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Initialization using Laplace transforms

By PETER LYNCH

Irish Meteorological Service, Glasnevin Hill, Dublin 9*

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SUMMARY

The initialization of limited area models is complicated by the difficulty of determining the linear normal modes and of allowing for general boundary conditions. A new method of initialization is devised, which does not require explicit knowledge of the normal modes. The method is based on a filtering procedure which uses a modified inverse Laplace transform. The efficacy of the method is demonstrated by application to a one-dimensional model, and the rationale for application to a general forecasting model is discussed. The method is closely related to the nonlinear normal mode method of initialization.

1. INTRODUCTION

The primitive equations used for numerical weather prediction have solutions representing two distinct types of motion. The solutions of meteorological significance have low frequencies, are generally close to geostrophic balance and have small divergence. There are also very fast gravity–inertia wave solutions, with phase speeds of hundreds of metres per second and large divergence; these do not interact strongly with the rotational motions, and are generally regarded as noise. A subtle state of balance exists in the atmosphere between the wind and pressure fields, ensuring that the gravity–inertia waves have much smaller amplitude than the rotational part of the flow. Forecasts made with the primitive equations generally contain large high frequency oscillations unless the initial fields are adjusted to reflect this balance. The process of adjustment is called initialization. The principal goal of initialization is to define the initial fields in such a way that the gravity–inertia waves remain small throughout the forecast.

(*a*) *Some initialization methods*

In the early days of numerical weather prediction forecasts were made using governing equations which were modified in such a way that they filtered out gravity–inertia waves altogether. However, the approximations made in deriving the filtered equations are not always valid, and the primitive equations are now generally used. The first efforts to control the high frequency motions consisted of defining various diagnostic relationships between the mass and wind fields (Charney 1955; Phillips 1960). These *static initialization* methods were only partially successful in reducing the noise and are unsatisfactory for a number of other reasons. An alternative approach to balancing the fields is to use the forecast model to integrate in a forward–backward cycle with a damping time-scheme (Miyakoda and Moyer 1968). This is known as *dynamic initialization*. The method is computationally expensive, no irreversible processes can be included because of the backward integrations, and the remaining noise is still a problem.

Recently, methods of initialization based upon the linear normal modes of the forecast model have been proposed. The first approach was to analyse the initial conditions into their slow rotational and fast gravity–inertia components and to set to zero the coefficients of the latter (Dickinson and Williamson 1972). This was partially successful but because of the nonlinearity of the equations the high frequency modes are rapidly re-excited. A more sophisticated approach is to set the initial rate of change of the fast modes to zero; this method is called *nonlinear normal mode* initialization (Baer 1977; Machenhauer 1977). It has been found to be superior to the simpler, linear method

* On temporary leave as Visiting Scientist at KNMI, De Bilt, Netherlands

and is currently regarded as the most satisfactory initialization method, despite certain disadvantages. The method has been reviewed comprehensively by Daley (1981).

To apply the nonlinear normal mode method we must know the linear normal modes of the model equations. For a global domain this is no particular problem: the equations can be linearized in such a way that the spatial variable dependencies are separable and the horizontal boundary conditions are those of periodicity or regularity of the solution. The vertical structure equation is easily solved for simple mean temperature profiles. For forecasts over a limited area the situation is dramatically different. Here the dependent variables are specified on the boundary of the forecast domain and are allowed to vary in time. It is difficult, if not impossible, to define the normal modes in a completely general way. If the area is of an irregular shape the horizontal variables may not be separable. Non-separability also results if the Coriolis parameter is a function of both variables, as in the case of a rotated latitude/longitude grid whose poles are not on the earth's axis.

Some progress has been made recently in adapting the normal mode approach to limited area modelling. Brière (1982) has applied the method to a fine mesh model in Cartesian coordinates with constant boundary conditions. Bijlsma and Hafkenscheid (1983) have incorporated the effects of sphericity, and their method apparently allows for variable boundary conditions. Both applications have been successful in reducing the noise in short range forecasts. However, in each case it has been necessary to make simplifying assumptions in order to derive the linear normal modes: the analysis has been done with a constant Coriolis parameter. It is not obvious if these assumptions can be relaxed. Nor is it obvious whether or not they have a degrading effect on the specification of the initial fields. Ballish (1979) has demonstrated that failure to include the β term in the derivation of the linear normal modes may lead to noise problems. Bourke and McGregor (1983) have developed a vertical mode initialization scheme for a limited area model: they project the initial fields onto the model vertical eigenfunctions and they set to zero either the tendencies of divergence and geopotential or the tendencies of divergence and departure from linear balance for the modes with large equivalent depths. Their method effectively suppresses spurious gravity wave activity; however, since the meteorologically significant rotational components of the flow have non-vanishing divergence, the application of this filtering condition must modify them to some degree. The extent to which this may affect the forecast is unclear. Thus, a totally satisfactory resolution of the problem of initialization for limited area models is still outstanding.

(*b*) *A new approach*: *The Laplace transform technique*

In this report the following question is addressed: Given a nonlinear prediction model, how can the initial conditions be specified in such a way that no component of the flow will evolve with high frequency gravity–inertia type timescales? A general technique is developed which provides the answer to this question. The technique involves the use of a filtering method based on a modification of the inversion formula for the Laplace transform. This filter is used in an iterative procedure which is applied to a general nonlinear system with multiple timescales. *Provided the procedure converges*, the result is a function which (i) satisfies the system equations and (ii) evolves slowly in time. The initial value of this function provides the answer to the question posed above.

The practical implementation of the procedure necessitates some approximations: (a) to reduce the computational effort; (b) to avoid numerical instabilities associated with the Laplace transform inversion. If the nonlinear terms are small (a necessary

condition for convergence anyway, and true for atmospheric motions) these approximations are quite acceptable, and lead to a simple initialization procedure. The procedure is seen to bear a close relationship to the linear and nonlinear normal mode initialization methods. A point of vital *practical* importance is that the new technique does *not* require knowledge of the linear normal modes of the system. Thus, it can be applied more generally than the previously proposed methods.

The feasibility of the Laplace transform technique is demonstrated by application to a simple one-dimensional model similar to that used by Baer (1977). A model run, starting from geostrophically balanced initial conditions, results in large amplitude, high frequency oscillations. These oscillations are most clearly seen in the evolution of the divergent kinetic energy. Application of the first approximant of the new technique (equivalent to linear normal mode initialization) reduces this noise considerably, but some small oscillations remain. The nonlinear procedure, with just one iteration, leads to virtually total extinction of the noise. These results are in complete agreement with those obtained by Baer with his technique, and demonstrate that the present method is an effective means of controlling the high frequency oscillations in the simple primitive equation model.

The application of the technique in the context of a general baroclinic (limited area or global) primitive equation model is discussed. The proposed method is essentially the same as that used in the case of normal mode initialization: the vertical and horizontal structures are separated, and the horizontal structure functions are initialized separately for each vertical mode or equivalent depth. The technique would appear to be as flexible as normal mode methods in its ability to handle orography, model physics, etc. It is more flexible insofar as it can also be applied to a limited area model, where the horizontal variables need not be separable, where the boundary conditions may vary in time and where the linear normal modes are unknown.

2. Theoretical discussion

We consider the following question: given a nonlinear system with multiple time-scales, how can the initial conditions be specified in such a way that the rapidly varying components of the solution are completely supressed *for all time*? We formulate a general procedure which, given convergence, provides a solution satisfying the original nonlinear equation and evolving slowly in time. It is shown how this procedure may be applied in an approximate form to specify initial conditions which guarantee a slowly evolving solution.

(*a*) *Filtering with Laplace transforms*

Consider a function $f(t)$ with components of various frequencies

$$f(t) = \sum_{n=1}^{\infty} a_n \exp(\mathrm{i}\omega_n t). \tag{1}$$

Suppose we wish to isolate that part of f which varies with frequency less than γ. Knowing the expansion (1), it is trivial to set a_n equal to zero whenever $|\omega_n| > \gamma$; the remaining components provide the solution. Now suppose that $f(t)$ is unknown but that its Laplace transform is given by

$$\mathfrak{L}\{f(t)\} = \hat{f}(s) = \sum_{n=1}^{\infty} a_n/(s - \mathrm{i}\omega_n). \tag{2}$$

The filtering of f can again be done by setting $a_n = 0$ when $|\omega_n| > \gamma$, and inverting the sum of the remaining terms.

Next, suppose we know $\hat{f}(s)$ to be of the form (2) but do not know the precise values (a_n, ω_n). The original function $f(t)$ may be obtained from the inverse transform (Doetsch 1971):

$$f(t) = \mathfrak{L}^{-1}\{\hat{f}\} = \frac{1}{2\pi i}\int_{\mathbf{C}} e^{st}\hat{f}(s)\,ds \tag{3}$$

where **C** is a line parallel to the imaginary axis and to the right of all singularities of $\hat{f}$. The contribution to this integral from the pole of $\hat{f}(s)$ at $i\omega_n$ is just $a_n \exp(i\omega_n t)$. Without knowing (a_n, ω_n), we can still eliminate all components with frequencies greater than γ by changing the contour **C** in (3) to a circle **C*** of radius γ centred at the origin of the s plane (see Fig. 1):

$$f^*(t) = \mathfrak{L}^*\{\hat{f}\} = \frac{1}{2\pi i}\oint_{\mathbf{C}^*} e^{st}\hat{f}(s)\,ds. \tag{4}$$

The function f^* is just the sum of the components resulting from those poles which fall within **C***, i.e. the components with frequency less than γ. This is exactly what is required. Thus, knowledge of the transformed function on the contour **C*** is sufficient to perform the desired filtering.

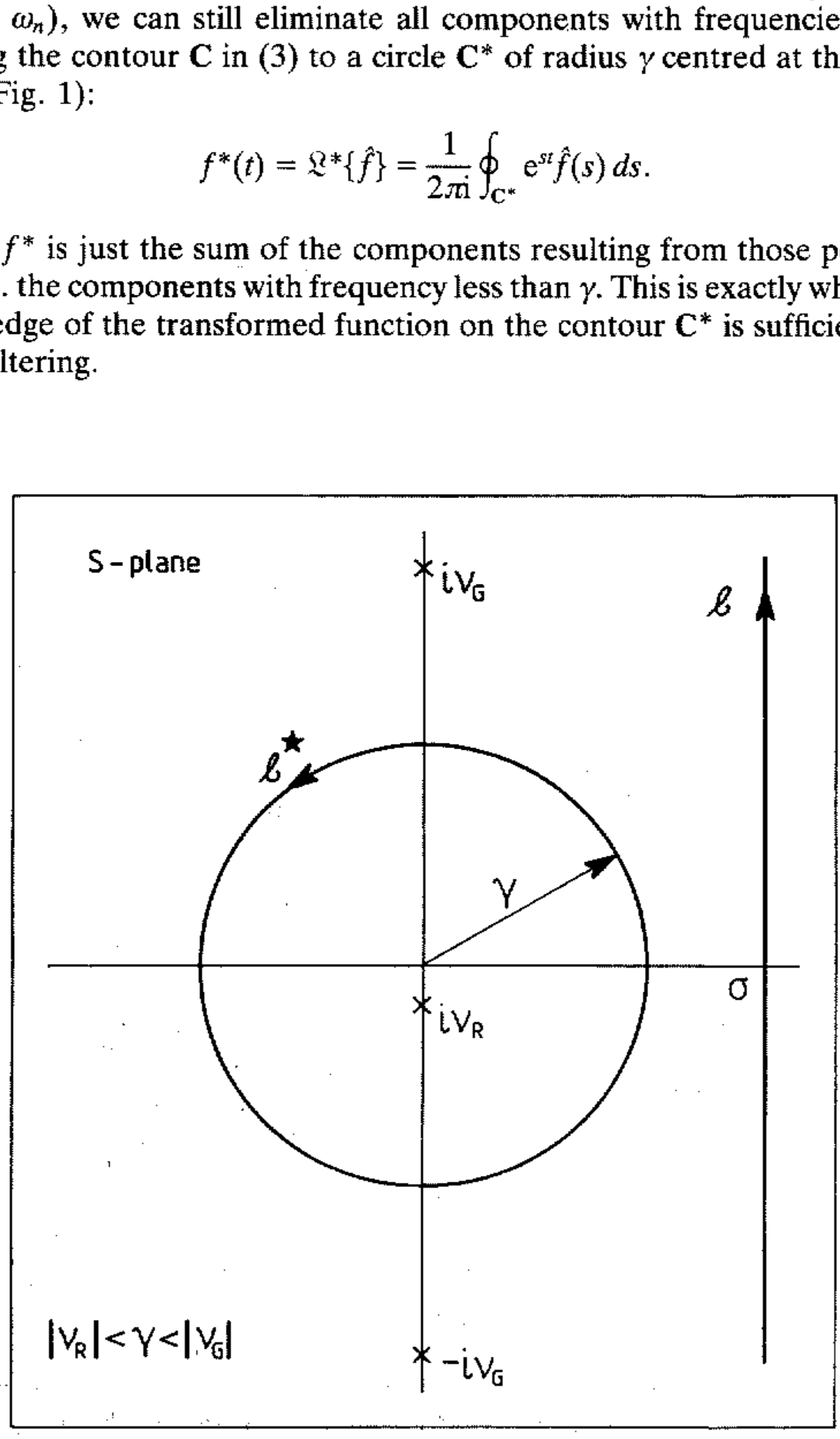


Figure 1. Contours in the s plane used for the regular and modified inverse Laplace transform. The value of γ is chosen to separate the rotational frequencies (ν_R) and the gravity wave frequencies (ν_G).

(*b*) *Separation of scales in a general nonlinear system*

Consider a system whose state at time t is specified by the vector $\mathbf{X}(t)$. For example, $\mathbf{X}$ might comprise the values of the dependent variables at all the gridpoints of a forecast model. We assume that the evolution of $\mathbf{X}$ is governed by a nonlinear equation of the form

$$d\mathbf{X}/dt + \mathbf{L}\,\mathbf{X} + \mathbf{N}(\mathbf{X}) = 0 \tag{5}$$

where $\mathbf{L}$ is a constant linear operator and $\mathbf{N}$ is a nonlinear vector function of $\mathbf{X}$. If the system is in the state $\mathbf{X}_0$ at $t = 0$ then the Laplace transform of this equation may be written

$$\mathbf{M}\hat{\mathbf{X}} + \hat{\mathbf{N}} = \mathbf{X}_0 \tag{6}$$

where we define the matrix $\mathbf{M} = \mathbf{M}(s) = (s\mathbf{I} + \mathbf{L})$, with $\mathbf{I}$ the identity matrix.

If the nonlinear terms in (6) are ignored the transformed solution is just

$$\hat{\mathbf{X}} = \mathbf{M}^{-1}\mathbf{X}_0\,. \tag{7}$$

Inverting this we recover the solution $\mathbf{X}(t)$. Suppose we are interested only in the slowly evolving part of $\mathbf{X}$, i.e. that part which oscillates with frequency less than γ. Then we may apply the operator $\mathfrak{L}^*$ to obtain

$$\mathbf{X}^*(t) = \mathfrak{L}^*\{\hat{\mathbf{X}}(s)\}. \tag{8}$$

This will have the desired property. Alternatively, if we define

$$\mathbf{X}^*(0) = \mathfrak{L}^*\{\hat{\mathbf{X}}(s)\}|_{t=0} \tag{9}$$

and use this as the initial state, the linear evolution of the system will again contain only low frequency components.

If $\mathbf{X}^*(0)$ is used as the initial state for the nonlinear equations (5) the high frequency components will be re-excited immediately by the nonlinear term $\mathbf{N}(\mathbf{X})$. We wish to avoid this happening by modifying the initial state. We can solve (6) for $\hat{\mathbf{X}}$ by an iterative procedure analogous to that used by Machenhauer (1977). We assume that the nonlinear terms are small and take as the first approximant the linear solution

$$\hat{\mathbf{X}}_1 = \mathbf{M}^{-1}\mathbf{X}_0\,. \tag{10}$$

The first estimate of the slowly varying solution is then

$$\mathbf{X}_1^* = \mathfrak{L}^*\{\hat{\mathbf{X}}_1\}. \tag{11}$$

When this is known the nonlinear term $\mathbf{N}_1 = \mathbf{N}(\mathbf{X}_1^*)$ can be obtained and its transform evaluated. Further approximations are given by the iterative procedure

$$\hat{\mathbf{X}}_{n+1} = \mathbf{M}^{-1}[\mathbf{X}_n^*(0) - \hat{\mathbf{N}}_n] \tag{12}$$

$$\mathbf{X}_{n+1}^* = \mathfrak{L}^*\{\hat{\mathbf{X}}_{n+1}\}. \tag{13}$$

Suppose the procedure converges: then we can write

$$\hat{\mathbf{X}}_\infty = \mathbf{M}^{-1}[\mathbf{X}_\infty^*(0) - \hat{\mathbf{N}}_\infty]; \qquad \mathbf{X}_\infty^* = \mathfrak{L}^*\{\hat{\mathbf{X}}_\infty\} \tag{14}$$

which, when inverted, shows that the vector $\mathbf{X}^* = \mathbf{X}_\infty^*$ has the following properties: (a) it satisfies the original equation (5); (b) it contains only slowly evolving components. This is exactly the solution which was desired. We note in passing that, if $\mathfrak{L}^{-1}$ is used instead of $\mathfrak{L}^*$ above, the full solution to the nonlinear problem is obtained.

(c) *Application to initialization*

The procedure developed above is difficult to apply in practice for two reasons. First, it involves the calculation of the Laplace transform of the nonlinear term at each iteration, which implies a formidable amount of work. Second, the Laplace transform is notoriously difficult to invert numerically, because of the factor $\exp(st)$; this also applies to the modified inverse transform, $\mathfrak{L}^*$. We circumvent these problems by assuming that the nonlinear term varies so slowly that it may be considered constant and its transform approximated by

$$\hat{\mathbf{N}}_n = \mathbf{N}(\mathbf{X}_n^*(0))/s. \tag{15}$$

Furthermore, since we are interested in the appropriate initial values to use, the operator $\mathfrak{L}^*$ need only be evaluated at time $t = 0$. The approximate procedure is outlined below.

First approximant:

$$\hat{\mathbf{X}}_1 = \mathbf{M}^{-1}\mathbf{X}_0 \tag{16}$$

$$\mathbf{X}_1^*(0) = \mathfrak{L}^*(\hat{\mathbf{X}}_1)\big|_{t=0} \tag{17}$$

Iterative procedure:

$$\hat{\mathbf{N}}_n = \mathbf{N}\{\mathbf{X}_n^*(0)\}/s \tag{18}$$

$$\hat{\mathbf{X}}_{n+1} = \mathbf{M}^{-1}[\mathbf{X}_n^*(0) - \hat{\mathbf{N}}_n] \tag{19}$$

$$\mathbf{X}_{n+1}^*(0) = \mathfrak{L}^*(\hat{\mathbf{X}}_{n+1})\big|_{t=0}\,. \tag{20}$$

The final iteration of Eq. (20) gives us the required initial conditions. In practice it is found that one or two iterations are sufficient to reduce the amplitude of the high frequency components to a negligible level. Note that the matrix **M** may be pre-calculated, inverted and stored for a set of values of s lying on $\mathbf{C}^*$; thus, the method is reasonably economical.

(d) *Relationship to normal mode initialization*

From the definition (4) of the modified inverse Laplace transform, it is clear that the first approximant in the above technique is equivalent to linear normal mode initialization. That is, application of (16) and (17) with an appropriate value of γ has the same effect as spectral resolution into normal modes followed by removal of all modes with frequency greater than γ. Note that if the equations have been linearized about a general mean flow $\bar{u} = \bar{u}(\phi, Z)$ its effects are felt by the linear normal modes. The eigenvalues may be complex and there may be a continuous spectrum in the case of a varying mean flow (Dikiy and Katayev 1971).

The full nonlinear technique yields a solution which evolves slowly for all time. The essential approximation in the practical implementation of the method is the assumption that the nonlinear terms evolve so slowly that they may be approximated by their initial values (see Eq. (15)). This is the same as the assumption made by Machenhauer (1977) (see his Eqs. (14) and (15)). It is thus clear that the method proposed here is closely related to that of Machenhauer. This relationship is examined in detail in the appendix.

3. Application to a one-dimensional model

(a) *Formulation of the method*

In order to test the feasibility of the Laplace transform method in the simplest context it has been used to initialize the data for a one-dimensional model. The model is similar to that used by Baer (1977) and a full description is given in Lynch (1984a). The basic equations are

$$\zeta_t + (u\zeta)_x + f\delta + \beta v = 0 \tag{21}$$

$$\delta_t + (u\delta)_x - f\zeta + \beta u' + \Phi'_{xx} = 0 \tag{22}$$

$$\Phi'_t + (u\Phi')_x - fuv + \Phi\delta = 0. \tag{23}$$

Here x is distance eastward, t is time, $u = \bar{u} + u'$ is the zonal velocity with $\bar{u}$ the constant mean wind speed, v the northward velocity, and $\Phi = \bar{\Phi} + \Phi'$ the geopotential. The vorticity and divergence are $\zeta = v_x$ and $\delta = u_x$. The Coriolis parameter f and its meridional derivative β are assumed constant.

Energy equations are derived in the usual manner (see e.g. Pedlosky 1979). The rate of change of eddy kinetic *plus* available potential energy is given by

$$\frac{d}{dt}\int \{\tfrac{1}{2}\rho(u'^2 + v^2)\Phi + \tfrac{1}{2}\rho\Phi'^2\}\,dx = -\int [\rho v\{\tfrac{1}{2}(u'^2 + v^2) + \Phi'\}\partial\bar{\Phi}/\partial y]\,dx. \tag{24}$$

Clearly, if the mean flow vanishes ($\bar{u} = \bar{\Phi}_y = 0$) the total eddy energy remains constant. The eddy kinetic energy can be split into contributions due to the rotational and divergent motions:

$$K = K_\psi + K_\chi; \qquad K_\psi = \int \tfrac{1}{2}\rho v^2\Phi\,dx; \qquad K_\chi = \int \tfrac{1}{2}\rho u'^2\Phi\,dx.$$

The values of these are calculated at each timestep and give valuable information about the dynamics of the motion being considered.

In order to clarify the relative magnitude of the various terms in the equations of motion it is convenient to nondimensionalize the equations by defining characteristic scales for length, time and velocity. It is also convenient numerically to have the principal terms of order unity. We introduce length and velocity scales L and V and scale time by f^{-1} and geopotential by fLV. We define some nondimensional combinations:

$$Ro = (V/fL); \quad R_\beta = (\beta L/f) \sim (L/a); \quad R_F = \bar{\Phi}/(fL)^2 = (L_R/L)^2.$$

Here Ro is the Rossby number; R_β is a measure of the importance of the β effect, determined by the scale of the motion; R_F is the reciprocal of the Froude number, and relates the length scale of the motion to the *Rossby radius of deformation*, $L_R = (\sqrt{\bar{\Phi}})/f$. The equations of motion, (21), (22) and (23), may now be written in non-dimensional form

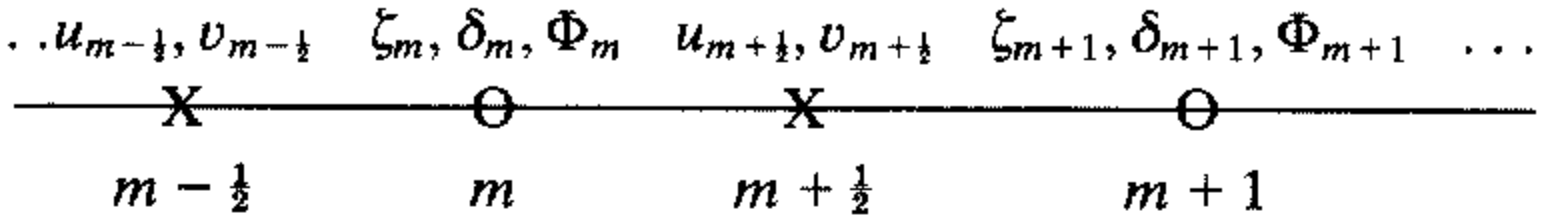
$$\zeta_t + Ro(u\zeta)_x + \delta + R_\beta v = 0 \tag{25}$$

$$\delta_t + Ro(u\delta)_x - \zeta + R_\beta u' + \Phi'_{xx} = 0 \tag{26}$$

$$\Phi'_t + Ro(u\Phi')_x - Ro\,u_0 v + R_F\delta = 0. \tag{27}$$

The relationship between the velocities (u, v) and the prognostic variables (ζ, δ) suggests that we specify them at alternate points of a grid staggered in space. The

velocities are specified at 'half-points' and the vorticity, divergence and perturbation geopotential at 'whole points':



Quantities not available directly are obtained by averaging. We define some finite difference operators in the usual way:

$$(q_m)_x = (q_{m+\frac{1}{2}} - q_{m-\frac{1}{2}})/\Delta x; \qquad (\overline{q}_m) = \tfrac{1}{2}(q_{m-\frac{1}{2}} + q_{m+\frac{1}{2}}).$$

A consistent (spatial) finite difference approximation to the equations may now be written as follows:

$$\partial(v_m)_x/\partial t + (u_m)_x + R_\beta(\overline{v}_m) = -Ro\{(u\overline{v}_x)_m\}_x \tag{28}$$

$$\partial(u_m)_x/\partial t - (v_m)_x + R_\beta(\overline{u}'_m) + (\Phi_m)_{xx} = -Ro\{(u\overline{u}_x)_m\}_x \tag{29}$$

$$\partial(\Phi_m)/\partial t - Ro\, u_0(\overline{v}_m) + R_F(u_m)_x = -Ro\{(u\overline{\Phi})_m\}_x. \tag{30}$$

All dependent quantities are assumed to have period $(N \cdot \Delta x)$. The state of the system at any time is completely defined by the vector

$$\mathbf{X} = (v_{\frac{1}{2}}, u_{\frac{1}{2}}, \Phi_1, \ldots, v_{m-\frac{1}{2}}, u_{m-\frac{1}{2}}, \Phi_m, \ldots, v_{N-\frac{1}{2}}, u_{N-\frac{1}{2}}, \Phi_N).$$

We assume in the following that the mean flow u_0 vanishes; this assumption is made to simplify the discussion and can easily be relaxed.

From the initial velocities and geopotential we assemble the vector

$$\Xi_0 = (\zeta_1^0, \delta_1^0, \Phi_1^0, \ldots, \zeta_m^0, \delta_m^0, \Phi_m^0, \ldots, \zeta_N^0, \delta_N^0, \Phi_N^0)$$

where $\zeta_m^0 = (v_{m+\frac{1}{2}}^0 - v_{m-\frac{1}{2}}^0)/\Delta x$ and similarly for δ_m^0. Eqs. (28), (29) and (30) are now Laplace-transformed to give

$$s(\hat{v}_m)_x + (\hat{u}_m)_x + R_\beta(\hat{\overline{v}}_m) = \zeta_m^0 - \mathfrak{L}[Ro\{(u\overline{v}_x)_m\}_x] \tag{31}$$

$$s(\hat{u}_m)_x - (\hat{v}_m)_x + R_\beta(\hat{\overline{u}}'_m) + \hat{\Phi}_{xx} = \delta_m^0 - \mathfrak{L}[Ro\{(u\overline{u}_x)_m\}_x] \tag{32}$$

$$s(\hat{\Phi}_m) + R_F(\hat{u}_m)_x = \Phi_m^0 - \mathfrak{L}[Ro\{(u\overline{\Phi})_m\}_x]. \tag{33}$$

(The Laplace transform of f is denoted by $\hat{f}$.) To cast this system into matrix form let us define some 3×3 matrices as follows:

$$\mathbf{D} = \begin{vmatrix} (-s/\Delta x + \frac{1}{2}R_\beta) & -1/\Delta x & 0 \\ 1/\Delta x & (-s/\Delta x + \frac{1}{2}R_\beta) & -2/\Delta x^2 \\ 0 & -R_F/\Delta x & s \end{vmatrix}$$

$$\mathbf{R} = \begin{vmatrix} (s/\Delta x + \frac{1}{2}R_\beta) & 1/\Delta x & 0 \\ -1/\Delta x & (s/\Delta x + \frac{1}{2}R_\beta) & 1/\Delta x^2 \\ 0 & R_F/\Delta x & 0 \end{vmatrix}$$

$$\mathbf{L} = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 1/\Delta x^2 \\ 0 & 0 & 0 \end{vmatrix}.$$

We assemble these into the periodic, block tri-diagonal matrix of order $3N$:

$$\mathbf{M} = \begin{vmatrix} \mathbf{D}\,\mathbf{R}\,\mathbf{0} & \ldots & \mathbf{L} \\ \mathbf{L}\,\mathbf{D}\,\mathbf{R} & \ldots & \mathbf{0} \\ \mathbf{0}\,\mathbf{L}\,\mathbf{D} & \ldots & \mathbf{0} \\ \vdots\ \vdots\ \vdots & & \vdots \\ \mathbf{R}\,\mathbf{0}\ \mathbf{0} & \ldots & \mathbf{D} \end{vmatrix}.$$

The system of equations (31)–(33) may now be written in the form

$$\mathbf{M}\hat{\mathbf{X}} = \boldsymbol{\Xi}_0 - \hat{\mathbf{N}}(\mathbf{X})\,. \tag{34}$$

Here the nonlinear terms have been collected into $\hat{\mathbf{N}}$, which is a transformed nonlinear vector function of the state vector $\mathbf{X}$. Equation (34) is formally identical to Eq. (6) in section 2 and the method of initialization described there can now be applied to the present model.

(*b*) *Computational results*

A number of model runs were carried out, with various parameter values and initial conditions. For all runs described here the nondimensional numbers had the following values:

$$Ro = 10^{-1}; \qquad R_\beta = 1{\cdot}6\times10^{-1}; \qquad R_F = 10\,.$$

The channel length was $L = 10^7$ m., where $N_x\Delta x = L$ with $N_x = 20$ and $\Delta x = 500$ km. *One day* forecasts were of duration $N_T\Delta t = 10^5$ s (=27·8 hours) where $N_T = 1000$ and $\Delta t = 100$ s. An Adams–Bashforth timestepping scheme was used. For the given parameters the maximum Rossby wave frequency and minimum gravity wave frequency can be calculated, and have the (nondimensional) values

$$|\nu_{\mathrm{R}}|_{\max} = 0{\cdot}203; \qquad |\nu_{\mathrm{G}}|_{\min} = 2{\cdot}224\,.$$

Any value of γ lying between these values should serve to separate the timescales. The value $\gamma = 1$ was chosen; the matrices $\mathbf{M}(s)$ were calculated for the centre points (s_k) of the sides (Δs_k) of a polygon inscribed in the unit circle and the integral in (4) was approximated by a sum:

$$\oint_{\mathrm{C}^*} \hat{f}(s)\,ds \doteq \sum_{k=1}^{K} \hat{f}(s_k)\cdot\Delta s_k\,.$$

K was set to 24, although $K = 12$ was found to give virtually identical results.

The initial conditions were defined by setting

$$\Phi_m^0 = \sum_{l=1} \cos\{(2\pi lm\Delta x/L) + \phi_l\}$$

where the phases ϕ_l were chosen randomly, and deriving the geostrophic winds

$$v_{m-\frac{1}{2}}^0 = (\Phi_m^0 - \Phi_{m-1}^0)/\Delta x; \qquad u_{m-\frac{1}{2}}^0 = 0\,.$$

We consider three different cases of initialization as follows:

NIL: No initialization	geostrophic ICs as above
LIN: Linear initialization	ICs from (17)
NLI: Nonlinear initialization	ICs from (20), one iteration.

The final values of the v and Φ fields were very similar in all three cases. However, without initialization (NIL) the time evolution at a central point had small, high frequency fluctuations superimposed on the slow development. With linear initialization (LIN) these disappeared. The evolution of Φ is shown in Fig. 2(a) for the two cases. The zonal wind field evolved noisily before, but smoothly after, initialization (Fig. 2(b)). The divergence field behaved in a similar manner (not shown).

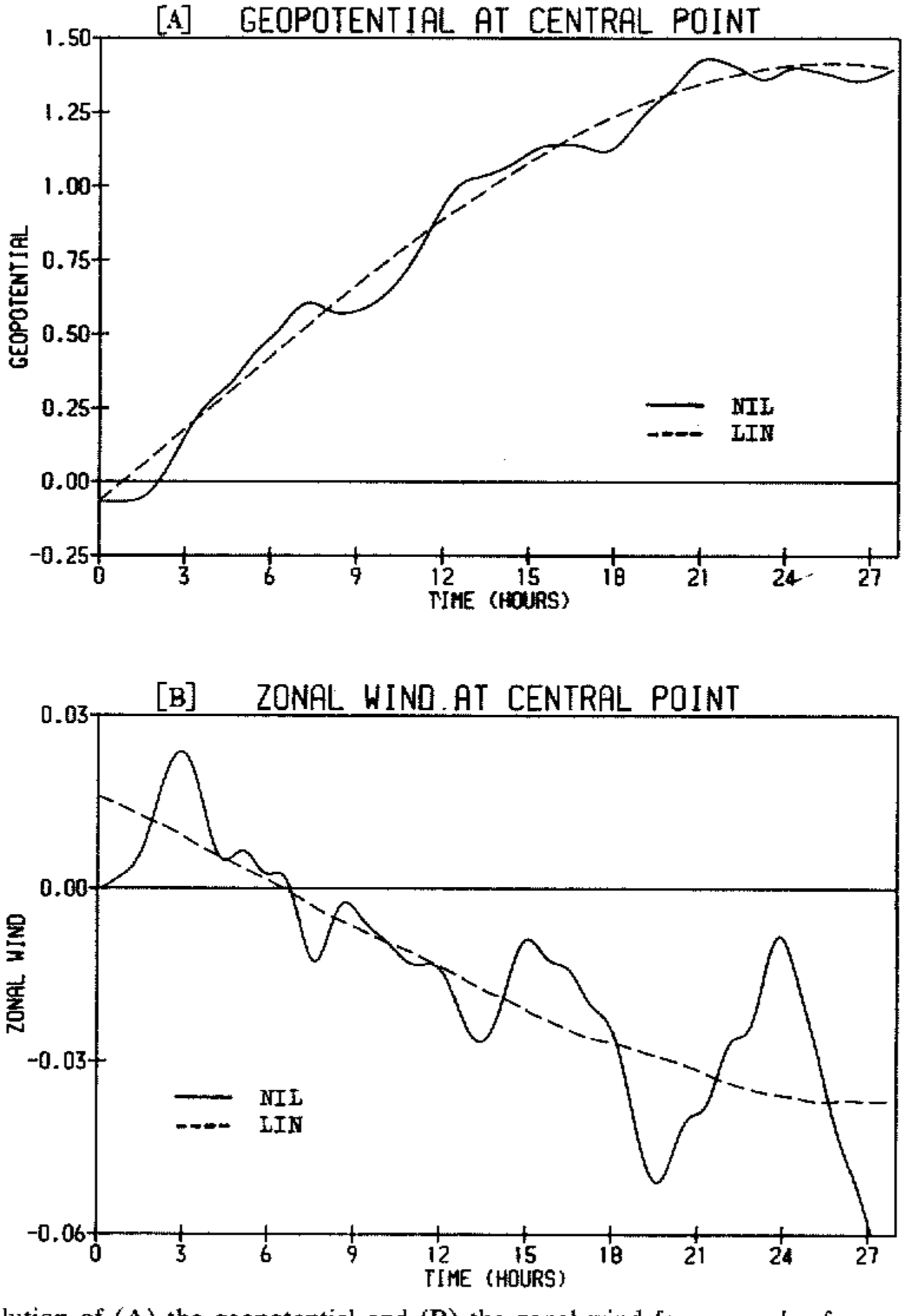


Figure 2. Evolution of (A) the geopotential and (B) the zonal wind for a *one-day* forecast starting from geostrophic initial winds (NIL, solid) and linearly initialized fields (LIN, dashed).

Since the gravity waves have relatively large divergence, we would expect the divergent kinetic energy to be a good indicator of their presence (Baer 1977). In Fig. 3(a) the divergent energy, K_χ, as a function of time is shown for the case of geostrophic initial winds (NIL) and linear initialization (LIN). The oscillations in K_χ are extremely large before initialization; the linear initialization reduces them dramatically, but does not remove them completely. The linear case is reproduced in Fig. 3(b) with a much

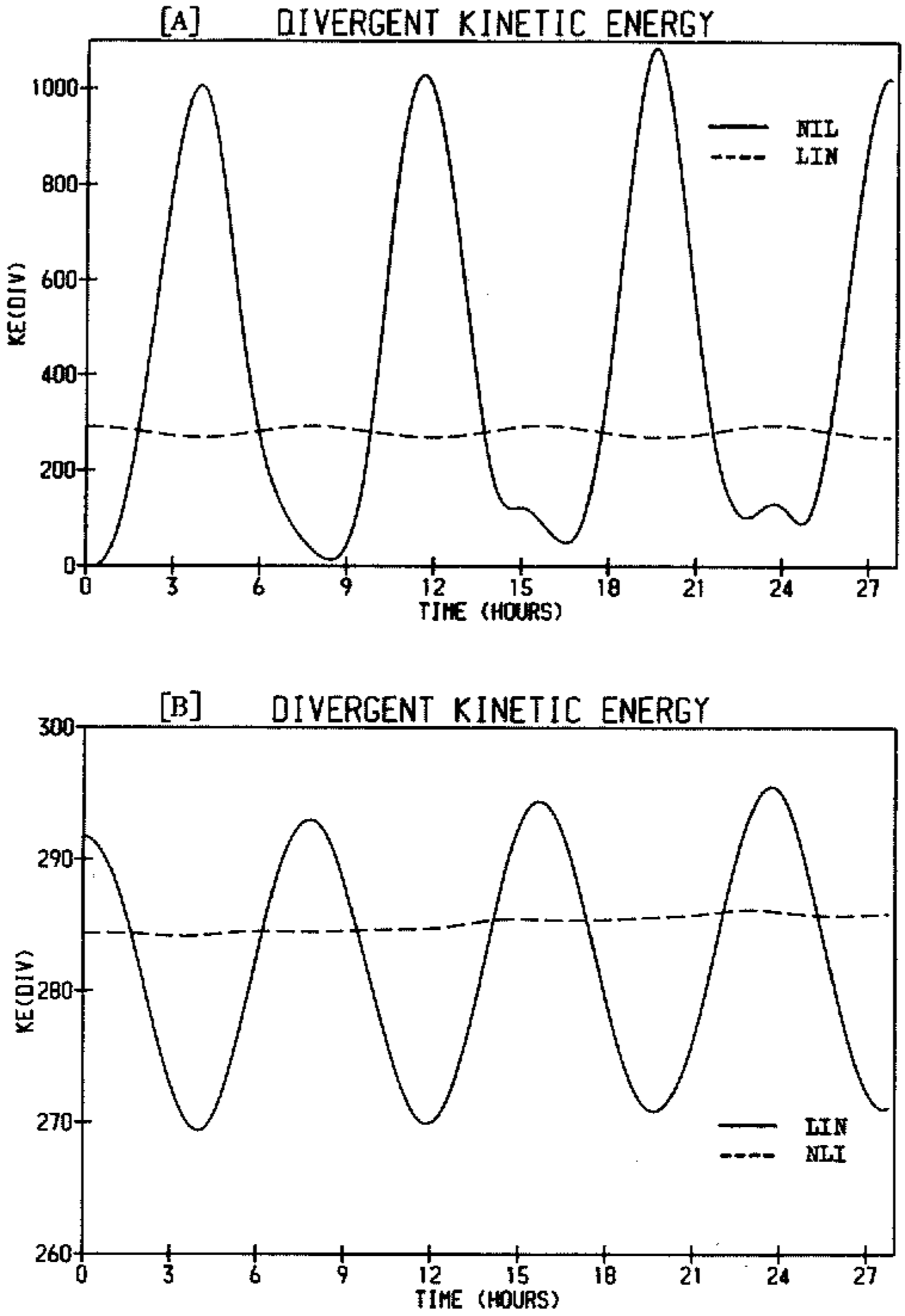


Figure 3. Plot of the divergent kinetic energy v. time starting from: (A) geostrophic initial winds (NIL, solid) and linearly initialized fields (LIN, dashed); (B) linearly initialized fields (LIN, solid) and nonlinearly initialized fields, one nonlinear iteration (NLI, dashed).

expanded vertical scale. The gravity oscillations remaining after linear initialization (LIN) are clear to see. Also shown is the evolution of K_χ, on the same scale, after a single nonlinear iteration of the initialization (NLI). The gravity wave noise is almost completely eliminated. This result is in full agreement with the results of Baer for his nonlinear normal mode method, and demonstrates the effectiveness of the present method in dealing with the problem of high frequency oscillations.

Several other model runs confirmed the efficacy of the method in controlling the gravity wave noise. A number of *ten day* (10^6 s) forecasts showed that there is no tendency for the noise to return during this period.

In all cases the convergence was very rapid, with little change after the first nonlinear iteration. A small residue of noise is not removed by further iterations; presumably, this

residue is associated with the approximation (15). Modification of (15) to allow $\mathbf{N}(\mathbf{X}_n^*)$ to vary linearly in time did not lead to any significant improvement. However, the residue is too small to cause any practical problems.

4. Principles of application to a general forecasting model

The application of the Laplace transform technique to a one-dimensional model was straightforward. For a general baroclinic model the state vector might contain, typically, about ten thousand elements. The inversion of the resulting gigantic matrices $\mathbf{M}(s)$ is an almost impossible task on present-day computers. Therefore, the problem must be formulated so as to produce matrices of manageable size.

Let us assume that initial fields are required for a limited area model with vertical coordinate $\sigma = p/p_s$, where p_s is the surface pressure. The vertical structure can be separated out if we linearize about a motionless state with mean temperature $\overline{T} = \overline{T}(\sigma)$. To perform the separation we introduce a new dependent variable, $P = \Phi' + R\overline{T} \ln p_s$, where Φ' is the perturbation geopotential (see e.g. Kasahara and Puri 1981). If the model has K levels, the vertical structure equation yields K eigenvalues, or equivalent depths, and K corresponding eigenfunctions, $\Psi_r(\sigma_k)$. For each equivalent depth, the horizontal structure is governed by a set of three equations for u, v and P which are of the form of the Laplace tidal equations.

For global and many limited area models the horizontal structure is separable into dependencies upon longitude λ and latitude ϕ. However, for a transformed longitude/latitude grid, whose poles are removed from the earth's axis, the Coriolis parameter is a function of both horizontal variables, x and y, and separation of variables no longer obtains. We consider a method of applying the Laplace transform initialization technique in this case.

The Laplace transform of the horizontal equations, for a given equivalent depth, D, may be written:

$$\left.\begin{aligned} s\hat{u} - f\hat{v} + \hat{P}_x &= u_0 \\ s\hat{v} + f\hat{u} + \hat{P}_y &= v_0 \\ \varepsilon^{-1}\nabla.\hat{\mathbf{V}} + s\hat{P} &= P_0 \end{aligned}\right\} \tag{35}$$

(where $\varepsilon = (2\Omega a)^2/gD$). Let the domain be discretized into $I \times J$ points and derivatives approximated by finite differences in the usual way. For simplicity, we consider a non-staggered grid where all prognostic variables are specified at all gridpoints. The values of $\hat{u}$, $\hat{v}$ and $\hat{P}$ on a single row (excluding end-points) are assembled in a vector

$$\hat{\mathbf{X}}_j = (\hat{u}_{2j}, \hat{v}_{2j}, \hat{P}_{2j}, \ldots, \hat{u}_{ij}, \hat{v}_{ij}, \hat{P}_{ij}, \ldots, \hat{u}_{I-1,j}, \hat{v}_{I-1,j}, \hat{P}_{I-1,j})$$

where $\hat{u}_{ij} = \hat{u}(x_i, y_j)$, etc. The system (35) may now be written formally as

$$\mathbf{A}_j\hat{\mathbf{X}}_{j-1} + \mathbf{B}_j\hat{\mathbf{X}}_j + \mathbf{C}_j\hat{\mathbf{X}}_{j+1} = \mathbf{D}_j \tag{36}$$

where $\mathbf{A}_j$, $\mathbf{B}_j$, $\mathbf{C}_j$ and $\mathbf{D}_j$ are block tridiagonal matrices and a column vector whose elements depend upon the values of the coefficients of the equation. The lateral boundary values $\hat{u}_{1j}$, $\hat{u}_{Ij}$, etc. occur in the vector $\mathbf{D}_j$.

Lindzen and Kuo (1969) have described a reliable direct method for solving systems of the form (36). We introduce a set of intermediate matrices and vectors, $\boldsymbol{\alpha}_j$ and $\boldsymbol{\beta}_j$, such that

$$\hat{\mathbf{X}}_j = \boldsymbol{\alpha}_j\hat{\mathbf{X}}_{j+1} + \boldsymbol{\beta}_j\,. \tag{37}$$

When (37) is used to eliminate $\hat{\mathbf{X}}_{j-1}$ from (36) a solution for $\boldsymbol{\alpha}_j$ and $\boldsymbol{\beta}_j$ is apparent in the form

$$\begin{aligned} \boldsymbol{\alpha}_j &= -(\mathbf{M}_j)^{-1}\mathbf{C}_j \\ \boldsymbol{\beta}_j &= -(\mathbf{M}_j)^{-1}(\mathbf{A}_j\,\boldsymbol{\beta}_{j-1} - \mathbf{D}_j) \end{aligned} \tag{38}$$

where $\mathbf{M}_j = (\mathbf{A}_j\,\boldsymbol{\alpha}_{j-1} + \mathbf{B}_j)$. The boundary conditions for $j = 1$ are used to obtain $\boldsymbol{\alpha}_1$, $\boldsymbol{\beta}_1$ and from (38) we then get $\boldsymbol{\alpha}_j$, $\boldsymbol{\beta}_j$. The boundary conditions at $j = J$ give us $\hat{\mathbf{X}}_J$ and the solution $\hat{\mathbf{X}}_j$ is obtained from (37). The crucial point is that the matrices $\mathbf{M}_j$ are independent of the boundary values and of the forcing terms; therefore, they can be inverted *once for all* and stored.

The first step in the initialization is now performed. The operator $\mathfrak{L}^*$, with a suitably chosen value of γ, is applied to $\hat{\mathbf{X}}_j$ and the new initial values are obtained by inverting the vertical separation of variables

$$\{\mathbf{X}^*(0)\}_{ijk} = \sum_{r=1}^{K} \mathfrak{L}^*(\hat{\mathbf{X}}_j)_i|_{t=0}\,\boldsymbol{\Psi}_r(\sigma_k)\,.$$

For linear initialization these are the required starting values.

The nonlinear terms can be calculated using the new values. Usually, this is equivalent to performing a single model timestep. We then form the combination

$$[\mathbf{X}^*(0) - \mathbf{N}\{\mathbf{X}^*(0)\}/s]$$

and perform the vertical separation of variables again (assuming for simplicity that the eigenfunctions $\boldsymbol{\Psi}_k$ are orthogonal (see Kasahara and Shigehisa 1983)). We can now calculate updated values for the vector $\mathbf{D}_j$ in (36). The system (36) is solved as before and the cycle repeated for the required number of iterations. Note that the matrices $\mathbf{M}_j$ do not change, so no further matrix inversion is required.

Since the operator $\mathfrak{L}^*$ is applied separately for each vertical mode a different value of γ (the cutoff frequency) can be chosen in each case. Furthermore, not all the vertical modes need be initialized.

Although the above method seems to be computationally feasible, there may be heavy storage requirements for the matrices $\mathbf{M}_j$. They are of order $3(I-2)$ and there are J of them for each of K vertical modes. Thus, if S values of s are chosen on the inversion contour $\mathbf{C}^*$ the total storage is about $9I^2JKS$. Assuming $S = I$ this is of the same order as the requirement to store the normal modes when the horizontal structure is nonseparable. Separability reduces the requirement by a factor I.

5. Concluding remarks

The new method of initialization described above has been shown to be very effective in controlling the gravity wave noise in a one-dimensional model. The methodology for applying the method to a more general model has been outlined. The method is closely related to the nonlinear normal mode method of initialization but has the advantage that it does not require knowledge of the model linear normal modes. This allows it to be applied in cases where the normal mode approach is difficult to use. The method allows for incorporation of orography and model physics in the same way as the normal mode method. The boundary conditions, which are included in the forcing terms, may be specified in a completely general way.

The Laplace transform technique is currently being applied to the initialization of an operational limited area baroclinic forecasting model. Preliminary results with a one-level version of the model (Lynch 1984b) are encouraging, and suggest that the method may provide a satisfactory answer to the problem of initialization for limited area models.

APPENDIX

The relationship between the Laplace transform technique and nonlinear normal mode initialization

Suppose that the linear normal modes of the system governed by Eq. (5) are known and that they span the space $\mathfrak{X}$. Then $\mathbf{X}$ may be expressed as a sum of these modes and $\mathbf{L}$ becomes a diagonal matrix $\mathbf{\Lambda}$:

$$\mathbf{\Lambda} = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3, \ldots) = (\mathbf{\Lambda}^*, \mathbf{\Lambda}^\dagger)$$

where the eigenfrequencies are arranged in order of ascending absolute value and split into slow modes ($\mathbf{\Lambda}^*$) and fast modes ($\mathbf{\Lambda}^\dagger$). We can define projection operators onto the slow and fast subspaces of $\mathfrak{X}$ (see e.g. Halmos 1958):

$$\mathfrak{X} = \mathbf{P}^*\mathfrak{X} \oplus \mathbf{P}^\dagger\mathfrak{X} = \mathfrak{X}^* \oplus \mathfrak{X}^\dagger$$

$$\mathbf{P}^* = \mathfrak{Q}^*\mathfrak{Q}; \qquad \mathbf{P}^\dagger = \mathfrak{Q}^\dagger\mathfrak{Q} = (\mathfrak{Q}^{-1} - \mathfrak{Q}^*)\,\mathfrak{Q}; \qquad \mathbf{P}^* + \mathbf{P}^\dagger = \mathbf{I}$$

and separate the solution into slow and fast parts:

$$\mathbf{X} = \mathbf{P}^*\mathbf{X} + \mathbf{P}^\dagger\mathbf{X} = \begin{vmatrix} \mathbf{X}^* \\ \mathbf{X}^\dagger \end{vmatrix}.$$

If the nonlinear terms are decomposed in the same way

$$\mathbf{N}(\mathbf{X}) = \mathbf{P}^*\mathbf{N} + \mathbf{P}^\dagger\mathbf{N} = \begin{vmatrix} \mathbf{N}^*(\mathbf{X}^*, \mathbf{X}^\dagger) \\ \mathbf{N}^\dagger(\mathbf{X}^*, \mathbf{X}^\dagger) \end{vmatrix}$$

we can separate the system (5) into slow and fast parts:

$$\begin{aligned} \dot{\mathbf{X}}^* + \mathbf{\Lambda}^*\mathbf{X}^* + \mathbf{N}^* &= \mathbf{0} \\ \dot{\mathbf{X}}^\dagger + \mathbf{\Lambda}^\dagger\mathbf{X}^\dagger + \mathbf{N}^\dagger &= \mathbf{0}\,. \end{aligned} \tag{A1}$$

The initialization is iterated by using Eq. (19), which here becomes

$$\hat{\mathbf{X}}_{n+1} = (s\mathbf{I} + \mathbf{\Lambda})^{-1}[\mathbf{X}_n^*(0) - \mathbf{N}_n/s]\,. \tag{A2}$$

Using partial fractions to split the diagonal elements we can write the matrix inversion:

$$(s\mathbf{I} + \mathbf{\Lambda})^{-1}/s = -(s\mathbf{I} + \mathbf{\Lambda})^{-1}\mathbf{\Lambda}^{-1} + \mathbf{\Lambda}^{-1}/s\,. \tag{A3}$$

Equation (A2) can then be written in the form

$$\hat{\mathbf{X}}_{n+1} = (s\mathbf{I} + \mathbf{\Lambda})^{-1}[\mathbf{X}_n^*(0) + \mathbf{\Lambda}^{-1}\mathbf{N}_n] - \mathbf{\Lambda}^{-1}\mathbf{N}_n/s\,. \tag{A4}$$

To get the new initial conditions we apply $\mathfrak{Q}^*$ (at $t = 0$) to (A4). Since the matrices are diagonal, the components can be considered individually. For slow modes $|\lambda| < \gamma$, the cutoff frequency, and all the terms in (A4) contribute to the solution. After cancellation we find

$$\mathbf{X}_{n+1}^*(0) = \mathbf{X}_n^*(0)\,. \tag{A5}$$

Thus, the coefficients of the slow modes remain unchanged. For the fast modes $|\lambda| > \gamma$ and the terms in (A4) involving $(s\mathbf{I} + \mathbf{\Lambda})^{-1}$ contribute nothing to the integral $\mathfrak{L}^*$. Thus, we have

$$\mathbf{X}^{\dagger}_{n+1}(0) = -(\mathbf{\Lambda}^{\dagger})^{-1}\mathbf{N}^{\dagger}_{n}. \tag{A6}$$

This is precisely the iterative solution of the equation

$$\mathbf{\Lambda}^{\dagger}\mathbf{X}^{\dagger} + \mathbf{N}^{\dagger}(\mathbf{X}^*, \mathbf{X}^{\dagger}) = \mathbf{0}$$

which is obtained by setting $\dot{\mathbf{X}}^{\dagger} = \mathbf{0}$ in (A1). Equations (A5) and (A6) show the equivalence between the Laplace transform technique and the nonlinear normal mode method as formulated by Machenhauer (1977).

The relationship between the two initialization methods may be illustrated by a simple example: consider a system with only one slow mode X^* and one fast mode $X^{\dagger}$ satisfying the equations

$$\left.\begin{aligned} \dot{X}^* + \mathrm{i}\omega^* X^* + N^*(X^*, X^{\dagger}) &= 0 \\ \dot{X}^{\dagger} + \mathrm{i}\omega^{\dagger} X^{\dagger} + N^{\dagger}(X^*, X^{\dagger}) &= 0 \end{aligned}\right\} \tag{A7}$$

where $|\omega^{\dagger}| \geqslant |\omega^*|$. In Machenhauer's scheme the tendency of the fast mode, $\dot{X}^{\dagger}$, is set to zero, resulting in the iteration process

$$\left.\begin{aligned} X^*_{n+1} &= X^*_0 \\ X^{\dagger}_{n+1} &= -N^{\dagger}(X^*_0, X^{\dagger}_n)/\mathrm{i}\omega^{\dagger}. \end{aligned}\right\} \tag{A8}$$

If the nonlinear terms are assumed constant, the Laplace transform of (A7) may be written

$$\hat{X}^* = (s + \mathrm{i}\omega^*)^{-1}[X^*_0 - N^*(X^*, X^{\dagger})/s]$$

$$\hat{X}^{\dagger} = (s + \mathrm{i}\omega^{\dagger})^{-1}[X^{\dagger}_0 - N^{\dagger}(X^*, X^{\dagger})/s]$$

where X^*_0 and $X^{\dagger}_0$ are the initial values. Using partial fractions these may be rewritten as

$$\hat{X}^* = [X^*_0 + N^*(X^*, X^{\dagger})/\mathrm{i}\omega^*]/(s + \mathrm{i}\omega^*) - [N^*(X^*, X^{\dagger})/\mathrm{i}\omega^*]/s$$

$$\hat{X}^{\dagger} = [X^{\dagger}_0 + N^{\dagger}(X^*, X^{\dagger})/\mathrm{i}\omega^{\dagger}]/(s + \mathrm{i}\omega^{\dagger}) - [N^{\dagger}(X^*, X^{\dagger})/\mathrm{i}\omega^{\dagger}]/s.$$

Applying the operator $\mathfrak{L}^*$ iteratively (with $|\omega^*| < \gamma < |\omega^{\dagger}|$) yields

$$X^*_{n+1} = X^*_0$$

$$X^{\dagger}_{n+1} = -N^{\dagger}(X^*_0, X^{\dagger}_n)/\mathrm{i}\omega^{\dagger}$$

which are identical to the relations (A8).

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