Mamikon's Visual Calculus and the Hodograph

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A remarkable theorem, discovered in 1959 by Armenian astronomer Mamikon Mnatsakanian, allows problems in integral calculus to be solved by simple geometric reasoning, without calculus or trigonometry. Mamikon's theorem states that: 'The area of a tangent sweep of a curve is equal to the area of its tangent cluster.' We shall illustrate how this theorem can help to solve a range of integration problems.

The area of an annulus

The inspiration for his theorem came to Mamikon while, as an undergraduate, he was examining how to calculate the annular area A between two concentric circles, given only the length $2a$ of the chord tangent to the inner circle (Figure 1). Since $a^2 = R^2 - r^2$, it follows that $A = \pi(R^2 - r^2) = \pi a^2$. For a given a, this area is independent of the radii, r and R, of the inner and outer circles.

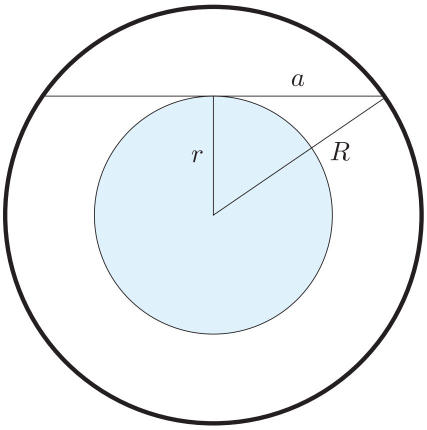

Figure 1: An annulus, the area between two concentric circles of radii r and R.

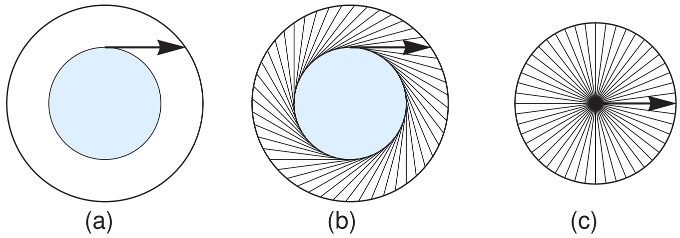

Figure 2: (a) A single tangent segment; (b) The tangent sweep *of a large collection of segments; (c) The* tangent cluster*, with all segments emanating from a common point.*

Mamikon considered a segment of length a tangent to the inner circle (Figure 2(a)). It is clear that the annulus is the area swept out by the tangent segment as it rotates around the inner circle, the *tangent sweep* (Figure 2(b)). Mamikon realised that this area, comprising the sum of numerous triangular regions, remains unchanged if all the regions are moved parallel to themselves so that all the tangent points coincide, forming the *tangent cluster* (Figure 2(c)). Since the area of the tangent cluster is πa^2, so is that of the tangent sweep, or annulus.

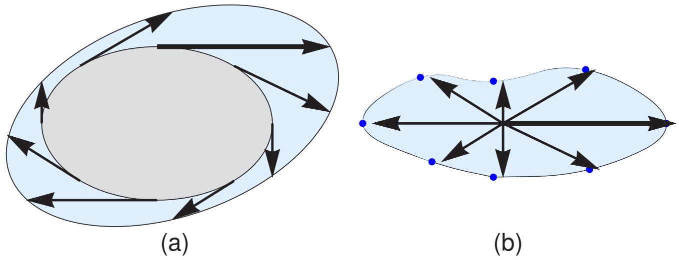

Figure 3: (a) The tangent sweep (shaded area) for an elliptic curve; (b) The tangent cluster formed when all tangents start from a common point.

Mamikon's theory can be applied to the motion of a turning bicycle.

Mamikon showed that the areas of the tangent sweep and tangent cluster are equal in much more general circumstances. The circles forming the annulus may be replaced by any smooth convex curves, closed or open, and the lengths of the tangents do not have to be constant. Figure 3 shows the (a) tangent sweep and (b) tangent cluster in a more general case. Mamikon's theorem ensures that the shaded areas in the two diagrams are equal.

Evaluating integrals

Tom Apostol, author of several influential textbooks on calculus, was a strong supporter of Mamikon's methods [1, 2]. In [1] he gave several examples of integral evaluation using Mamikon's theorem. The exponential function $y = \exp(x/b)$ and the parabola $y = cx^2$ can easily be integrated in this way. We remark that the quadrature of the parabola was first achieved by Archimedes, using a method that adumbrated integral calculus 2000 years before Newton and Leibniz. More generally, polynomial functions are easily integrated using Mamikon's approach. Indeed, Apostol discussed the quadrature of a wide selection of classical curves, the hyperbola, catenary, cardioid, tractrix and cycloid amongst them (details in [1]).

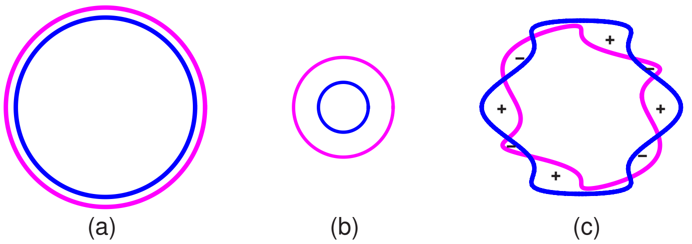

Figure 4: Bicycle moving in a circle. Steady bicycle (a and b) where the annular areas are equal and (c) wobbling bicycle.

Another interesting application is the calculation of the area confined between the front and rear tyre tracks of a bicycle. This is a generalisation of the tractrix curve, dubbed by Mamikon the *bicyclix curve*. A bicycle with the angle of the front wheel held constant moves in a circle while the front and rear wheels trace out concentric circles forming an annulus (Figure 4). Since the distance d between the axles is fixed, the area between the two circles is independent of the angle of the front wheel: the tangent cluster is a circle with radius d. More generally, the tracks cross each other as the front wheel angle varies, enclosing areas that may be counted as positive or negative (Figure 4(c)). However, the tangent segment from the rear wheel to the front one is of constant length d and sweeps out the region between the tracks. The total area is given by the area of the tangent cluster, which is a circular sector.

The cycloid

The cycloid is the locus of a point fixed to the rim of a circular disc that is rolling along a straight line (Figure 5). The parametric equations for the cycloid are

$$x = r(\theta - \sin\theta), \qquad y = r(1 - \cos\theta), \tag{1}$$

where θ is the angle through which the disc has rotated. The centre of the disc is at $(x, y) = (r\theta, r)$. The differentials of the coordinates (1) are

$$\mathrm{d}x = r(1 - \cos\theta)\,\mathrm{d}\theta, \qquad \mathrm{d}y = r\sin\theta\,\mathrm{d}\theta. \tag{2}$$

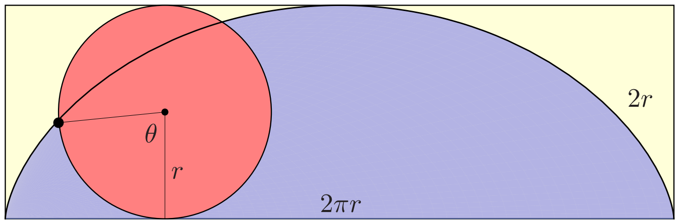

Figure 5: The cycloid, generated by a rolling disc. A single arch of the cycloid is shown, bounded within a rectangle of area $4\pi r^2$.

From these we find the increment of arc length $\mathrm{d}\ell$:

$$\mathrm{d}\ell = \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2} = 2r\sin\tfrac{1}{2}\theta\,\mathrm{d}\theta, \tag{3}$$

and the increment of area $\mathrm{d}A$:

$$\mathrm{d}A = y\,\mathrm{d}x = r^2(1 - \cos\theta)^2\,\mathrm{d}\theta. \tag{4}$$

The length L of an arch is easily computed by integrating (3):

$$L = \int_0^{2\pi} 2r\sin\tfrac{1}{2}\theta\,\mathrm{d}\theta = 8r.$$

We see that the arc length does not depend on π. The area under an arch is the integral of (4) over the interval $\theta \in [0, 2\pi]$, which is slightly more tricky:

$$A = \int_0^{2\pi} r^2(1 - \cos\theta)^2\,\mathrm{d}\theta = 3\pi r^2. \tag{5}$$

Thus, the area under an arch is three times the area of the generating circle or three-quarters of the area of the surrounding rectangle.

To get the area A, we required the parametric equations (1) for the cycloid and the evaluation of a definite integral (5). Using Mamikon's theorem, we shall now show that the area can be found by simple geometric reasoning, without any equations or integrations [1].

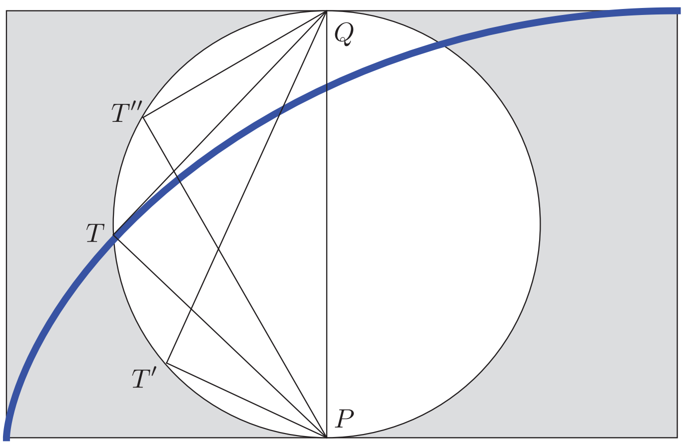

Figure 6: Tangent TQ is orthogonal to PT from the point of contact.

For rolling motion, the instantaneous centre of rotation of the disc is the point of contact P (Figure 6) and the vertical line PQ is the diameter of the disc. All points T, T' and T'' on the boundary of the disc move in directions orthogonal to the chords joining them to P. In particular, the angle PTQ is a right angle, so TQ

is a segment of the tangent to the cycloid. As the disc rolls, the point Q moves and the chords tangent to the cycloid sweep out the region above this curve.

We draw a set of tangents for the left-hand side of the arch in Figure 7(a). Now, moving all these segments so that they have a common upper point O (Figure 7(b)), we see that they fill a semicircle of radius r. The tangents for the right-hand side of the arch complete the circle. Thus, the area of the tangent cluster is πr^2, the same area as the generating disc.

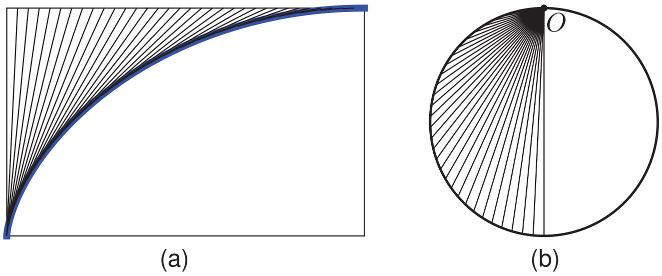

Figure 7: (a) Tangent sweep and (b) tangent cluster.

By Mamikon's theorem, the area of the rectangle above the arch – the tangent sweep – is equal to the area of the tangent cluster or πr^2. But the total area of the rectangle is $4\pi r^2$, so the area of the arch must be $3\pi r^2$, or three times the area of the generating disc.

Hamilton's hodograph

The hodograph is a vector diagram showing how velocity changes with position or time. It was made popular by William Rowan Hamilton, who, in 1847, gave an account of it in the *Proceedings of the Royal Irish Academy* [3]. The underlying idea is very simple: velocity vectors at different times or places are plotted with a common origin, or emanating from a single point. The hodograph is the locus of the arrow heads. Their varying directions and magnitudes make a pattern that can yield dynamical information in a visually clear way. Hamilton explained the origin of the word hodograph, from $o\delta o\varsigma$, a way and $\gamma\rho\alpha\varphi\omega$, to write.

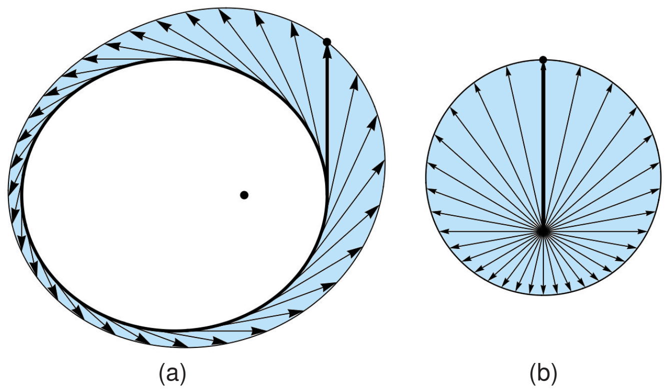

Figure 8: (a) Kepler orbit with velocity vectors and (b) circular hodograph.

Hodographs are valuable in fluid dynamics, astronomy and meteorology. Wind arrows at different levels, plotted on a polar diagram, illustrate the wind shear with height. Dynamically, vertical wind shear is linked to horizontal temperature gradients, through a relationship called the *thermal wind* [4]. Just as the wind blows with the low pressure to the left (in the northern hemisphere), the wind-shear vector has the low temperature to its left. So, the hodograph gives information about the location of warm and cold air and about temperature advection or heat transport.

In 1609, Kepler published his law of the ellipse, shattering the arguments of the ancient Greeks that circular orbits, being the epitome of perfection, must be found. However, the circle re-emerged some 237 years after Kepler, when Hamilton announced his law of the circular hodograph [3].

Figure 8(a) shows a Kepler orbit with velocity vectors. The velocity is scaled so that, at the pericentre, $v_P = a$ where a is the semimajor axis. The velocity v_A at the apocentre follows from conservation of angular momentum, $r_P v_P = r_A v_A$. Hamilton discovered the remarkable fact that if all velocity vectors are plotted from a common point, they trace out a circle of radius $\bar{v} = (v_P + v_A)/2$ (Figure 8(b)).

Hamilton's hodograph is also what Mamikon calls a tangent cluster. Since the area of the hodograph is $\pi\bar{v}^2$, Mamikon's theorem shows that the region swept out by the vectors around the Kepler orbit (shaded region in Figure 8(a)) also has this area.

Discussion

Mamikon Mnatsakanian worked with Tom Apostol in *Project Mathematics*, at the California Institute of Technology, to develop a series of educational videos and accompanying workbooks [5]. When combined with computer animation, his method provides a valuable teaching aid. It is applicable not only to plane curves and areas but to space curves, areas of ruled surfaces and solids of revolution in three dimensions.

Apostol [1] observed that the great contribution of Newton and Leibniz was to demonstrate the relationship between differentiation and integration. He remarked that Mamikon's method has some of the same ingredients, because 'it relates moving tangent segments with the areas of the regions swept out by those tangent segments'. Thus, the relationship between differentiation and integration is embedded in Mamikon's method.

However, it must be acknowledged that the method lacks the great generality of classical calculus. It is valuable for solving a large class of integration problems but, usually, the solutions are far from obvious and require special geometric insight, whereas the techniques of calculus are applicable in a transparent manner.

References

1 Apostol, T.M. (2000) A visual approach to calculus problems, *Eng. Sci.*, vol. 63, no. 3, pp. 22–31.

2 Mamikon, M. (1997) Annular rings of equal area, *Math. Horiz.*, vol. 5, no. 2, pp. 5–8.

3 Hamilton, W.R. (1847) The hodograph, or a new method of expressing in symbolical language the Newtonian law of attraction, *Proc. R. Ir. Acad.*, vol. 3, pp. 344–353.

4 Holton, J. and Hakim, G. (2012) *An Introduction to Dynamic Meteorology, 5th edition*, Academic Press, Cambridge.

5 Wikipedia, *Project Mathematics!*, `en.wikipedia.org/wiki/Project_Mathematics!`.