Magnums: Counting Sets with Surreals

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Abstract

How many odd numbers are there? How many even numbers? From Galileo to Cantor, the suggestion was that there are the same number of odd, even and natural numbers, because all three sets can be mapped in one-one fashion to each other. This jars with our intuition.

The class of *surreal numbers* $\mathbb{S}$ is the largest possible ordered field. All the basic arithmetical operations are defined, and sensible arithmetic can be carried out over $\mathbb{S}$. Using the surreals, we define the "magnum" for subsets of the natural numbers. The magnum of a proper subset of a set is strictly less than the magnum of the set itself.

1 Introduction

Cardinality is a *blunt instrument*: the natural numbers, rationals and algebraic numbers all have the same cardinality, so $\aleph_0$ fails to discriminate between them. Our objective is to define a number $m(A)$ for a subset A of the natural numbers that corresponds to our intuition about the size or magnitude of A.

The class of surreal numbers, denoted by **No** or $\mathbb{S}$, discovered by J. H. Conway around 1970, is the largest possible ordered field, with all the basic arithmetical operations, and sensible arithmetic can be carried out over $\mathbb{S}$. Moreover, quantities like $\omega/2$ and $\sqrt{\omega}$ are meaningful (where ω is the "first-born" infinite surreal number). Surreal numbers have attracted relatively little attention, mostly from recreational mathematicians. Given their appealing properties and their remarkable elegance, this is surprising.

Using the class of surreal numbers, we define the *magnum* for subsets of the natural numbers $\mathbb{N}$. If A is a finite set, then $m(A)$ is just the cardinality of A. The magnum of a proper subset of a set is strictly less than the magnum of the set itself.

There are difficulties evaluating limits in $\mathbb{S}$. However, the extension of many elementary functions from domain $\mathbb{R}$ to domain $\mathbb{S}$ can be done without difficulty. We know that for the real numbers, $0.999\cdots = 1$. For the surreals, this is not the case; $0.999\ldots = 1 - 10^{-\varpi} < 1$, where $\varpi = m(\mathbb{N})$.

We define the counting sequence and density for a set A and the magnum $m(A)$, a surreal number. We then find that $m(\text{Odd}) = m(\text{Even}) = \frac{1}{2}m(\mathbb{N}) = \varpi/2$. This is consistent with our intuition about the relative sizes of the sets. We shall find that $m(A)$ cannot always be defined, but it is possible for many interesting sets $A \subset \mathbb{N}$.

2 Background

The natural numbers, rationals and algebraic numbers all have the same cardinality, so, $\aleph_0$ fails to discriminate between them:

$$\mathrm{card}(\mathbb{N}) = \mathrm{card}(\mathbb{Q}) = \mathrm{card}(\mathbb{A}) = \aleph_0$$

Every set has a cardinal number. Every well-ordered set has an ordinal number. For infinite sets, there are many possible orderings, resulting in different ordinal numbers. For example

$$\mathrm{ord}(\{1, 2, 3, 4, \dots\}) = \omega \qquad \text{while} \qquad \mathrm{ord}(\{2, 3, 4, \dots, 1\}) = \omega + 1$$

but the two sets have precisely the same elements. The ordinals have the unattractive property of being non-commutative:

$$1 + \omega \neq \omega + 1$$

Worse still, $1 + \omega = \omega$ and one is tempted to subtract ω to get the nonsensical result $1 = 0$. This is not a good basis for a calculus of transfinite numbers.

The class of surreal numbers, denoted by **No** or $\mathbb{S}$, discovered by Conway around 1970 is the largest possible ordered field (Conway, 2001, referenced below as [ONAG]). The basic arithmetical operations — addition, subtraction, multiplication, division and roots — are defined in $\mathbb{S}$. The surreal sum is commutative:

$$1 + \omega = \omega + 1 > \omega$$

and sensible arithmetic can be carried out over $\mathbb{S}$. Moreover, quantities like $\omega/2$ and $\sqrt{\omega}$ are meaningful. *We shall use surreals to quantify the magnitude of subsets of* $\mathbb{N}$.

3 Desiderata

We should like to assign a number to a subset A of $\mathbb{N}$ that indicates a "size" of the set that accords with intuition. This is not always possible, but it can be done for many interesting subsets of $\mathbb{N}$. Let us denote the new measuring number of a set A by $m(A)$. We will call it the "magnitude number" or *magnum* of A.

We list here some desirable properties of the magnum function $m(A)$.

- For a finite subset A we require $m(A) = \mathrm{card}(A)$
- For the complete set of natural numbers, $m(\mathbb{N}) = \varpi$
- There should be a linear ordering of the m-numbers.
- A proper subset A of B should have a smaller m-number than B:

$$A \subsetneqq B \implies m(A) < m(B)\,.$$

- Removal of an element $a \in A$ from a set should reduce the magnum by one:

$$m(A \setminus \{a\}) = m(A) - 1$$

- Addition of a new element $a \notin A$ should increase the magnum by one:

$$m(A \cup \{a\}) = m(A) + 1$$

- The m-numbers of the union of two disjoint sets should add: if $A \cap B = \varnothing$, then

$$m(A \uplus B) = m(A) + m(B)$$

- The m-function should be additive on countable disjoint unions:

$$m(\uplus_i A_i) = \sum_i m(A_i)$$

- The odd and even non-negative numbers should behave *sensibly*:

$$\begin{array}{lcl} \mathbb{N}_O = \{1, 3, 5, \dots\} & \implies & m(\mathbb{N}_O) \approx \frac{1}{2} m(\mathbb{N}) \\ \mathbb{N}_E = \{2, 4, 6, \dots\} & \implies & m(\mathbb{N}_E) \approx \frac{1}{2} m(\mathbb{N}) \end{array}$$

  (where $\approx$ means that small terms may possibly be ignored), in such a way that

$$m(\mathbb{N}_E \uplus \mathbb{N}_O) = m(\mathbb{N}_E) + m(\mathbb{N}_O) = m(\mathbb{N})$$

- Similar requirements apply for multiples of 3, etc.

$$\begin{array}{lcl} \mathbb{T}_1 = \{1, 4, 7, \dots\} & \implies & m(\mathbb{T}_1) \approx \frac{1}{3} m(\mathbb{N}) \\ \mathbb{T}_2 = \{2, 5, 8, \dots\} & \implies & m(\mathbb{T}_2) \approx \frac{1}{3} m(\mathbb{N}) \\ \mathbb{T}_3 = \{3, 6, 9, \dots\} & \implies & m(\mathbb{T}_3) \approx \frac{1}{3} m(\mathbb{N}) \end{array}$$

- The size of the set of perfect squares should be 'reasonable:'

$$\mathrm{S} = \{1, 4, 9, 16, \dots\} \implies m(\mathrm{S}) \approx \sqrt{m(\mathbb{N})}$$

- Let $\mathbb{P}$ be the set of primes. From the prime number theorem, we should like to have

$$m(\mathbb{P}) \approx \frac{m(\mathbb{N})}{\log m(\mathbb{N})} = \frac{\varpi}{\log \varpi}$$

4 Extending Functions from the Reals to the Surreals

There are difficulties evaluating limits in $\mathbb{S}$. In ONAG (page 43), Conway remarks that, since there is a vast class of numbers greater than all elements of $\mathbb{N}$ and less than ω, one cannot conclude that the limit of the sequence $(1, 2, 3, \dots)$ is ω. Indeed, it seems that we cannot conclude that $m(\mathbb{N}) = \omega$, where ω is the first-born infinite surreal number, defined as $\{1, 2, 3, \dots \mid \}$. Therefore, we will write $m(\mathbb{N}) = \varpi$ (although we do not exclude the possibility that $\varpi = \omega$).

The extension of many elementary functions from domain $\mathbb{R}$ to domain $\mathbb{S}$ can be done without difficulty. For example, since powers are well defined over the surreals, we can extend the function

$$f : x \mapsto x^2 \,,\ x \in \mathbb{R} \qquad \text{to} \qquad f : x \mapsto x^2 \,,\ x \in \mathbb{S}$$

so we have $f(\varpi) = \varpi^2$ and so on. The extension is valid for all polynomials and rational functions and for the exponential and logarithmic functions. In the case of other functions, the interpretation of $f(\varpi)$ may need further consideration.

Some Examples

Let us examine a few special cases to see what the above extension entails.

$$f(n) = \left(\frac{n-1}{n}\right) = 1 - \frac{1}{n} \qquad \text{so that} \qquad f(\varpi) = 1 - \frac{1}{\varpi}$$

A general rational function of n is written

$$f(n) = \frac{a_p n^p + a_{p-i} n^{p-i} + \cdots + a_0}{n^q + b_{q-i} n^{q-i} + \cdots + b_0} \,, \qquad \text{so that} \qquad f(\varpi) = \frac{a_p \varpi^p + a_{p-i} \varpi^{p-i} + \cdots + a_0}{\varpi^q + b_{q-i} \varpi^{q-i} + \cdots + b_0} \,.$$

This is well defined, since division is defined for the surreals.

The value of $f(\varpi)$ may not be defined in all cases. For example

$$f(n) = (-1)^n \qquad \text{extends to} \qquad f(\varpi) = (-1)^\varpi$$

and it is not clear what the value of this should be. We introduce the notation

$$\Lambda \equiv (-1)^\varpi \tag{1}$$

without (yet) defining the value to be assigned to Λ. Evaluation of Λ corresponds to answering the question "Is ϖ an odd or an even number?", and it is related to the paradox of Thompson's Lamp ([Wiki1]) and to the behaviour of Grandi's (divergent) series

$$S = 1 - 1 + 1 - 1 + 1 - 1 + \cdots \,.$$

It seems probable that $\Lambda = 1$ may be *assumed* as an axiom without contradiction.

Numerical Examples

We know that for the real numbers, $0.999\dots = 1$. For the surreals, this is not the case:

$$f(n) = \underbrace{0.999\dots9}_{n\text{ terms}} = 1 - 10^{-n}\,, \qquad \text{therefore} \qquad f(\varpi) = 1 - 10^{-\varpi} < 1\,.$$

In a similar manner, we find for example that $0.333\dots = \frac{1}{3}(1 - 10^{-\varpi})$. We note that, in $\mathbb{S}$, $\frac{1}{3}$ and $0.333\dots$ are distinct numbers. The distinction makes sense: let us explicitly evaluate the fraction $\frac{1}{3}$ by division:

$$\forall n \in \mathbb{N}\,, \quad \frac{1}{3} = \underbrace{0.333\dots3}_{n\text{ terms}} + \frac{1}{3} \times 10^{-n}$$

so that, replacing n by ϖ we get

$$\frac{1}{3} = 0.\dot{3} + \frac{1}{3} \times 10^{-\varpi}\,, \qquad \text{consistent with} \qquad 0.333\dots = \frac{1}{3}(1 - 10^{-\varpi})\,.$$

It appears that $\frac{1}{3}$ has no decimal expansion. Of course, it has the ternary expansion $\frac{1}{3} = 0.1_3$. Many more examples could be given, such as

$$\frac{1}{4} = 0.25\,, \qquad \text{whereas} \qquad 0.24\dot{9} = \frac{1}{4} - 10^{-(\varpi+2)}\,.$$

It is more effective to consider the sum of a general geometric series

$$S_n = a + ar + ar^2 + ar^3 + \dots + ar^{n-1} = a\left(\frac{1 - r^n}{1 - r}\right)\,.$$

Substituting ϖ for n, we get the result

$$S_\varpi = a + ar + ar^2 + ar^3 + \dots = a\left(\frac{1 - r^\varpi}{1 - r}\right)\,.$$

which allows us to evaluate recurring decimal quantities. For example,

$$0.\overline{12} = 0.121212\dots = \frac{12}{10^2} + \frac{12}{10^4} + \dots = \frac{12}{99}(1 - 10^{-2\varpi})\,.$$

Another example, obtained by dividing 1 by 7, is

$$0.\overline{142857} = \frac{142{,}857}{1{,}000{,}000}\left[1 + 10^{-6} + 10^{-12} + \dots\right] = \frac{142{,}857}{999{,}999}\left[1 - 10^{-6\varpi}\right] = \frac{1}{7}\left[1 - 10^{-6\varpi}\right]\,.$$

5 Counting Sequence, Density and Magnum

We can define a general subset A of the natural numbers $\mathbb{N}$ by means of its *characteristic function* or indicator function

$$\chi_A(n) = \begin{cases} 1, & n \in A \\ 0, & \text{otherwise} \end{cases}$$

We assume that $\mathbb{N}$ has its standard order, $1 < 2 < 3 < \ldots$, and that all subsets of $\mathbb{N}$ have the ordering inherited from the natural numbers. Thus, if the set $A \subset \mathbb{N}$ is written

$$A = \{a_1, a_2, a_3, \ldots, a_n, \ldots\}$$

it will be assumed that the elements are in canonical order, $a_1 < a_2 < a_3 < \cdots < a_n < \ldots$.

Definition 1. *For any $A \subset \mathbb{N}$, we define the* <u>counting sequence</u> κ_A *to be the sequence of partial sums of the sequence $\{\chi_A(n)\}$:*

$$\kappa_A(n) = \sum_{k=1}^{n} \chi_A(k)$$

As the name suggests, $\kappa_A(n)$ counts the number of elements of A less than or equal to n. Clearly, $\kappa(n) \leq n$.

Definition 2. *If the value of the counting function $\kappa_A(n)$ is defined for argument ϖ, then the* <u>magnum</u> *of the set $A \subset \mathbb{N}$ is the value of the counting function at ϖ:*

$$m(A) = \kappa_A(\varpi)$$

Definition 3. *The* <u>density sequence</u> *for a set A is*

$$\rho_A(n) = \frac{\kappa_A(n)}{n} .$$

Note that $\rho_A(n)$ is the average value of $\{\chi_A(1), \chi_A(2), \ldots, \chi_A(n)\}$. Clearly, $\rho(n) \leq 1$.

Definition 4. *A sequence A has* <u>density $\rho_{\mathbb{R}}(A)$ in $\mathbb{R}$</u> *if the following limit exists:*

$$\rho_{\mathbb{R}}(A) = \lim_{n\to\infty} \frac{\kappa_A(n)}{n} = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_A(k) .$$

The *density sequence* always exists, but it may or may not have a limit in $\mathbb{R}$ (Tenenbaum, 1995).

Definition 5. *A sequence A has* <u>density $\rho(A)$ in $\mathbb{S}$</u> *if the following quantity is well defined:*

$$\rho(A) = \frac{\kappa_A(\varpi)}{\varpi} .$$

Principal Part of the Magnum

If A is a finite set, then $m(A)$ is just the cardinality of A. In the following, we assume, unless otherwise stated, that A is an infinite set. In that case, $m(A)$, if it exists, is also infinite. We denote by $M(A)$ the infinite part (principal part) of $m(A)$. According to ONAG (page 32), this can be done unambiguously by writing the surreal number $m(A)$ in its *normal form*. Then $m(A) - M(A)$ is finite and, using the normal form of the surreal number $m(A)$, the partitioning

$$m(A) = \underbrace{M(A)}_{\text{Infinite}} + \underbrace{\big(m(A) - M(A)\big)}_{\text{Finite}}$$

can be done in a canonical manner.

To compute the magnum of specific sets, we aim to write the counting sequence in the form

$$\kappa_A(n) = \pi_A(n) + \big(\kappa_A(n) - \pi_A(n)\big)$$

and evaluate this for $n = \varpi$. Then $M(A) = \pi_A(\varpi)$ (if this exists).

6 A Set without a Magnum

We now construct a set that does not have any limiting density. We define the set U by means of its indicator function $\chi_U(n)$, as follows:

$$\begin{aligned} \chi_U(1) &= 0 \\ \chi_U(n) &= \frac{1 + (-1)^k}{2} \quad \text{if } 10^k < n \leq 10^{k+1}\,, \quad k \geq 0 \end{aligned}$$

Thus, the indicator function is

$$\chi_U(n) = (0, \underbrace{1, \ldots, 1}_{9 \text{ values}}, \underbrace{0, \ldots, 0}_{90 \text{ values}}, \underbrace{1, \ldots, 1}_{900 \text{ values}}, \underbrace{0, \ldots, 0}_{9000 \text{ values}}, \ldots)$$

The set U may also be defined as *the set of natural numbers having an odd number of decimal digits* (Bajnok, 2013).

Now we calculate some values of the density of U,

$$\rho_U(N) = \frac{1}{N} \sum_{n=1}^{N} \chi_U(n)$$

and get the following results:

$$\begin{aligned} \rho_U(1) &= 0.0 \\ \rho_U(10) &= 0.9 \\ \rho_U(100) &= 0.09 \\ \rho_U(1000) &= 0.909 \\ \rho_U(10000) &= 0.0909 \end{aligned}$$

We see that $\rho_U(n)$ oscillates between values greater than 0.9 and values less than 0.1. It is clear that the density of U cannot approach a limit (in $\mathbb{R}$) as n increases, so no definite value can be assigned to $\rho_U(\varpi)$.

7 Intuition about Odd and Even Numbers

How many odd numbers are there? How many even numbers? We denote these sets by

$$\mathbb{N}_O = \{1, 3, 5, \dots\} \qquad \text{and} \qquad \mathbb{N}_E = \{2, 4, 6, \dots\}\,.$$

From Galileo to Cantor, the suggestion was that there are the same number of odd, even and natural numbers, because all three sets can be mapped in one-one fashion to each other. While this is technically correct, it jars with our intuition, which suggests that there are *twice as many natural numbers as odd or even numbers.*

How do we 'know' that the set $\mathbb{N}_E$ of even numbers is half the size of the set $\mathbb{N}$ of natural numbers? We do not. But we have a 'feeling' or intuitive notion that this is the case. Why?

The characteristic sequence for the *odd numbers* $\mathbb{N}_O$ is

$$\chi_O(n) = (1, 0, 1, 0, 1, 0, \dots)$$

and the counting sequence for the odd numbers is

$$\kappa_O(n) = (1, 1, 2, 2, 3, 3, \dots)$$

We can write the characteristic function and the counting function as

$$\chi_O(n) = \frac{1-(-1)^n}{2} \qquad \text{and} \qquad \kappa_O(n) = \tfrac{1}{2}\left[n + \frac{1-(-1)^n}{2}\right]$$

Evaluating the counting function at ϖ we get

$$m(\mathbb{N}_O) \equiv \kappa_O(\varpi) = \frac{\varpi}{2} + \frac{1}{4}\left[1-(-1)^{\varpi}\right] = \frac{\varpi}{2} + \frac{1}{4} - \frac{\Lambda}{4}\,. \tag{2}$$

Now, repeating this procedure for the even numbers, we get

$$m(\mathbb{N}_E) \equiv \kappa_E(\varpi) = \frac{\varpi}{2} - \frac{1}{4}\left[1-(-1)^{\varpi}\right] = \frac{\varpi}{2} - \frac{1}{4} + \frac{\Lambda}{4}\,. \tag{3}$$

Noting that $\mathbb{N}_E$ and $\mathbb{N}_O$ are disjoint and $\mathbb{N}_E \uplus \mathbb{N}_O = \mathbb{N}$, it is refreshing to observe that

$$m(\mathbb{N}_O) + m(\mathbb{N}_E) = \varpi = m(\mathbb{N})\,.$$

8 Some Simple Theorems

Adding Zeros at the Beginning

From a sequence A defined by the characteristic function $\chi_A(n)$, we can form the sequence B, shifted one place to the right. Then we have

Theorem 1. *Suppose the set A has magnum $m(A)$ and density $\rho(A)$. Then the shifted sequence B defined by*

$$\chi_B(1) = 0\,, \qquad \chi_B(n) = \chi_A(n-1)\,, \quad n > 1$$

has magnum and density

$$m(B) = m(A) - \chi_A(\varpi) \qquad \textit{and} \qquad \rho(B) = \rho(A) - \frac{\chi_A(\varpi)}{\varpi}\,.$$

Proof. Let the sets A and B have counting sequences $\kappa_A(n)$ and $\kappa_B(n)$. Then

$$\kappa_B(n) = \kappa_A(n) = \kappa_A(n) - \chi_A(n)$$

Evaluating this at ϖ we get

$$m(B) = m(A) - \chi_A(\varpi) \tag{4}$$

Dividing by ϖ we also have

$$\rho(B) = \rho(A) - \frac{\chi_A(\varpi)}{\varpi}\,.$$

□

Corollary. If the sequence B is shifted from A by k places, we have

$$m(B) = m(A) - \sum_{j=1}^{k} \chi_A(\varpi + 1 - j)$$

We can evaluate this if we have a formula for $\chi_A(n)$ that can be extended to $\mathbb{S}$.

Example. Let A be the set of odd numbers. Then

$$\chi_A(2n-1) = 1\,, \qquad \chi_A(2n) = 0$$

Defining the set B by shifting indices, we have

$$\chi_B(2n-1) = 0\,, \qquad \chi_B(2n) = 1$$

so $B = \mathbb{N}_E$, the even numbers. We saw in §7 that

$$m(A) = m(\mathbb{N}_O) = \frac{\varpi}{2} + \frac{1}{4} - \frac{\Lambda}{4}\,.$$

We can define the characteristic function of A as $\chi_O(n) = [1 - (-1)^n]/2$ so that, using (1) we have $\chi_O(\varpi) = (1 - \Lambda)/2$. Then

$$m(B) = m(\mathbb{N}_E) = m(\mathbb{N}_O) - \chi_O(\varpi) = \frac{\varpi}{2} - \frac{1}{4} + \frac{\Lambda}{4}\,.$$

as obtained already in §7, Eqn. (3).

General Arithmetic Sequence

We now consider the arithmetic sequence

$$A_0 = \{1, 1+d, 1+2d, \dots\}$$

The characteristic sequence is

$$\chi(A_0) = (\underbrace{1, 0, \dots, 0}_{d \text{ terms}}, \underbrace{1, 0, \dots, 0}_{d \text{ terms}}, \dots)$$

and the sequence of partial sums of this sequence is

$$\kappa(A_0) = (\underbrace{1, 1, \dots, 1}_{d \text{ terms}}, \underbrace{2, 2, \dots, 2}_{d \text{ terms}}, \dots)$$

We define the principal part as a sequence that increases steadily

$$\pi_{A_0}(n) = \left(\tfrac{1}{d}, \tfrac{2}{d}, \tfrac{3}{d}, \tfrac{4}{d}, \tfrac{5}{d}, \tfrac{6}{d}, \dots\right) = \tfrac{1}{d}(1, 2, 3, 4, 5, 6, \dots)$$

This represents the infinite part of the magnum of A_0. We find that $\pi_{A_0}(\varpi) = \varpi/d$. Now we write the counting sequence in the form

$$\kappa_{A_0}(n) = \pi_{A_0}(n) + \big(\kappa_{A_0}(n) - \pi_{A_0}(n)\big)$$

We express the finite part as the sum of a constant sequence and a sequence with zero mean:

$$\begin{aligned}
\big(\kappa_{A_0}(n) - \pi_{A_0}(n)\big) &= \left(1 - \tfrac{1}{d}, 1 - \tfrac{2}{d}, \dots, 1 - \tfrac{d}{d};\quad 2 - \tfrac{1}{d}, \dots\right) \\
&= \left(\tfrac{d-1}{2d}, \tfrac{d-1}{2d}, \dots, \tfrac{d-1}{2d};\quad \tfrac{d-1}{2d}, \dots\right) + \left(\tfrac{d-1}{2d}, \tfrac{d-3}{2d}, \dots, -\tfrac{d-1}{2d};\quad \tfrac{d-1}{2d}, \dots\right) \\
&= \mu_{A_0}(n) + \nu_{A_0}(n)
\end{aligned}$$

While $\mu_{A_0}(n)$ is equal to a constant $(d-1)/2d$, the sequence ν_{A_0} oscillates in saw-tooth fashion. We disregard this term to arrive at the result for the magnum of the arithmetic sequence A_0:

$$m(A_0) = \frac{\varpi}{d} + \left(\frac{d-1}{2d}\right)$$

We now use Theorem 1 to generalize this result. The general arithmetic sequence $A = \{a, a+d, a+2d, a+3d, \dots\}$ is formed by puting $a-1$ zeros in front of the above sequence A_0. The density of A_0 is $1/d$ so the theorem implies that the magnum of A is

$$m(A) = m(A_0) - (a-1)\rho_{A_0} = \frac{\varpi}{d} + \left(\frac{d-1}{2d}\right) - \left(\frac{a-1}{d}\right) = \frac{\varpi}{d} + \left(\frac{d+1-2a}{2d}\right)$$

We can express this as a theorem:

Theorem 2. *The magnum of the arithmetic sequence $A = \{a, a+d, a+2d, a+3d, \dots\}$ is*

$$m(s) = \frac{\varpi}{d} + \left(\frac{d+1-2a}{2d}\right) \tag{5}$$

It is easy to show that (2) and (3) follow from (5). As an exercise, show that

$$\begin{aligned} m(\mathbb{T}_1) = m(\{1,4,7,\dots\}) &= \frac{\varpi}{3} + \frac{1}{3} \\ m(\mathbb{T}_2) = m(\{2,5,8,\dots\}) &= \frac{\varpi}{3} \\ m(\mathbb{T}_3) = m(\{3,6,9,\dots\}) &= \frac{\varpi}{3} - \frac{1}{3} \end{aligned}$$

Squares of Natural Numbers

We now consider the set of squares of natural numbers $S = \{1,4,9,16,\dots\}$. The characteristic sequence is

$$\chi_S(n) = (1,\ \underbrace{0,0}_{\text{2 zeros}};\ 1,\underbrace{0,0,0,0}_{\text{4 zeros}};\ 1,\underbrace{0,0,0,0,0,0}_{\text{6 zeros}};\ 1,\dots)$$

and the sequence of partial sums of this sequence is

$$\kappa(n) = (\underbrace{1,1,1}_{\text{3 terms}},\underbrace{2,2,2,2,2}_{\text{5 terms}},\underbrace{3,3,3,3,3,3,3}_{\text{7 terms}},\dots)$$

Observing that $\kappa(n) = \lfloor\sqrt{n}\rfloor$, we define the function $\pi(n) = \sqrt{n}$, so that $\{\pi(n^2)\}$ is the sequence $\mathbb{N}$ of natural numbers. We write

$$\kappa(n) = \pi(n) - \big(\pi(n) - \kappa(n)\big)$$

and note that $\pi(n) - \kappa(n)$ is bounded in $[0,1)$. It can be expressed over the range $[n^2,(n+1)^2-1]$ as a mean value and a bounded zero-mean sequence. We approximate the average of $\pi(k)$ by an integral:

$$\frac{1}{2n+1}\sum_{k=n^2}^{(n+1)^2-1}\pi(k) \approx \frac{1}{2n+1}\int_{n^2}^{(n+1)^2}\sqrt{x}\,\mathrm{d}x = \frac{2n^2+2n+\frac{2}{3}}{2n+1}$$

Now subtract n, the value of κ on the range $[n^2,(n+1)^2-1]$ to get

$$\overline{\big(\pi(n)-\kappa(n)\big)} \approx \frac{n+\frac{2}{3}}{2n+1} \approx \frac{1}{2} + \frac{1}{12n}$$

Then, ignoring small terms we have

$$\kappa(n) = \sqrt{n} - \frac{1}{2} - \frac{1}{12n}\,.$$

Theorem 3. *The magnum of the sequence of squares $S = \{1,4,9,16,\dots\}$ is*

$$m(S) = \sqrt{\varpi} - \tfrac{1}{2} + \text{HOT}\,.$$

General Geometric Sequence

We now consider the general geometric sequence

$$G = \{\beta r, \beta r^2, \beta r^3, \dots\}$$

The characteristic sequence is

$$\chi_G(n) = (\underbrace{0,0,0,\dots,0}_{\beta r - 1 \text{ terms}};\ \underbrace{1,0,0,\dots,0}_{\beta r(r-1) \text{ terms}};\ \underbrace{1,0,0,\dots,0}_{\beta r^2(r-1) \text{ terms}}, \dots)$$

and the sequence of partial sums of this sequence is

$$\kappa_G(n) = (\underbrace{0,0,0,\dots,0}_{\beta r - 1 \text{ terms}};\ \underbrace{1,1,1,\dots,1}_{\beta r(r-1) \text{ terms}};\ \underbrace{2,2,2,\dots,2}_{\beta r^2(r-1) \text{ terms}}, \dots)$$

We define a sequence

$$\pi(x) = (\log_r x - \log_r \beta) = \left(\frac{\ln x - \ln\beta}{\ln r}\right) = \frac{\ln x}{\ln r} - \gamma$$

where $\gamma = \ln\beta/\ln r$. Then for $x = k \in \mathbb{N}$ we have

$$\beta r^{\pi(k)} = k$$

So this is the sequence $\mathbb{N}$ of natural numbers. The counting sequence is

$$\kappa(k) = \lfloor \pi(k) \rfloor \qquad \text{and} \qquad \big(\pi(k) - \kappa(k)\big) = \big(\pi(k) - \lfloor \pi(k) \rfloor\big) = \{\pi(k)\}$$

where $\{x\}$ is the fractional part of x. Now we can write

$$\kappa(n) = \pi(n) - \big(\pi(n) - \kappa(n)\big)$$

We note that $\big(\pi(n) - \kappa(n)\big)$ is bounded in $[0,1)$. We assume that it can be expressed as a mean value and a bounded zero-mean term

$$\big(\pi(n) - \kappa(n)\big) = \big(h + h'(n)\big)$$

There are heuristic reasons indicating that $h \approx \frac{1}{2}$. Then, ignoring the bounded zero-mean term and assuming $h = \frac{1}{2}$, we have

$$\kappa(n) = \frac{\ln n}{\ln r} - \left(\frac{\ln\beta}{\ln r} + \frac{1}{2}\right) .$$

Theorem 4. *The magnum of the geometric sequence $G = \{\beta r, \beta r^2, \beta r^3 \dots\}$ is*

$$m(G) = \frac{\ln\varpi}{\ln r} - \left(\frac{\ln\beta}{\ln r} + \frac{1}{2}\right) + \text{HOT} .$$

9 Concluding Remarks

Since their invention/discovery around 1970, surreal numbers have attracted relatively little attention. In *The Princeton Companion to Mathematics*, a comprehensive review of mathematics extending to more than 1,000 pages, there is no index entry for 'surreal numbers'. J.H.Conway's name is indexed, but all four references are to his work in group theory.

A small number of researchers have worked on surreal numbers, and they have also interested recreational mathematicians. Given their appealing properties and their remarkable elegance, this is surprising. In a 31-page section of the Notices of the American Mathematical Society for September 2018 (NAMS18), thesis titles were listed for more than 2,000 doctoral degrees awarded in the mathematical sciences between July 1, 2016 and June 30, 2017, as reported by 275 departments in 202 universities in the United States. There were no occurrences of the word 'surreal' in this list.

* * *

The development of analysis on $\mathbb{S}$ is the next big step in the application of surreal numbers. Progress on analysis has been faultering, with several failures. The paper *Analysis on Surreal Numbers* by Rubinstein-Salzedo and Swaminathan (2014), attempts to extend the application of surreals to functions, limits, derivatives, power series and integrals. The authors introduce a new definition of surreal numbers. They present a formula for the limit of a sequence and characterize convergent sequences. They define a new topology on $\mathbb{S}$ and, although the surreals are not Cauchy complete, they prove an Intermediate Value Theorem.

We have defined the magnum of a set as $m(A) = \kappa_A(\varpi)$. However, what is very much desired is what Conway calls a *genetic definition.* That is, given a set A we should like to define two sets L_A and R_A such that the surreal number $\{L_A \mid R_A\}$ is the magnum of A. Attempts have been made, but a satisfactory definition has not yet been found.

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Appendix: Definition of the Surreal Numbers

The Surreal numbers $\mathbb{S}$ are constructed inductively. We give a brief sketch, without proofs or details, of $\mathbb{S}$. A full account is given in Conway (2001 [ONAG]) and an elementary discussion in Knuth (1974).

Every number x is defined as a pair of sets, the left set and the right set:

$$x = \{x^L \mid x^R\}$$

where every element of x^L is les than every element of x^R, and x is the *simplest* or earliest appearing number between these two sets. We start with 0, defined as $\{\ \mid\ \}$. Then

$$\{0 \mid\ \} = 1 \qquad \{1 \mid\ \} = 2 \qquad \{2 \mid\ \} = 3 \qquad \dots$$

Negative numbers are defined inductively as $-x = \{-x^R \mid x^L\}$. So

$$\{\ \mid 0\} = -1 \qquad \{\ \mid 1\} = -2 \qquad \{\ \mid 2\} = -3 \qquad \dots$$

The dyadic fractions (rationals of the form $m/2^n$) appear as

$$\{0 \mid 1\} = \tfrac{1}{2} \qquad \{1 \mid 2\} = \tfrac{3}{2} \qquad \{0 \mid \tfrac{1}{2}\} = \tfrac{1}{4} \qquad \{\tfrac{1}{2} \mid 1\} = \tfrac{3}{4} \qquad \dots$$

After an infinite number of stages, when all dyadic fractions have emerged, all the remaining real numbers appear. The first infinite number ω is defined as

$$\omega = \{0, 1, 2, 3, \dots\ \mid\ \}$$

We can also introduce

$$\omega + 1 = \{0, 1, 2, 3, \dots\ \omega \mid\ \} \qquad \omega - 1 = \{0, 1, 2, 3, \dots\ \mid\ \omega\}$$

$$2\omega = \{0, 1, 2, 3, \dots\ \omega, \omega + 1, \omega + 2, \dots\ \mid\ \} \qquad \tfrac{1}{2}\omega = \{0, 1, 2, 3, \dots\ \mid\ \omega, \omega - 1, \omega - 2, \dots\}$$

and many other more exotic numbers. They behave beautifully: $\mathbb{S}$ is a totally oordered field, indeed, the largest such field. Since we are considering subsets of $\mathbb{N}$, we are concerned primarily with surreals less than or equal to ω.