The High-power Hypar *

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The capacity of mathematics to provide general, unifying structures is one of its most powerful characteristics. Maths frequently shows us surprising and illuminating connections between physical systems that are not obviously related: the analysis of one system often turns out to be ideally suited for describing another.

To illustrate this, we will show how a surface in three dimensional space — the hyperbolic paraboloid, or *hypar* — pops up in unexpected ways and in many different contexts. In the process we find unexpected connections between architecture, tennis balls, weather forecasting and the snack food called Pringles.

Curves and surfaces

In two dimensions a point (x, y) is given by the two coordinates x and y. Each is free to vary independently; we say the point has two degrees of freedom. If we now specify an equation $f(x, y) = 0$, the dimension is usually reduced by one if the equation has solutions: instead of the whole plane, we have a one-dimensional subset, a curve. In the special case where the equation is linear, the curve is a straight line.

Moving up a notch, a point (x, y, z) in 3-space is given by three coordinates x, y and z. If we specify an equation $g(x, y, z) = 0$, the point is confined to a two-dimensional surface. In the special case of a linear equation, that surface is a plane in 3-space.

To describe a *curve* in 3-space, we need to reduce the dimension once more, by giving a second equation, $h(x, y, z) = 0$. If both equations are linear, they describe two planes, whose intersection is typically a straight line. More generally, they are nonlinear, and provided they intersect they describe a one-dimensional curve.

The most ancient and best-understood curves are the conic sections, the ellipse, parabola and hyperbola, arising from the intersection of a plane and a cone. Intensively studied since ancient times, they apply to an enormous range of physical systems, ranging from radar scanners to planetary orbits.

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*A more technical version of this article has been filed on arxiv.org [1]

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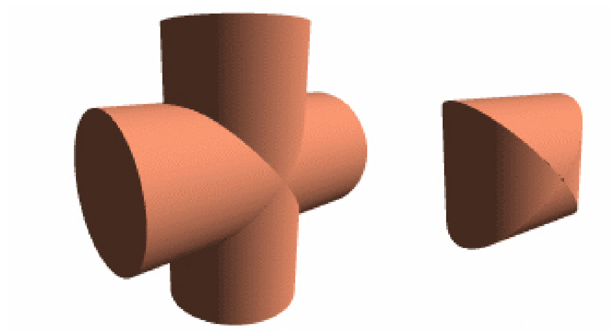

Figure 1: The intersection of two perpendicular cylinders consists of two ellipses (left). The volume of the intersection is called a bicylinder (right). [**Image needs to be redrawn**]

Curves in space

All the examples described so far are flat: the curves lie in a plane. But if both intersecting surfaces are nonlinear then the intersecting curve can twist around in 3-space like a roller-coaster. Let's consider the case of two cylinders, each with a circular cross-section, whose axes are at right angles and intersect in a point.

If the equations for the two cylinders are added, we obtain the equation of an *oblate spheroid*, a sphere flattened like an orange. If they are subtracted, we obtain an equation representing two planes. The actual intersection of the cylindrical surfaces comprises two ellipses. This may seem abstract, or even abstruse, but two perpendicular barrel vaults in a classical building intersect in this way, and we can clearly see the elliptical curves in the resulting groin vault.

The volume common to two cylinders of equal radii with orthogonal intersecting axes, called a *bicylinder*, was known to Archimedes, and also to the Chinese mathematician Tsu Ch'ung-Chih. In the fifth century, Tsu Ch'ung-Chih used it to calculate the volume of a sphere.

Now let's flatten the two cylinders so that they have elliptical cross sections and displace them along the axis that is orthogonal to both of them, in opposite directions. Their equations are

$$2y^2 + (z+d)^2 = R^2 \qquad\qquad 2x^2 + (z-d)^2 = R^2 \,.$$

[**What are R and d?**] Adding and subtracting these two equations, we get

$$x^2 + y^2 + z^2 = a^2 \qquad\qquad x^2 - y^2 = 2dz \,.$$

These are the equation for a sphere of radius a (where $a^2 = R^2 - d^2$), and another surface called a *hyperbolic paraboloid*. For brevity, let's call this a *hypar*.

In addition to its use in classical buildings, the hypar has proved useful in modern architecture. The advent of shell construction in the 20th century and the mathematical theory of surfaces allowed very thin, strong vaults to be constructed using the hypar form.

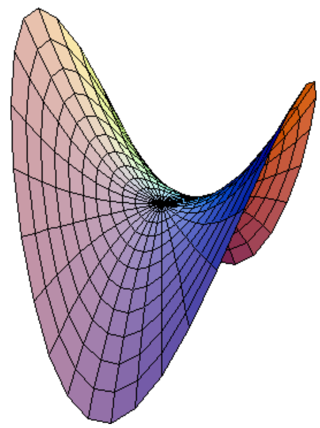

Figure 2: The hypar. **[Image needs to be redrawn. This one is from http://upload.wikimedia.org/wikipedia/commons/4/4a/HyperbolicParaboloid.png]**

Because it is a 'ruled surface', generated by straight lines, saddle-shaped roofs of this form are easily constructed from straight sections.

In figure 3, we plot the curve determined by the intersection of the sphere and hypar. It resembles the seam on a baseball or the groove on a tennis ball. The hyperbolic paraboloid is also the shape of the snack food called pringles and the edge of a pringle is like the tennis ball curve.

From tennis balls to weather forecasts

The groove on a tennis ball curve is not defined explicitly, and may be approximated in many ways. The official rules of the game are not much help, stating only that: 'The ball shall have a uniform outer surface of a fabric cover and shall be white or yellow in colour. If there are any seams, they shall be stitchless' (ITF Rules of Tennis 2012). The challenge is to construct a cover for the spherical ball from two flat pieces of felt. The great mathematician Carl Friedrich Gauss showed that to do this exactly is impossible: there is no exact mapping from a plane to a sphere. But, in practice, the felt flats are shaped like peanuts and, with a little stretching, fit snugly on the ball.

Many models of the tennis ball curve have been proposed. Indeed, the ingenious and versatile mathematician John Conway formulated a conjecture: No two definitions of 'the correct curve' will give the same answer unless their equivalence is obvious from the start [**What does that mean?**]. Indeed, the curve found on tennis balls is well approximated by a combination of four circular arcs. While this solution may appeal to engineers, it is unattractive to mathematicians, for the composite curve does not have nice analytical properties. The curve formed from the intersection of the sphere and hypar has an elegant mathematical equation [**So has the hypar been used on tennis balls?**].

The tennis ball curve arises from the practical need to cover the ball with flat felt. But the resulting partition of the sphere turns out to have another very practical use. In

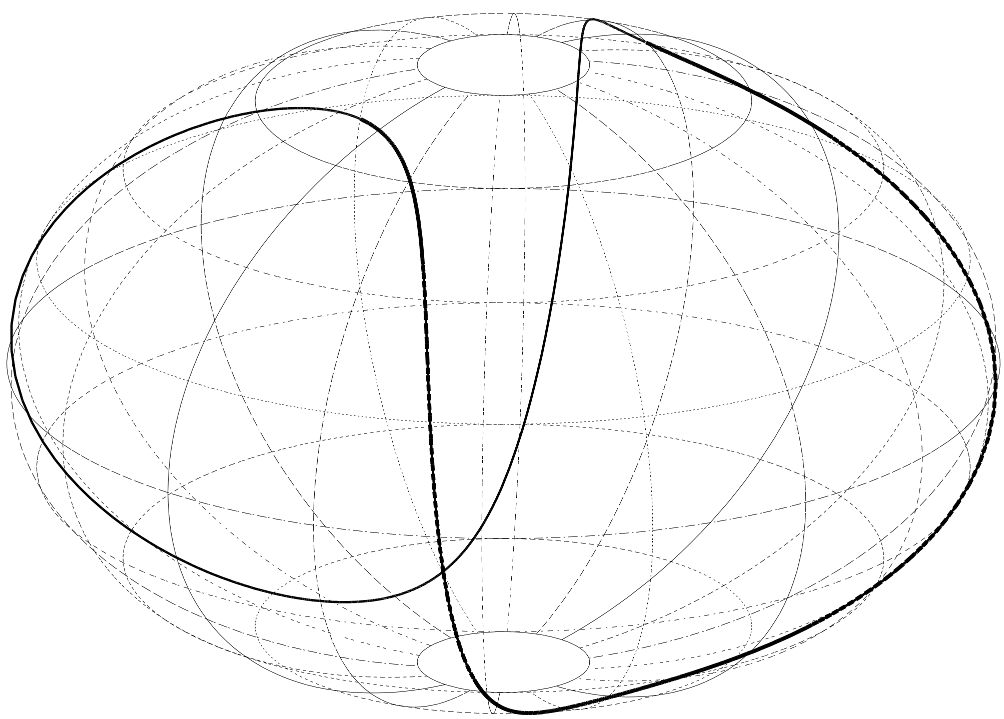

Figure 3: The Tennis Ball curve, the intersection of two offset elliptic cylinders, and also of a sphere and hyperbolic paraboloid or hypar.

weather forecasting, we have to represent the atmosphere using a grid of points that cover the globe. The usual geographical latitude and longitude coordinates cause big problems: the meridians converge towards the poles, so the coverage with a latitude/longitude grid is highly non-uniform. By dividing the sphere into two parts by means of the tennis ball curve, and using a separate grid on each part, we avoid the difficulties. This solution is called the Yin-Yang Grid, as it is reminiscent of the ancient Chinese symbol of that name.

Further Reading

[1] Lynch, Peter (2013). Another parameterization of the baseball and tennis-ball curves. [`arxiv.org`. Full ref. to follow].

[2] Kiang, Tao (1972). An old Chinese way of finding the volume of a sphere. *Math. Gazette*, **56**, 88–90. Reprinted in *The Changing Shape of Geometry*, Ed. C. Pritchard, Camb. Univ. Press (2003).

[3] Banks, Robert B (1999). *Slicing Pizzas, Racing Turtles, and Further Adventures in Applied Mathematics.* Princeton Univ. Press, 286pp.