the components were saved and the elements of the instantaneous ellipse were calculated. The time

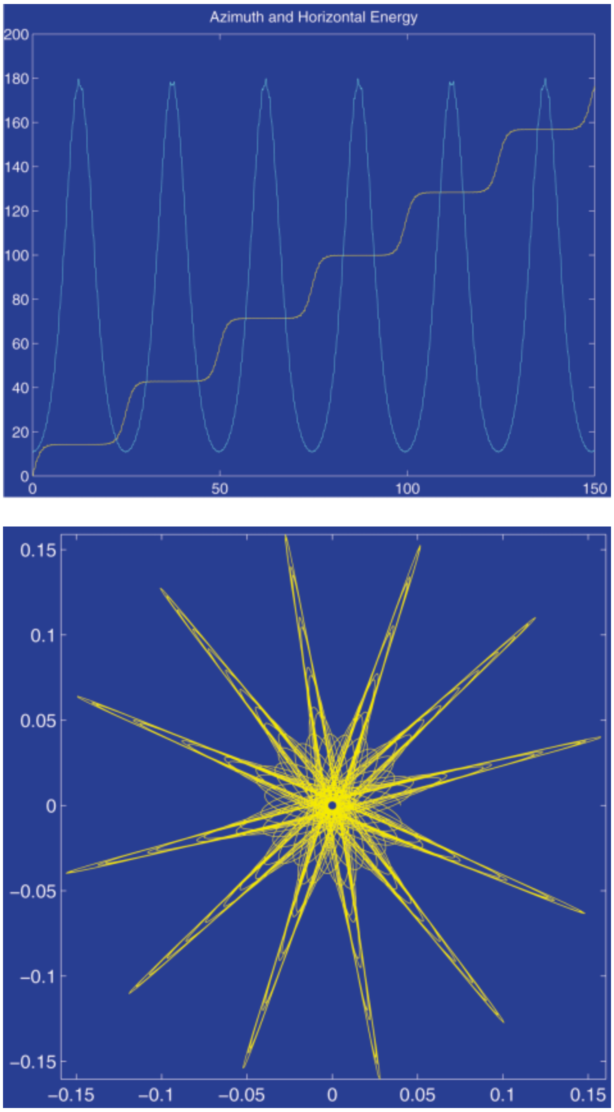

FIG. 7. (top) Azimuth θ (yellow) and horizontal energy (cyan) of spring solution; (bottom) horizontal projection of spring solution, y vs x.

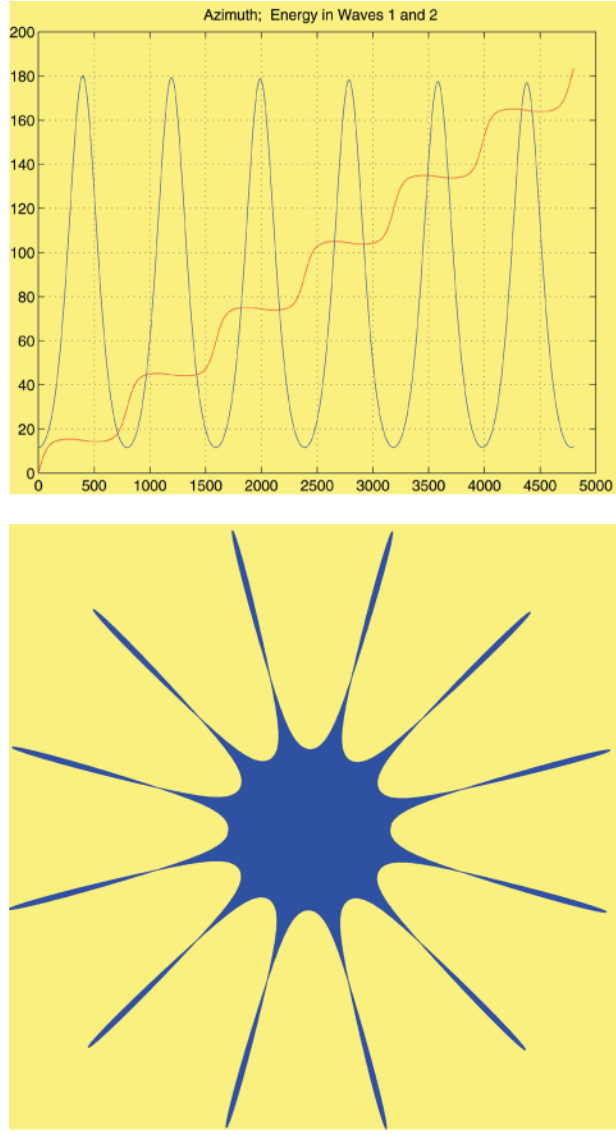

FIG. 8. (top) Azimuth θ of instantaneous ellipse (see text) and energy of waves 1 and 2. (bottom) Polar plot of A_{maj} vs θ. The starlike pattern is similar to that found for the spring (Fig. 7).

variation of θ is similar to that found for the spring (cf. Figs. 7 and 8), exhibiting a characteristic stepwise precession. Figure 8 (lower panel) shows a polar plot of A_{maj} versus θ. We immediately see the starlike pattern, similar to that found for the spring (Fig. 7). The precession angle, the change in azimuth between successive maxima of A_{maj}, is again about 30°. This is remarkable, and illustrates the value of the analogy. Phase precession for Rossby wave triads has not been noted before and is an example of the insight coming from the mathematical equivalence of the two systems.

The precession has implications for the predictability of atmospheric motion. A flow dominated by a single Rossby wave may be unstable and, if so, will be rapidly distorted due to inevitable perturbations. Triad resonance is the primary mechanism for this breakdown. However, the resulting pattern is highly sensitive to details of minute perturbations, which are impossible to determine accurately. Drastically different patterns can result from states, which are initially very similar. To investigate this sensitivity for realistic amplitudes, we consider a primary wave of amplitude 60 m and secondary waves whose amplitude is 1% of this value, and integrate (1) over a period of 4 days. Figure 9 shows the initial and final fields for two integrations; the initial fields differ only in the sign of the perturbation. Considering the upper panels of Fig. 9 as two weather "analyses," we would have to regard them as practically identical (the perturbation amplitude is only 1% of that of the primary wave). Yet, the resulting "forecasts" in the two lower panels differ drastically. The center point is marked by a yellow plus sign. In one case (lower-left-hand panel), it is close to a high pressure center (red); in the other (lower-right-hand panel) it is close to a low (blue). Thus, the forecasts from almost identical initial conditions diverge significantly within a matter of a few days. Because the three-wave equations are integrable, this sensitivity cannot be described in the usual terms of chaos (the solutions of these equations are regular). We, therefore, have a chaoslike phenomenon in an integrable system. Thus, predictability may be severely limited even in systems that are not chaotic.

The task of forecasting Rossby wave breakdown may be compared to that of trying to predict the emergence of a growing perturbation in a baroclinically unstable flow. Because the location of the perturbation is unknown, the phase of the developing baroclinic wave cannot be anticipated before it has grown to a detectable amplitude. In the case of the unstable Rossby wave, although accurate knowledge of the primary wave phase is available, it is of no help in estimating the phases of the growing perturbations that soon dominate the flow. The forecaster's task is even harder than might have been imagined.

For realistic Rossby waves, the small-amplitude approximation is invalid, and we would expect the flow to become chaotic. The nature of the transition from the integrable solutions of the three-wave equations to irregular chaotic flow is worthy of attention. We do not undertake a detailed study but show a single example in Fig. 10. The component amplitudes are scaled by 2.5 relative to those upon which Fig. 3 is based. But now something interesting happens: large and small peaks alternate, suggesting that the period for energy exchange between the wave components has doubled. This period-doubling bifurcation is a well-known path to chaos (Ott 1993), and this preliminary evidence should encourage a more detailed investigation to confirm if the period-doubling mechanism is at work here.

SUMMARY AND DISCUSSION. We have considered the interactions between resonant triads of Rossby waves by integrating the barotropic potential vorticity equation from appropriately chosen initial conditions. The behavior for a small amplitude is consistent with that predicted by a perturbation analysis: the total energy of the triad is constant, but the energy is exchanged on a slow timescale between the components. The perturbation analysis leads to the three-wave equations, an integrable system. These same equations govern the small-amplitude dynamics of an elastic pendulum or swinging spring. This equivalence allows us to deduce properties, not otherwise evident, of atmospheric flow from the behavior of the mechanical system. In particular, we have seen that the stepwise precession found for the spring is also a characteristic of triad interactions.

When a single wave, the primary wave, dominates the initial conditions, the subsequent development is found to depend sensitively on the details of the perturbation. Because in general, these details are unobservable, accurate prediction of the

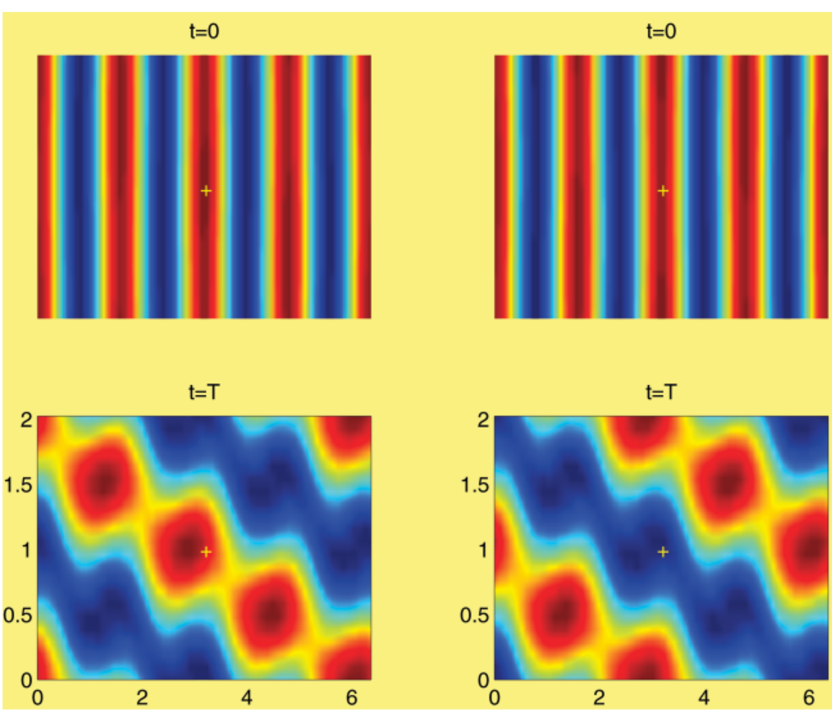

Fig. 9. Initial and final fields for two 4-day integrations of Eq. (1). The initial fields (top panels) differ only in the sign of the perturbation. The resulting forecasts are shown in the two lower panels. The center of the domain is marked by a yellow cross.

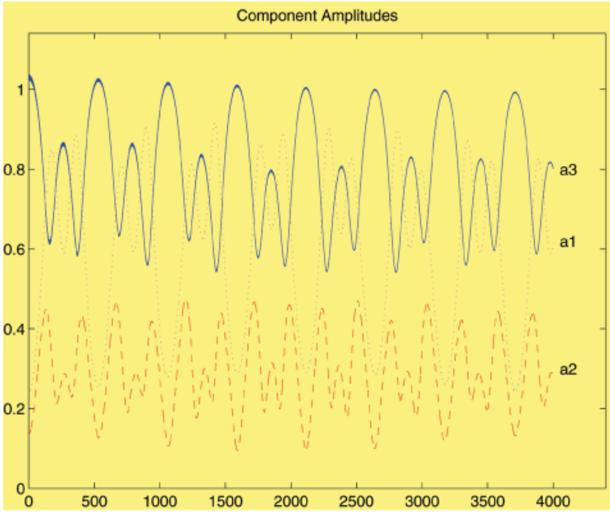

FIG. 10. Evolution of the coefficients of the three components for initial conditions scaled by 2.5, relative to those in Fig. 3. The periodic exchange of energy between the components indicates a period-doubling route to chaos.

flow is difficult or impossible.[4] The question of the extent to which these findings apply to more complex situations, such as atmospheric flow, depends on how spherical and baroclinic effects influence the dynamics. For flow on the sphere, a single large-scale Rossby wave may be stable, because it may not be possible to find a resonant triad such that the scale of this primary wave is intermediate between those of the other two components. Evidence from elsewhere suggests that the largest-scale Rossby waves are indeed stable (Lorenz 1972; Hoskins 1973).

The MATLAB code for solving the barotropic potential vorticity equation is available online at www.maths.tcd.ie/~plynch/Rossby_Wave_Triads/triad.html and the code to integrate the swinging spring equations may be found at www.maths.tcd.ie/~plynch/Rossby_Wave_Triads/spring.html. These programs may be used to pursue the study of the equivalence between the two systems. For example, two-dimensional planar motion of the spring corresponds to triad interactions for which the two secondary wave envelopes are locked in phase and proportional in amplitude. The elliptic–parabolic modes of the spring discussed by Lynch (2002a) must also have counterparts for triad dynamics. The transition to turbulence for triad motions merits a more detailed study.

[4] This situation corresponds to spring oscillations, which are initially quasi-vertical. It is virtually impossible to predict the direction of the first horizontal excursion.

Finally, we mention that the implications of the spring dynamics are much wider than discussed above. For example, drift waves in magnetic confinement devices, such as tokamaks, are believed to dominate the turbulent transport of energy. They are also relevant for the dynamics of the ionosphere and magnetosphere. The three-wave equations are central to the small-amplitude dynamics of these systems.

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REFERENCES

Cartwright, D. E., 1999: *Tides: A Scientific History*. Cambridge University Press, 292 pp.

Charney, J. G., 1948: On the scale of atmospheric motions. *Geofys. Publ.,* **17,** 3–17.

——, 1971: Geostrophic turbulence. *J. Atmos. Sci.,* **28,** 1087–1095.

——, and M. E. Stern, 1962: On the stability of internal baroclinic jets in a rotating atmosphere. *J. Atmos. Sci.,* **19,** 159–172.

Fjørtoft, R., 1953: On the changes in the spectral distribution of kinetic energy for two-dimensional nondivergent flow. *Tellus,* **5,** 225–230.

Hasegawa, A., and K. Mima, 1977: Pseudo-three-dimensional turbulence in magnetized nonuniform plasmas. *Phys. Fluids,* **21,** 87–92.

Holm, D. D., and P. Lynch, 2002: Stepwise precession of the resonant swinging spring. *SIAM J. Appl. Dyn. Syst.,* **1,** 44–64.

Horton, W., and A. Hasegawa, 1994: Quasi-two-dimensional dynamics of plasmas and fluids. *Chaos,* **4,** 227–251.

Hoskins, B. J., 1973: Stability of the Rossby–Haurwitz wave. *Quart. J. Roy. Meteor. Soc.,* **99,** 723–745.

Kraichnan, R., 1967: Inertial ranges in two-dimensional turbulence. *Phys. Fluids,* **10,** 1417–1423.

Lewis, J. M., 1992: Carl-Gustaf Rossby: A study in mentorship. *Bull. Amer. Meteor. Soc.,* **73,** 1425–1438.

Lindzen, R. S., E. N. Lorenz, and G. W. Platzman, Eds., 1990: *The Atmosphere—A Challenge: The Science of Jule Gregory Charney.* Amer. Meteor. Soc., xviii + 321 pp.

Lorenz, E. N., 1972: Barotropic instability of Rossby wave motion. *J. Atmos. Sci.,* **29,** 258–264.

Lynch, P., 2002a: Resonant motions of the three-dimensional elastic pendulum. *Int. J. Nonlinear Mech.,* **37,** 345–367.

——, 2002b: The swinging spring: A simple model for atmospheric balance. *Geometric Methods and Models,* J. Norbury and I. Roulstone, Eds., Vol. 2, *Large-Scale Atmosphere–Ocean Dynamics,* Cambridge University Press, 64–108.

Nezlin, M. V., and E. N. Snezhkin, 1993: *Rossby Vortices, Spiral Structures, Solitons: Astrophysics and Plasma Physics in Shallow Water Experiments.* Springer-Verlag, 223 pp.

Obukhov, A. M., 1949: On the question of the geostrophic wind (in Russian). *Izv. Akad. Nauk. SSSR. Ser. Geogr. Geofiz.,* **13,** 281–306. (Translated by J. C. Howcroft, University of Chicago, 1962.)

Ott, E., 1993: *Chaos in Dynamical Systems.* Cambridge University Press, 385 pp.

Pedlosky, J., 1987: *Geophysical Fluid Dynamics.* 2d ed. Springer, 710 pp.

Phillips, N. A., 1990: The emergence of quasi-geostrophic theory. *The Atmosphere—A Challenge: The Science of Jule Gregory Charney,* R. S. Lindzen et al., Eds., Amer. Meteor. Soc., 177–206.

——, 1998: Carl-Gustaf Rossby: His times, personality, and actions. *Bull. Amer. Meteor. Soc.,* **79,** 1097–1112.

——, W. Blumen, and O. Coté, 1960: Numerical weather prediction in the Soviet Union. *Bull. Amer. Meteor. Soc.,* **41,** 599–617.

Platzman, G. W., 1962: The analytical dynamics of the spectral vorticity equation. *J. Atmos. Sci.,* **19,** 313–328.

——, 1968: The Rossby Wave. Symons Memorial Lecture, 3 April, 1968. *Quart. J. Roy. Meteor. Soc.,* **94,** 225–248.

Rossby, C. G., and Coauthors, 1939: Relations between variations in the intensity of the zonal circulation of the atmosphere and the displacements of the semi-permanent centers of action. *J. Mar. Res.,* **2,** 38–55.

Synge, J. L., and B. A. Griffith, 1959: *Principles of Mechanics.* McGraw-Hill, 552 pp.

Vitt, A., and G. Gorelik, 1933: Kolebaniya uprugogo mayatnika kak primer kolebaniy dvukh parametricheski svyazannykh linejnykh sistem. *Zh. Tekh. Fiz.,* **3,** 294–307. (Available in English: Oscillations of an elastic pendulum as an example of the oscillations of two parametrically coupled linear systems. Translated by L. Shields, Met Eireann Historical Note No. 3, 15 pp.)