AweSums

Marvels and Mysteries of Mathematics

•

LECTURE 8

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Evening Course, UCD, Autumn 2021

Outline

Outline

Meaning and Content of Mathematics

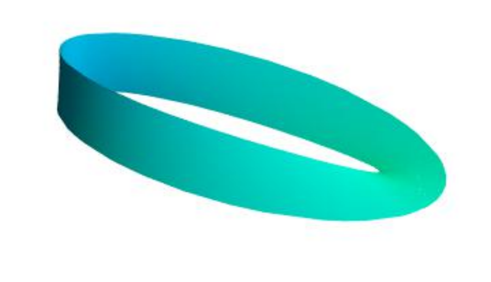
The word Mathematics comes from Greek $\mu\alpha\theta\eta\mu\alpha$ (máthéma), meaning "knowledge" or "lesson" or "learning".

It is the study of topics such as

- Quantity: [Numbers. Arithmetic]
- Structure: [Patterns. Algebra]
- Space: [Geometry. Topology]
- Change: [Analysis. Calculus]

Outline

The Möbius Band



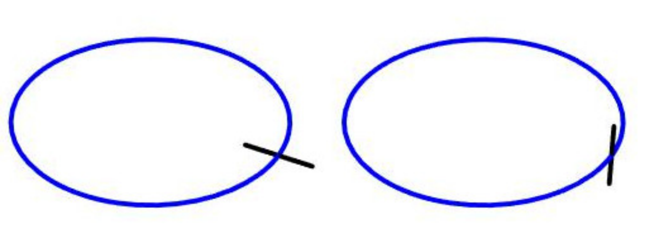
You may be familiar with the Möbius strip or Möbius band. It has one side and one edge.

It was discovered independently by August Möbius and Johann Listing in the same year, 1858.

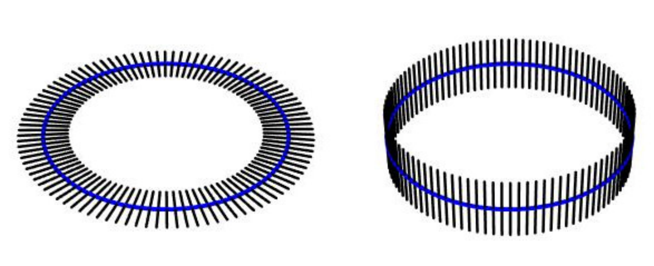
Building the Band

It is easy to make a Möbius band from a paper strip.

For a geometrical construction, we start with a circle and a small line segment with centre on this circle.



Now move the line segment around the circle:



To show the boundary of the surface, we color one end of the line segment red and the other magenta.

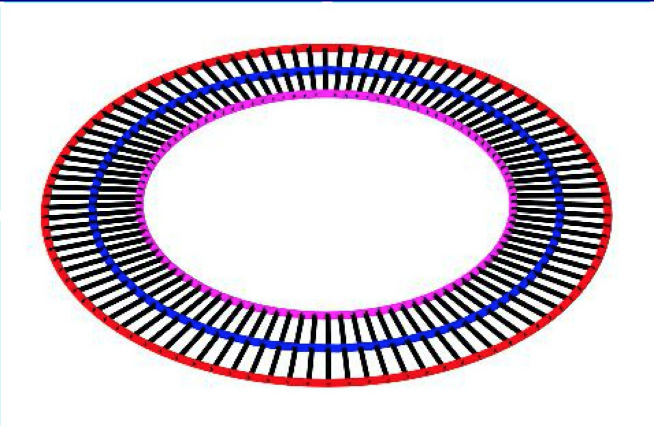


Figure: The boundary comprises two unlinked circles

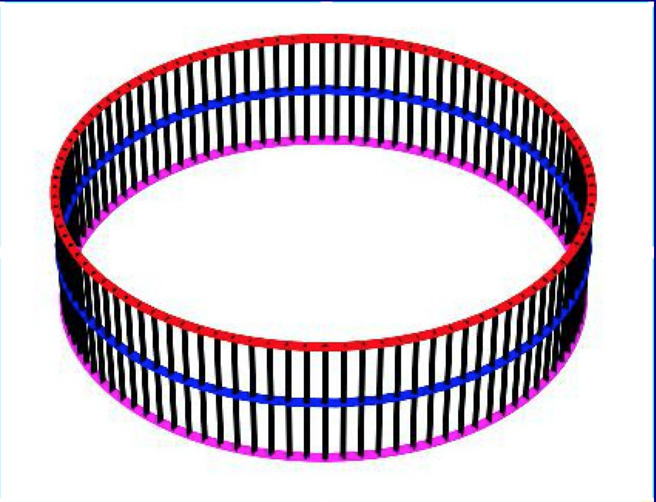
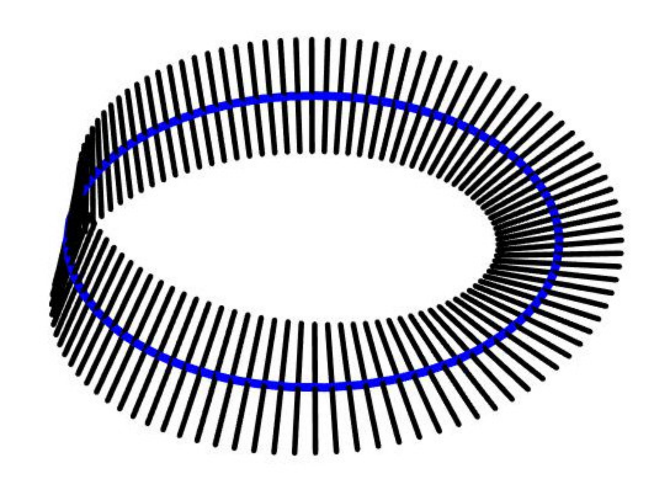


Figure: The boundary comprises two unlinked circles

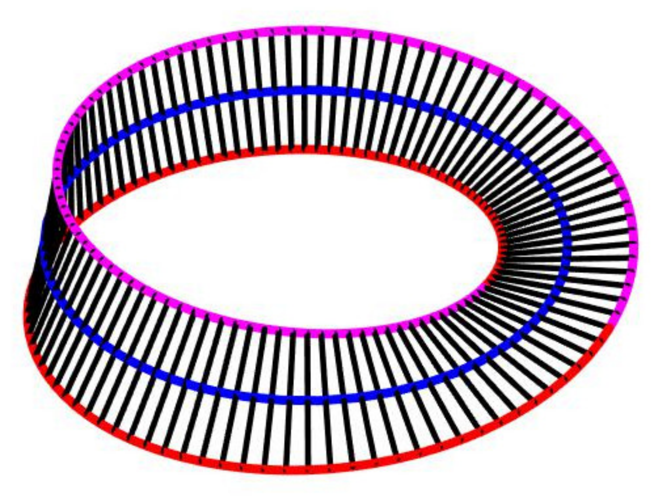
The Möbius Band

Now, as the line moves, we give it a half-twist:



The Möbius Band

The two boundary curves now join up to become one:



The Möbius Band

The Möbius Band has only one side.

It is possible to get from any point on the surface to any other point without crossing the edge.

The surface also has just one edge.

Band with a Full Twist

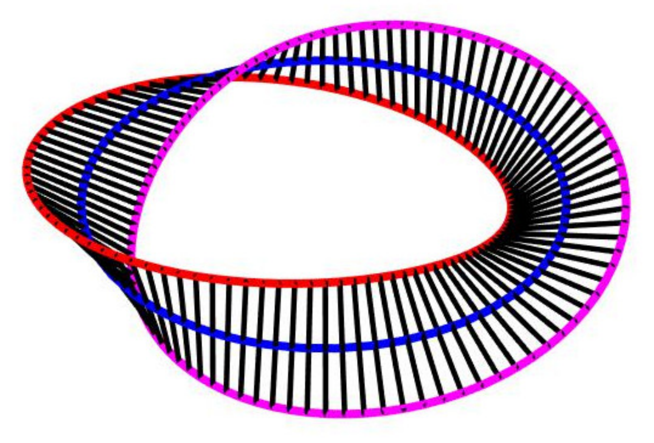


Figure: The boundary comprises two linked circles

Band with Three Half-twists

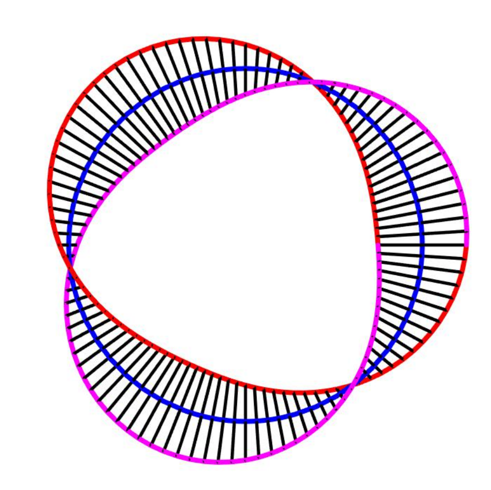


Figure: One side and one edge. What shape is the edge?

Band with Three Half-twists

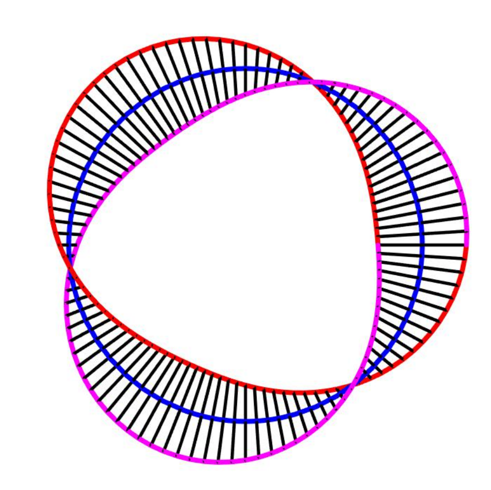
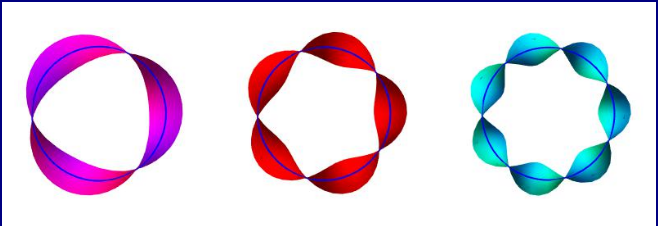


Figure: One side and one edge. What shape is the edge?

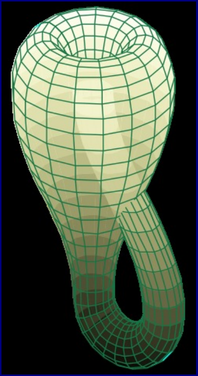
The boundary is a knot, a trefoil curve



Tadashi Takieda Video

https://www.youtube.com/watch?v=wKV0GYvR2X8

Two Möbius Bands make a Klein Bottle



A mathematician named Klein
Thought the Möbius band was divine.
Said he: “If you glue
The edges of two,
You’ll get a weird bottle like mine.”

Equations for the Möbius Band

The process of moving the line segment around the circle leads us to the equations for the Möbius band.

In cylindrical polar coordinates the circle is $(r, \theta, z) = (a, \theta, 0)$**.**

The tip of the segment, relative to its centre, is

$$(r, \theta, z) = (b \cos \phi, 0, b \sin \phi)$$

where $b = \frac{1}{2}\ell$ **is half the segment length and** $\phi = \alpha\theta$**, with** α **determining the amount of twist.**

The tip of the line has $(r, z) = (a + b \cos \alpha\theta, b \sin \alpha\theta)$**.**

Equations for the Möbius Band

In cartesian coordinates, the equations become

$$
\begin{aligned}
x &= (a + b\cos\alpha\theta)\cos\theta \\
y &= (a + b\cos\alpha\theta)\sin\theta \\
z &= (b\sin\alpha\theta)
\end{aligned}
$$

These are the parametric equations for the twisted bands, with $\theta \in [0, 2\pi]$ and $b \in [-\ell, \ell]$.

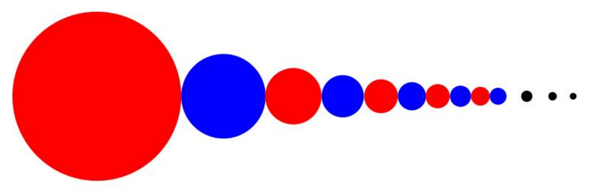
For the Möbius band, $\alpha = \frac{1}{2}$.

Outline

A Surprising Result

Let us consider an infinite row of cookies each smaller than the previous one.

Assume that the radius of the n-th cookie is $1/n$. Then the surface area is π/n^2.



A Surprising Result

The total length of the row of cookies is

$$2\left(1+\frac{1}{2}+\frac{1}{3}+\dots\right)=2\sum_{n=1}^{\infty}\frac{1}{n}$$

This is the harmonic series, which diverges.

A Surprising Result

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$$2\left(1+\frac{1}{2}+\frac{1}{3}+\dots\right) = 2\sum_{n=1}^{\infty}\frac{1}{n}$$

This is the harmonic series, which diverges.

The total surface area of the cookies is

$$\sum_{n=1}^{\infty}\pi\times\left(\frac{1}{n}\right)^2 = \pi\left(\sum_{n=1}^{\infty}\frac{1}{n^2}\right) = \frac{\pi^3}{6}$$

The series is known as the Basel series, and it is convergent, with sum $\pi^2/6$.

Outline

Alfred Moessner's Conjecture

Aus den Sitzungsberichten der Bayerischen Akademie der Wissenschaften
Mathematisch-naturwissenschaftliche Klasse 1951 Nr. 3

Eine Bemerkung über die Potenzen der natürlichen Zahlen

Von Alfred Moessner in Gunzenhausen

Vorgelegt von Herrn O. Perron am 2. März 1951

A Remark on the Powers of the Natural Numbers

Moessner's Construction: n=2

We start with the sequence of natural numbers:

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 . . .

Now we delete every second number and calculate the sequence of partial sums:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1		4		9		16		25		36		49		64	

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1		4		9		16		25		36		49		64	

The result is the sequence of perfect squares:

$$1^2 \quad 2^2 \quad 3^2 \quad 4^2 \quad 5^2 \quad 6^2 \quad 7^2 \quad 8^2 \quad \ldots$$

Moessner's Construction: n=3

Now we delete every third number and calculate the sequence of partial sums.

Then we delete every second number and calculate the sequence of partial sums:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	3		7	12		19	27		37	48		61	75		91
1			8			27			64			125			216

The result is the sequence of perfect cubes:

$$1^3 \quad 2^3 \quad 3^3 \quad 4^3 \quad 5^3 \quad 6^3 \quad \ldots$$

Moessner’s Construction: n=4

The Moessner Construction also works for larger n:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	3	6		11	17	24		33	43	54		67	81	96	
1	4			15	32			65	108			175	256		
1				16				81				256			

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1	4			15	32			65	108			175	256		
1				16				81				256			

The result is the sequence of fourth powers:

$$1^4 \quad 2^4 \quad 3^4 \quad 4^4 \quad \ldots$$

Moessner's Constructions

Remark:

Using Moessner's construction, we can generate a table of squares, cubes or higher powers.

The only arithmetical operations used are counting and addition!

Moessner's Constructions

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Using Moessner's construction, we can generate a table of squares, cubes or higher powers.

The only arithmetical operations used are counting and addition!

Are there any other sequences generated in this way?

Moessner's Construction for n!

We begin by striking out the **triangular numbers**, $\{1, 3, 6, 10, 15, 21, \ldots\}$ and form partial sums.

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Next, we delete the final entry in each group and form partial sums. This process is repeated indefinitely:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
	2		6	11		18	26	35		46	58	71	85		101
			6			24	50			96	154	225			326
						24				120	274				600
										120					720

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This yields the factorial numbers:

1! 2! 3! 4! 5! 6! . . .

Wikipedia Mathematics Portal

Topics in mathematics edit

General
- Mathematicians
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- Philosophy of mathematics
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- Outline of mathematics
- List of mathematical symbols
- Wikipedia Books: Mathematics

Foundations
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    - Gödel's incompleteness theorems
  - Model theory
  - Recursion theory
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  - Axiomatic set theory
- Category theory (portal)
  - Topos theory

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- Analytic number theory
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  - Fundamental theorem of arithmetic
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  - Natural numbers
  - Prime numbers
  - Rational numbers
  - Algebraic numbers

Discrete mathematics
- Discrete mathematics (portal)
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  - Coding theory
  - Combinatorial design
  - Enumerative combinatorics
  - Combinatorial optimization
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- Order theory
  - Lattice theory
- Digital Signal Processing

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  - Ring theory
  - Field theory
  - Commutative algebra
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- Linear algebra
  - Matrix theory
  - Multilinear algebra
- Universal algebra
- Fundamental theorem of algebra

Analysis
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  - Fundamental theorem of calculus
  - Vector calculus
  - Geometric calculus
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- Complex analysis
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  - Partial differential equations
- Integral equations
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- Functional analysis
- Operator theory

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- Convex geometry
- Discrete geometry
- Algebraic geometry
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  - Geometric topology
  - Differential topology

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  - Control theory
    - Calculus of variations
- Optimization
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  - Game theory
  - Mathematical finance
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  - Computability theory
  - Complexity theory
- Cryptography (portal)
- Information theory

Index of mathematics articles edit

Outline

Source of Some Puzzles

Mathematical Lateral Thinking Puzzles
by
Paul Slone & Des MacHale

Slicing a Cake with One Cut

Bake a cake that you can slice into 6 equal pieces with one cut?

Hint: The cake can be any shape you like

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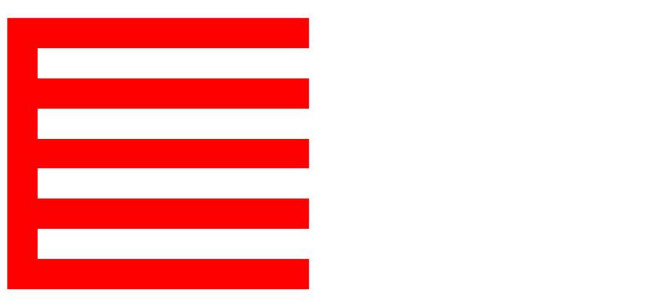

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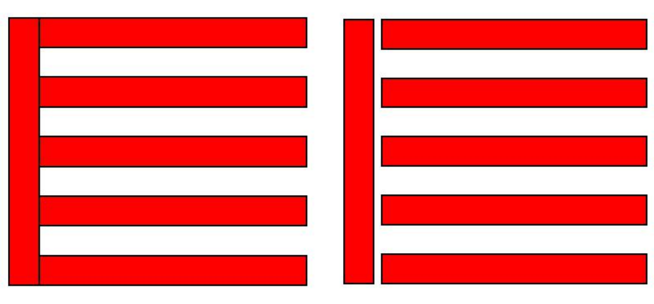

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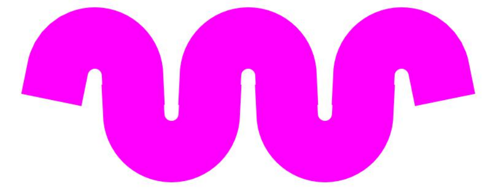

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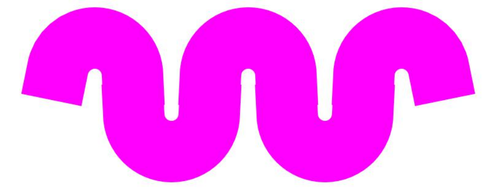

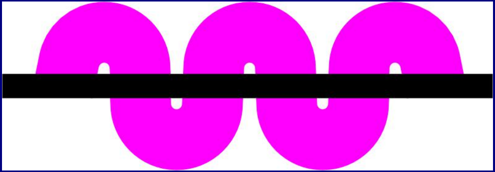

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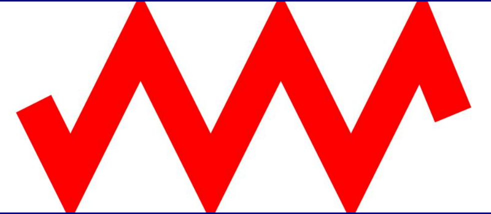

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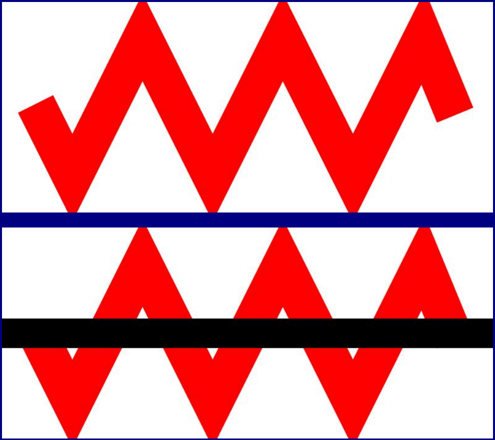
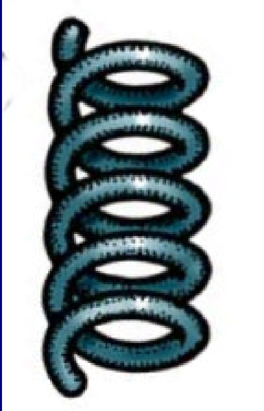


A Three-dimensional Cake

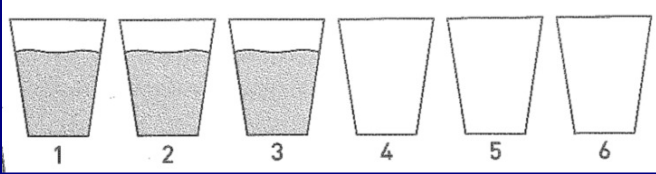


Cake in the form of a helix.

This is like twist . . .

. . . pastry twisted round
a stick and cooked over a
camp-fire.

Rearrange Six Glasses



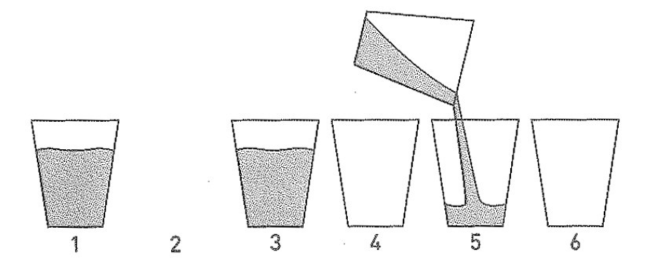
There are six glasses in a row.

Glasses 1, 2 and 3 are full.
Glasses 4, 5 and 6 are empty.

How can you arrange for the full and empty glasses to alternate, moving only one glass?

Rearrange Six Glasses

First, pour water from Glass 2 into glass 5:



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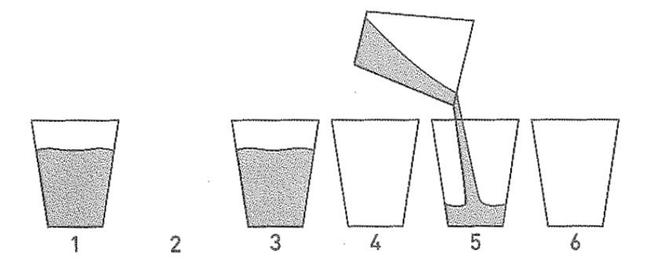
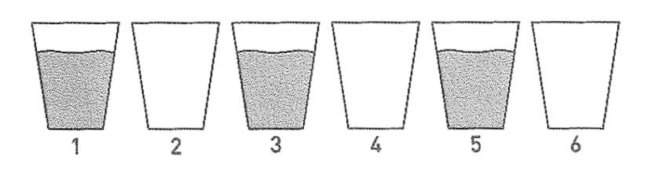
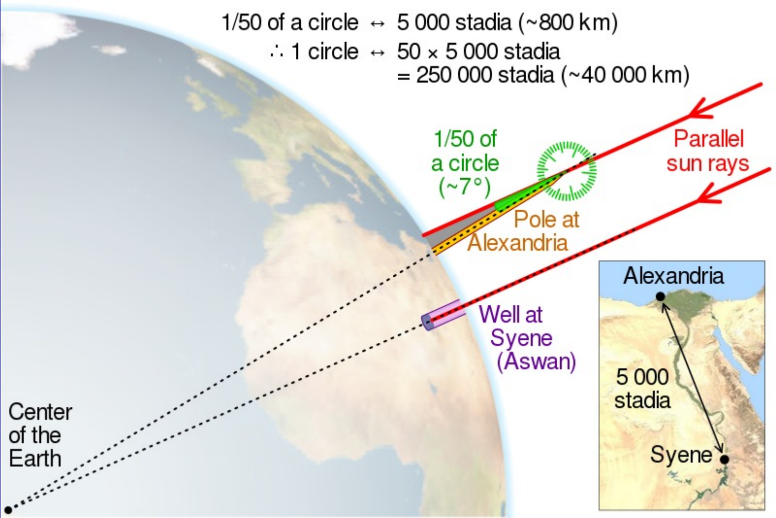


Then, place Glass 2 in its original position:



Outline

Eratosthenes Measured the Earth



The Sieve of Eratosthenes

Eratosthenes was the Librarian in Alexandria when Archimedes flourished in Syracuse.

They were "pen-pals".

Eratosthenes estimated size of the Earth.

He devised a systematic procedure for generating the prime numbers: the Sieve of Eratosthenes.

The Sieve of Eratosthenes

The idea:

- List all natural numbers up to n.
- Circle 2 and strike out all multiples of two.
- Move to the next number, 3.
- Circle it and strike out all multiples of 3.
- Continue till no more numbers can be struck out.

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- Continue till no more numbers can be struck out.

The numbers that have been circled are the prime numbers. Nothing else survives.

It is sufficient to go as far as $\sqrt{n}$.

The Sieve of Eratosthenes

	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

The Sieve of Eratosthenes

	2	3		5		7		9	
11		13		15		17		19	
21		23		25		27		29	
31		33		35		37		39	
41		43		45		47		49	
51		53		55		57		59	
61		63		65		67		69	
71		73		75		77		79	
81		83		85		87		89	
91		93		95		97		99	

The Sieve of Eratosthenes

	2	3		5		7			
11		13				17		19	
		23		25				29	
31				35		37			
41		43				47		49	
		53		55				59	
61				65		67			
71		73				77		79	
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	2	3		5		7			
11		13				17		19	
		23						29	
31						37			
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The Sieve of Eratosthenes

	2	3		5		7			
11		13				17		19	
		23						29	
31						37			
41		43				47			
		53						59	
61						67			
71		73						79	
		83						89	
						97			

The Sieve of Eratosthenes

	2	**3**	4	**5**	6	**7**	8	9	10
11	12	**13**	14	15	16	**17**	18	**19**	20
21	22	**23**	24	25	26	27	28	**29**	30
31	32	33	34	35	36	**37**	38	39	40
41	42	**43**	44	45	46	**47**	48	49	50
51	52	**53**	54	55	56	57	58	**59**	60
61	62	63	64	65	66	**67**	68	69	70
71	72	**73**	74	75	76	77	78	**79**	80
81	82	**83**	84	85	86	87	88	**89**	90
91	92	93	94	95	96	**97**	98	99	100

Is There a Pattern in the Primes?

It is a simple matter to make a list of all the primes less that 100 or 1000.

It becomes clear very soon that there is no clear pattern emerging.

The primes appear to be scattered at random.

Figure: Prime numbers up to 100

The grand challenge is to find patterns in the sequence of prime numbers.

This is an enormously difficult problem that has taxed the imagination of the greatest mathematicians for centuries.

Outline

David Hilbert (1862–1943)

David Hilbert, from a contemporary postcard.

Hilbert's Problems

In August 1900, David Hilbert addresed the International Congress of Mathematicians in the Sorbonne in Paris:

> ***"Who of us would not be glad to lift the veil behind which the future lies hidden; to cast a glance at the next advances of our science and at the secrets of its development during future centuries?"***

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Hilbert presented 23 problems that challenged mathematicians through the twentieth century.

Hilbert's Problems

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
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MATHEMATICAL PROBLEMS

DAVID HILBERT

Lecture delivered before the International Congress of Mathematicians at Paris in 1900.

Hilbert's eighth problem concerned itself with what is called the Riemann Hypothesis (RH).

RH is generally regarded as the deepest and most important unproven mathematical problem.

Anyone who can prove it is assured of lasting fame.

Why is RH Important?

A large number of mathematical theorems (1000's) depend for their validity on the RH.

Were RH to turn out to be false, many of these mathematical arguments would simply collapse.

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In 2000, industrialist Landon Clay donated $7M, with $1M for each of 7 problems in mathematics.

The Riemann hypothesis is one of these problems.

http://www.claymath.org/millennium-problems

Why is RH Important?

Whoever proves Riemann's hypothesis will have completed thousands of theorems that start like this:

"Assuming that the Riemann hypothesis is true . . . ".

He or she will be assured of lasting fame.

Those who establish fundamental mathematical results probably come closer to immortality than almost anyone else.

Outline

Sources to Continue your Interest

- YouTube
- Plus Magazine
- Quanta Magazine
- Mathigon.org
- Wolfram Alpha
- MoMath.org
- Gresham College
- Wikipedia

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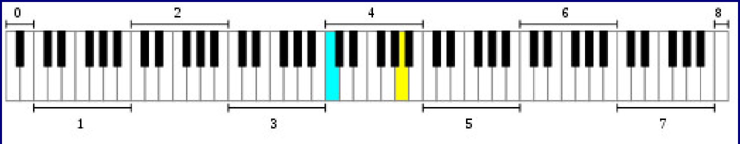
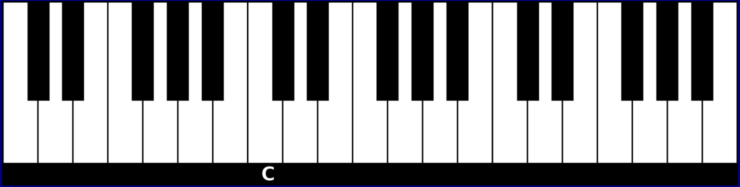
The Piano Keyboard





Middle C

C is the first note of the C major scale.
Middle C is the ‘central note on the piano.

It is commonly pitched at 261.63 Hz.

The standard frequency of the note A4 is 440 Hz.

$$261.63 \times 2^{9/12} = 440$$

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$$261.63 \times 2^{9/12} = 440$$

Where does the peculiar factor $2^{9/12}$ come from?

We will look at well-tempered scales later.

Bernstein: *I Like to be in America*

https://en.wikipedia.org/wiki/America_(West_Side_Story_song)

Music: Leonard Bernstein. Lyrics: Stephem Sondheim.

Brubeck: *Blue Rondo a la Turk*

https://musescore.com/fierabrass/scores/286641

Thank you