# AweSums

## Marvels and Mysteries of Mathematics

•

## LECTURE 7


Peter Lynch
School of Mathematics & Statistics
University College Dublin


Evening Course, UCD, Autumn 2021

# Outline

# Outline

# Meaning and Content of Mathematics

The word **Mathematics** comes from Greek $\mu\alpha\theta\eta\mu\alpha$ (máthéma), meaning “knowledge” or “lesson” or “learning”.

It is the study of topics such as

- Quantity: [Numbers. Arithmetic]
- Structure: [Patterns. Algebra]
- Space: [Geometry. Topology]
- Change: [Analysis. Calculus]

# Outline

# Carl Friedrich Gauss (1777–1855)

# Carl Friedrich Gauss (1777–1855)

**A German mathematician who made profound contributions to many fields of mathematics:**

- Number theory
- Algebra
- Statistics
- Analysis
- Differential geometry
- Geodesy & Geophysics
- Mechanics & Electrostatics
- Astronomy

**One of the greatest mathematicians of all time.**

# Gauss Outsmarts his Teacher

Gauss was a genius. He was known as

**The Prince of Mathematicians.**

# Gauss Outsmarts his Teacher

Gauss was a genius. He was known as

The Prince of Mathematicians.

When very young, Gauss outsmarted his teacher.

# Gauss Outsmarts his Teacher

Gauss was a genius. He was known as

The Prince of Mathematicians.

When very young, Gauss outsmarted his teacher.

I can now reveal a fact unknown to historians:

The teacher got his own back. Ho! ho! ho!

# Gauss Outsmarts his Teacher

**Gauss's school teacher tasked the class:**

- **Add up all the whole numbers from 1 to 100.**

# Gauss Outsmarts his Teacher

Gauss's school teacher tasked the class:

- Add up all the whole numbers from 1 to 100.

Gauss solved the problem in a flash.

He wrote the correct answer,

$$5,050$$

on his slate and handed it to the teacher.

# Gauss Outsmarts his Teacher

Gauss's school teacher tasked the class:

- Add up all the whole numbers from 1 to 100.

Gauss solved the problem in a flash.

He wrote the correct answer,

$$5,050$$

on his slate and handed it to the teacher.

How did Gauss do it?

## First, Gauss wrote the numbers in a row:

$$1 \quad 2 \quad 3 \quad \ldots \quad 98 \quad 99 \quad 100$$

**First, Gauss wrote the numbers in a row:**

$$1 \quad 2 \quad 3 \quad \ldots \quad 98 \quad 99 \quad 100$$

**Next he wrote them again, in reverse order:**

$$100 \quad 99 \quad 98 \quad \ldots \quad 3 \quad 2 \quad 1$$

**First, Gauss wrote the numbers in a row:**

$$1 \quad 2 \quad 3 \quad \ldots \quad 98 \quad 99 \quad 100$$

**Next he wrote them again, in reverse order:**

$$100 \quad 99 \quad 98 \quad \ldots \quad 3 \quad 2 \quad 1$$

**Then he added the two rows, column by column:**

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 2 | 3 | ... | 98 | 99 | 100 |
| 100 | 99 | 98 | ... | 3 | 2 | 1 |
| — — — | — — — | — — — | — — — | — — — | — — — | — — — |
| 101 | 101 | 101 | ... | 101 | 101 | 101 |

**Clearly, the total for the two rows is 10,100.**

First, Gauss wrote the numbers in a row:

$$1 \quad 2 \quad 3 \quad \ldots \quad 98 \quad 99 \quad 100$$

Next he wrote them again, in reverse order:

$$100 \quad 99 \quad 98 \quad \ldots \quad 3 \quad 2 \quad 1$$

Then he added the two rows, column by column:

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 2 | 3 | ... | 98 | 99 | 100 |
| 100 | 99 | 98 | ... | 3 | 2 | 1 |
| — — — | — — — | — — — | — — — | — — — | — — — | — — — |
| 101 | 101 | 101 | ... | 101 | 101 | 101 |

Clearly, the total for the two rows is 10,100.

But every number from 1 to 100 is counted twice.

$$\therefore \; 1 + 2 + 3 + \cdots + 98 + 99 + 100 = 5{,}050$$

# Triangular Numbers

**Gauss had calculated the 100-th triangular number.**

**Let us take a geometrical look at the sums of the first few natural numbers:**

1 •

1 •
2 • •

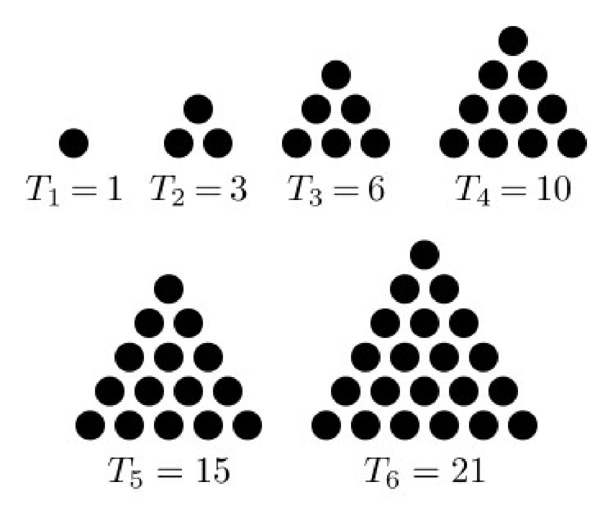
1 •
2 • •
3 • • •

1 •
2 • •
3 • • •
4 • • • •

**We see that the sums can be arranged as triangles.**

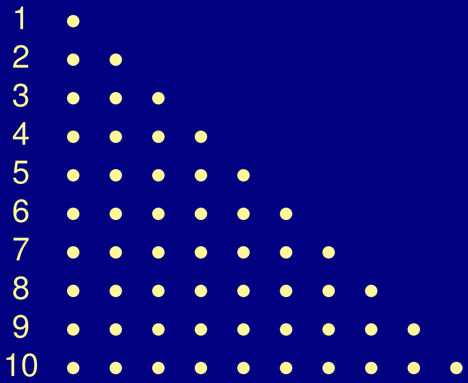
# Triangular Numbers

**The first few triangular numbers are $\{1, 3, 6, 10, 15, 21\}$.**



**Let's look at the 10th triangular number.**

**For $n = 10$ the pattern is:**




**How do we compute its value? Gauss's method!**

## It is easy to show that the *n*-th triangular number is

$$T_n = (1 + 2 + 3 + \cdots + n) = \tfrac{1}{2}n(n+1)$$

**It is easy to show that the *n*-th triangular number is**

$$T_n = (1 + 2 + 3 + \cdots + n) = \tfrac{1}{2}n(n+1)$$

**We do just as Gauss did, and list the numbers twice:**

$$\begin{array}{cccccc} 1 & 2 & 3 & \ldots & n-1 & n \\ n & n-1 & n-2 & \ldots & 2 & 1 \\ --- & --- & --- & \ldots & --- & --- \\ n+1 & n+1 & n+1 & \ldots & n+1 & n+1 \end{array}$$

**It is easy to show that the *n*-th triangular number is**

$$T_n = (1 + 2 + 3 + \cdots + n) = \tfrac{1}{2}n(n+1)$$

**We do just as Gauss did, and list the numbers twice:**

$$\begin{array}{cccccc}
1 & 2 & 3 & \ldots & n-1 & n \\
n & n-1 & n-2 & \ldots & 2 & 1 \\
--- & --- & --- & \ldots & --- & --- \\
n+1 & n+1 & n+1 & \ldots & n+1 & n+1
\end{array}$$

**There are $n$ columns, each with total $n+1$.**

**So the grand total is $n \times (n+1)$.**

It is easy to show that the $n$-th triangular number is

$$T_n = (1 + 2 + 3 + \cdots + n) = \frac{1}{2} n(n+1)$$

We do just as Gauss did, and list the numbers twice:

$$\begin{array}{cccccc} 1 & 2 & 3 & \dots & n-1 & n \\ n & n-1 & n-2 & \dots & 2 & 1 \\ --- & --- & --- & \dots & --- & --- \\ n+1 & n+1 & n+1 & \dots & n+1 & n+1 \end{array}$$

There are $n$ columns, each with total $n+1$.

So the grand total is $n \times (n+1)$.

Each number has been counted twice, so

$$T_n = \frac{1}{2} n(n+1)$$

**Let's check this for Gauss's problem of $n = 100$:**

$$T_{100} = (1 + 2 + 3 + \cdots + 100) = \frac{100 \times 101}{2} = 5,050$$

**Let's check this for Gauss's problem of $n = 100$:**

$$T_{100} = (1 + 2 + 3 + \cdots + 100) = \frac{100 \times 101}{2} = 5,050$$

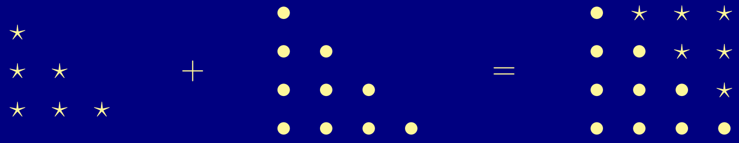
**Gauss's approach was to look at the problem from a new angle.**

**Such *lateral thinking* is very common in mathematics:**

**Problems that look difficult can sometimes be solved easily when tackled from a different angle.**

# Two Triangles Make a Square

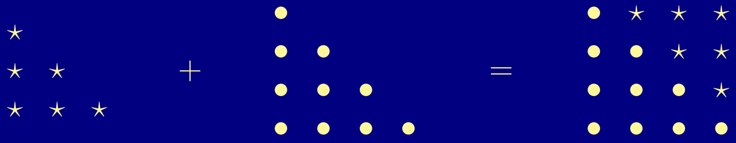
## A nice property of *consecutive* triangular numbers:



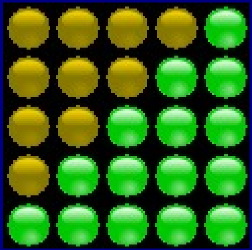
$$T_3 + T_4 = 6 + 10 = 16 = 4^2$$

# Two Triangles Make a Square

## A nice property of *consecutive* triangular numbers:



$$T_3 + T_4 = 6 + 10 = 16 = 4^2$$



# Triangular Numbers

We have seen, by means of geometry that the sum of two consecutive triangular numbers is a square.

Now let us prove this algebraically:

# Triangular Numbers

**We have seen, by means of geometry that the sum of two consecutive triangular numbers is a square.**

**Now let us prove this algebraically:**

$$\begin{aligned} T_n + T_{n+1} &= \tfrac{1}{2}n(n+1) + \tfrac{1}{2}(n+1)(n+2) \\ &= \tfrac{1}{2}(n+1)[n+(n+2)] \\ &= \tfrac{1}{2}(n+1)[2(n+1)] \\ &= (n+1)^2 \end{aligned}$$

# Triangular Numbers

We have seen, by means of geometry that the sum of two consecutive triangular numbers is a square.

Now let us prove this algebraically:

$$\begin{aligned} T_n + T_{n+1} &= \tfrac{1}{2}n(n+1) + \tfrac{1}{2}(n+1)(n+2) \\ &= \tfrac{1}{2}(n+1)[n+(n+2)] \\ &= \tfrac{1}{2}(n+1)[2(n+1)] \\ &= (n+1)^2 \end{aligned}$$

The result is a perfect square.

I leave you to check the details!

# Puzzle

**What is the sum of all the numbers from 1 up to 100 and back down again?**

# Puzzle

What is the sum of all the numbers
from 1 up to 100 and back down again?

The answer is in the video coming up now.

# Videos from the Museum of Mathematics


**VIDEOS on Beautiful Maths available at**

`http://momath.org/home/beautifulmath/`

**Video by James Tanton**

# The Teacher's Revenge

The teacher thought that he would have a half-hour of peace and quiet while the pupils grappled with the problem of adding up the first 100 numbers.

But he managed to outsmart the great Gauss.

# The Teacher's Revenge

The teacher thought that he would have a half-hour of peace and quiet while the pupils grappled with the problem of adding up the first 100 numbers.

But he managed to outsmart the great Gauss.

He was annoyed when Gauss came up almost immediately with the correct answer 5,050.

"So, you think you are so smart!
Multiply the first 100 numbers."

# The Teacher's Revenge

The teacher thought that he would have a half-hour of peace and quiet while the pupils grappled with the problem of adding up the first 100 numbers.

But he managed to outsmart the great Gauss.

He was annoyed when Gauss came up almost immediately with the correct answer 5,050.

"So, you think you are so smart!
Multiply the first 100 numbers."

CHALLENGE: Think about that! Can you do it?

# Outline

# Prime & Composite Numbers

**A prime number** is a number that cannot be broken into a product of smaller numbers.

The first few primes are 2, 3, 5, 7, 11, 13, 17 and 19.

There are 25 primes less than 100.

# Prime & Composite Numbers

A prime number is a number that cannot be broken into a product of smaller numbers.

The first few primes are 2, 3, 5, 7, 11, 13, 17 and 19.

There are 25 primes less than 100.

Numbers that are not prime are called composite. They can be expressed as products of primes.

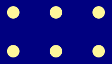
Thus, $6 = 2 \times 3$ is a composite number.

The number $1$ is neither prime nor composite.

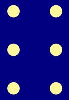
# Primes: Atoms of the Number System

A line of six spots

can be arranged in a rectangular array:



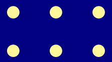
or



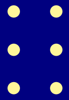
# Primes: Atoms of the Number System

A line of six spots

can be arranged in a rectangular array:



or



Note that

$$2 \times 3 = 3 \times 2$$

This is the **commutative law of multiplication.**

# The Atoms of the Number System

The primes play a role in mathematics analogous to the elements of Mendeleev's Periodic Table.

Just as a chemical molecule can be constructed from the 100 or so fundamental elements, any whole number be constructed by combining prime numbers.

The primes 2, 3, 5 are the hydrogen, helium and lithium of the number system.

# Some History

In 1792 Carl Friedrich Gauss, then only 15 years old, found that the proportion of primes less that $n$ decreased approximately as $1/\log n$.

Around 1795 Adrien-Marie Legendre noticed a similar logarithmic pattern of the primes, but it was to take another century before a proof emerged.

In a letter written in 1823 the Norwegian mathematician Niels Henrik Abel described the distribution of primes as *the most remarkable result in all of mathematics.*

# Percentage of Primes Less than $N$

**Table:** Percentage of Primes less than $N$

| | | |
|---|---|---|
| 10 | 4 | 40.0% |
| 100 | 25 | 25.0% |
| 1,000 | 168 | 16.8% |
| 1,000,000 | 78,498 | 7.8% |
| 1,000,000,000 | 50,847,534 | 5.1% |
| 1,000,000,000,000 | 37,607,912,018 | 3.8% |

**We can see that the percentage of primes is falling off with increasing size.**

**But the rate of decrease is very slow.**

# Prime counting function for $0 \leq n \leq 50$.

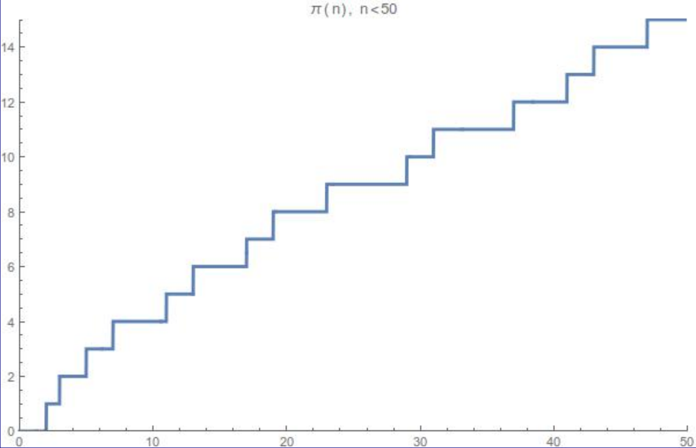



**Figure:** The prime counting function $\pi(n)$ for $0 \leq n \leq 50$.

# Prime counting function for $0 \leq n \leq 500$.

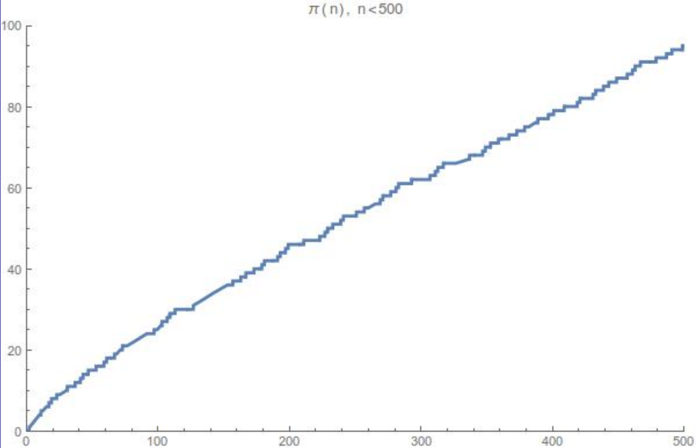



**Figure:** The prime counting function $\pi(n)$ for $0 \leq n \leq 500$.

# Is There a Pattern in the Primes?

It is a simple matter to make a list of all the primes less that 100 or 1000.

It soon becomes evident that no clear pattern is emerging.

The primes appear to be scattered at random.

**Figure:** Prime numbers up to 100

# Is There a Pattern in the Primes?

Do the primes settle down as $n$ becomes larger?

Between 9,999,900 and 10,000,000
(100 numbers) there are 9 primes.

Between 10,000,000 and 10,000,100
(100 numbers) there are just 2 primes.

# Is There a Pattern in the Primes?

Do the primes settle down as $n$ becomes larger?

Between 9,999,900 and 10,000,000
(100 numbers) there are 9 primes.

Between 10,000,000 and 10,000,100
(100 numbers) there are just 2 primes.

What kind of function could generate this behaviour?

We just do not know.

# Is There a Pattern in the Primes?

The gaps between primes are very irregular.

- Can we find a pattern in the primes?
- Can we find a formula that generates primes?
- How can we determine the hundreth prime?
- What is the thousandth? The millionth?

# WolframAlpha©

**WolframAlpha is a Computational Knowledge Engine.**

# WolframAlpha©

WolframAlpha is a Computational Knowledge Engine.

Wolfram Alpha is based on Wolfram's flagship product Mathematica, a computational platform or toolkit that encompasses computer algebra, symbolic and numerical computation, visualization, and statistics.

# WolframAlpha©

WolframAlpha is a Computational Knowledge Engine.

Wolfram Alpha is based on Wolfram's flagship product Mathematica, a computational platform or toolkit that encompasses computer algebra, symbolic and numerical computation, visualization, and statistics.

WolframAlpha is freely available via a web browser.

# Euler's Formula for Primes

**No mathematician has ever found a useful formula that generates all the prime numbers.**

**Euler found a beautiful little formula:**

$$n^2 - n + 41$$

**This gives prime numbers for $n$ between $1$ and $40$.**

# Euler's Formula for Primes

No mathematician has ever found a useful formula that generates all the prime numbers.

Euler found a beautiful little formula:

$$n^2 - n + 41$$

This gives prime numbers for $n$ between $1$ and $40$.

But for $n = 41$ we get

$$41^2 - 41 + 41 = 41 \times 41$$

a composite number.

# The Infinitude of Primes

Euclid proved that there is no finite limit to the number of primes.

His proof is a masterpiece of symplicity.

(See Dunham book or Wikipedia: *Euclid's Theorem.*)

# Some Unsolved Problems

**There appear to be an infinite number of prime pairs**

$$(2n-1, 2n+1)$$

**There are also gaps of arbitrary length:**

**for example, there are 13 consecutive composite numbers between 114 and 126.**

# Some Unsolved Problems

**There appear to be an infinite number of prime pairs**

$$(2n-1, 2n+1)$$

**There are also gaps of arbitrary length:**

**for example, there are 13 consecutive composite numbers between 114 and 126.**

**We can find gaps as large as we like:**

**Challenge: Show that *N! + 1* is followed by a sequence of *N - 1* composite numbers.**

## Primes have been used as markers of civilization.

In the novel *Cosmos*, by Carl Sagan,
the heroine detects a signal:

- First 2 pulses
- Then 3 pulses
- Then 5 pulses
- ...
- Then 907 pulses.

In each case, a prime number of pulses.

**Primes have been used as markers of civilization.**

In the novel *Cosmos*, by Carl Sagan,
the heroine detects a signal:

- First 2 pulses
- Then 3 pulses
- Then 5 pulses
- …
- Then 907 pulses.

In each case, a prime number of pulses.

**Could this be due to any natural phenomenon?**
**Is it evidence of extra-terrestrial intelligence?**

# Which Primes are Sums of Squares?

```
(* PRINT THE FIRST 100 PRIME NUMBERS  *)
primes = {};
For[i = 1, i < 100, i ++, AppendTo[primes, Prime[i]]]
Print["PRIMES"]
primes
PRIMES
Out[51]= {2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43,
 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, 101,
 103, 107, 109, 113, 127, 131, 137, 139, 149, 151,
 157, 163, 167, 173, 179, 181, 191, 193, 197, 199,
 211, 223, 227, 229, 233, 239, 241, 251, 257, 263,
 269, 271, 277, 281, 283, 293, 307, 311, 313, 317,
 331, 337, 347, 349, 353, 359, 367, 373, 379, 383,
 389, 397, 401, 409, 419, 421, 431, 433, 439, 443,
 449, 457, 461, 463, 467, 479, 487, 491, 499, 503,
 509, 521, 523}
(* PRINT THE FIRST 100 SQUARE NUMBERS  *)
squares = {};
```

UCD DUBLIN

# Which Primes are Sums of Squares?

```
509, 521, 523}
(* PRINT THE FIRST 100 SQUARE NUMBERS *)
squares = {};
For[i = 1, i < 25, i ++, AppendTo[squares, i ^ 2]]
Print["SQUARES"]
squares
SQUARES
Out[63]= {1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121,
 144, 169, 196, 225, 256, 289, 324, 361, 400,
 441, 484, 529, 576}
Prime[1 000 000 000]
Out[69]= 22 801 763 489
```

# Which Primes are Sums of Squares?

A Theorem of Fermat states that:

A prime number n may be expressed as a sum of squares if and only if

$$p \equiv 1 \pmod{4}$$

# Which Primes are Sums of Squares?

A Theorem of Fermat states that:

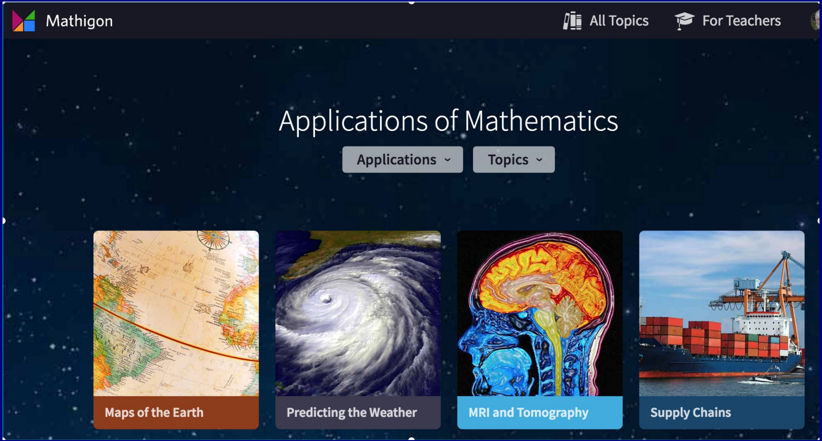
A prime number n may be expressed as
a sum of squares if and only if

$$p \equiv 1 \pmod 4$$

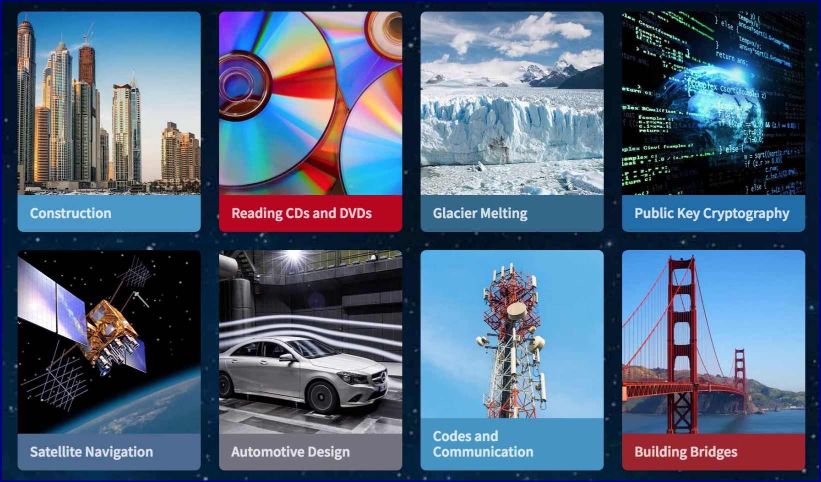
In plain language, if $n$ divided by 4 has remainder 1.

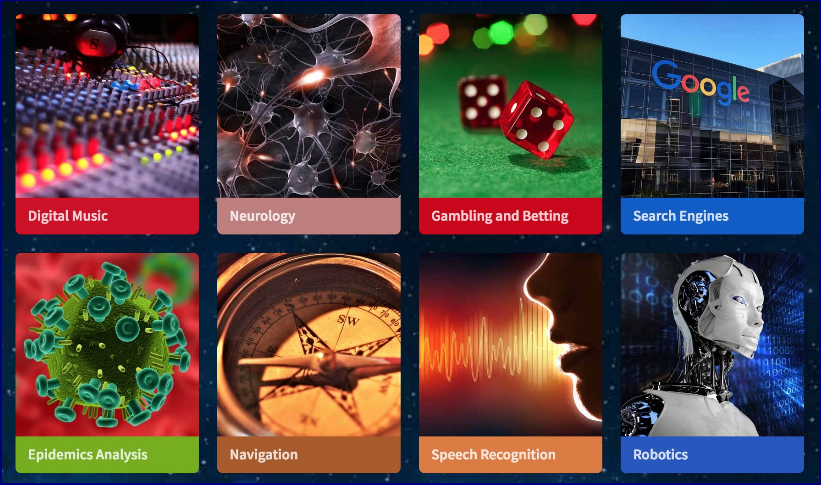
# Outline

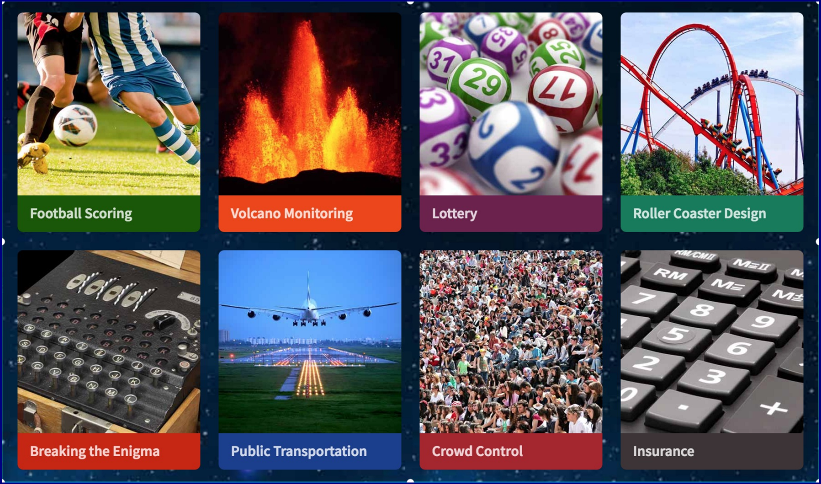
# Applications on mathigon.org

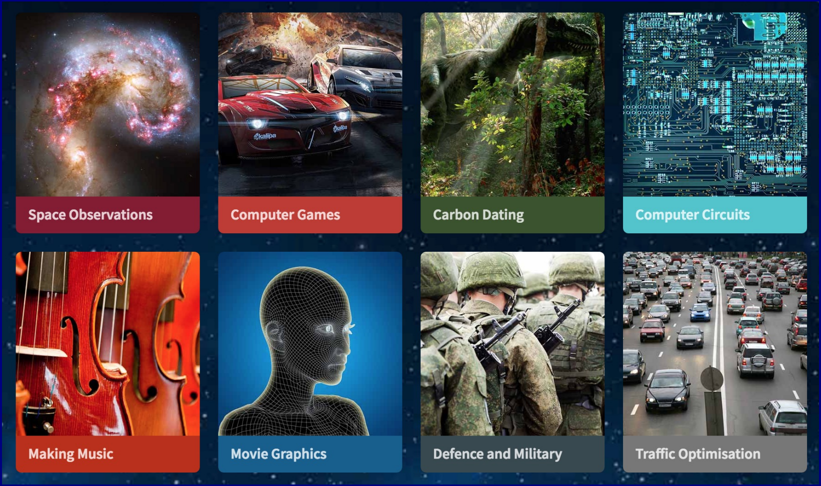


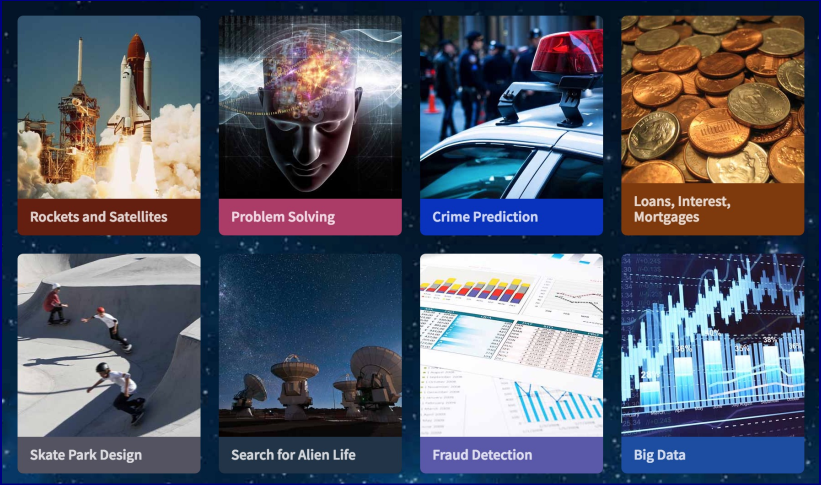

Maps of the Earth
Predicting the Weather
MRI and Tomography
Supply Chains
Finance and Banking
Internet and Phones
Cosmology
Computers

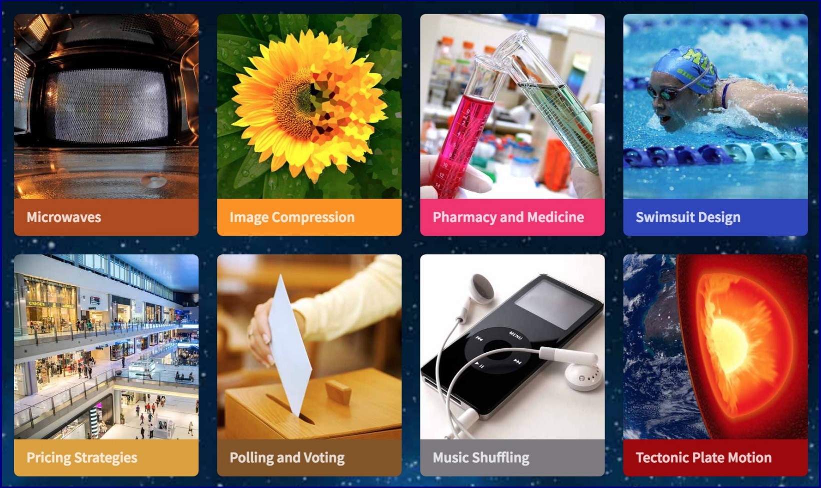

Construction
Reading CDs and DVDs
Glacier Melting
Public Key Cryptography
Satellite Navigation
Automotive Design
Codes and Communication
Building Bridges

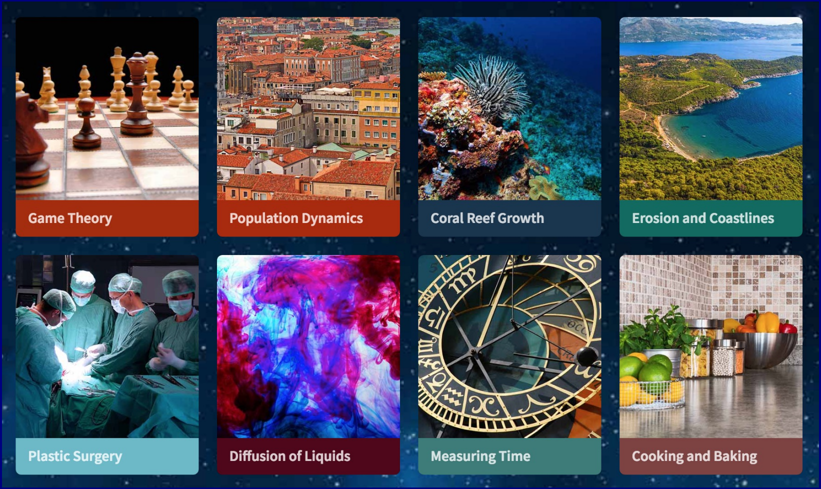

Google
Digital Music
Neurology
Gambling and Betting
Search Engines
Epidemics Analysis
Navigation
Speech Recognition
Robotics

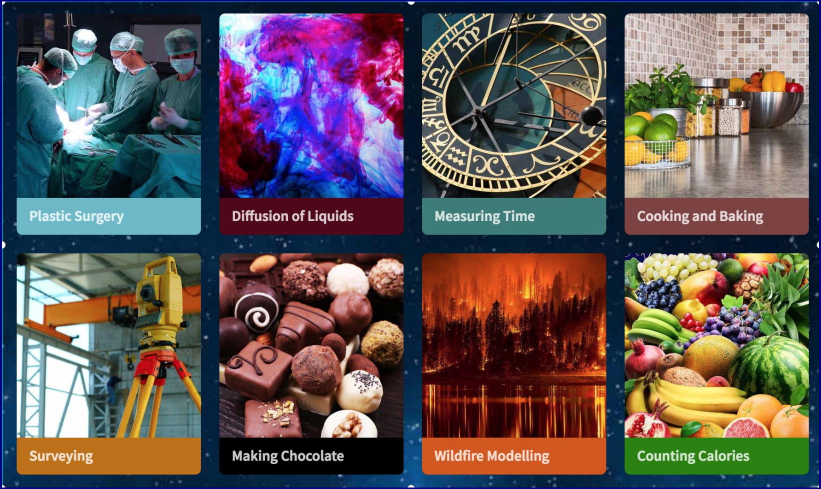

Football Scoring
Volcano Monitoring
Lottery
Roller Coaster Design
Breaking the Enigma
Public Transportation
Crowd Control
Insurance


Space Observations
Computer Games
Carbon Dating
Computer Circuits
Making Music
Movie Graphics
Defence and Military
Traffic Optimisation


Rockets and Satellites
Problem Solving
Crime Prediction
Loans, Interest, Mortgages
Skate Park Design
Search for Alien Life
Fraud Detection
Big Data


Microwaves
Image Compression
Pharmacy and Medicine
Swimsuit Design
Pricing Strategies
Polling and Voting
Music Shuffling
Tectonic Plate Motion


Game Theory
Population Dynamics
Coral Reef Growth
Erosion and Coastlines
Plastic Surgery
Diffusion of Liquids
Measuring Time
Cooking and Baking


Plastic Surgery
Diffusion of Liquids
Measuring Time
Cooking and Baking
Surveying
Making Chocolate
Wildfire Modelling
Counting Calories

# Outline

# The Irish Lotto Rollover

News | Irish News

The 'unwinnable' Lotto jackpot rolls over again for a 47th time

By **Lisa ODonnell** - 22/11/2021

# What is Randomness?

Randomness is a slippery concept,
defying precise definition.

Toss a coin and get a sequence like HTTHHHTHTT.
We can write this as a binary string 1001110100.

# What is Randomness?

Randomness is a slippery concept,
defying precise definition.

Toss a coin and get a sequence like HTTHHHTHTT.
We can write this as a binary string 1001110100.

Some uses of Random Numbers:

- Computer simulations of fluid flow.
- Interactions of subatomic particles.
- Evolution of galaxies.

To get random numbers, coin tossing is impractical.
We need more effective methods.

# Defining Randomness?

**Richard von Mises (1919):**
**A binary sequence is random if the proportion of zeros and ones approaches 50% and if this is also true for any sub-sequence.**
**Consider ( 0101010101 ).**

# Defining Randomness?

**Richard von Mises (1919):**
A binary sequence is random if the proportion of zeros and ones approaches 50% and if this is also true for any sub-sequence.
Consider ( 0101010101 ).

**Andrey Kolmogorov** defined the complexity of a binary sequence as the length of a computer program or algorithm that generates it.

The phrase a sequence of one million 1's completely defines a sequence.

**Non-random sequences are compressible.**
Randomness and incompressibility are equivalent.

# Pseudo-random versus Truly Random

**Pseudo-random number generators** are algorithms that use mathematical formulae to produce sequences of numbers.

The sequences appear completely random and satisfy various statistical conditions for randomness.

**Pseudo-random numbers** are valuable for many applications but they have serious difficiencies.

# Pseudo-random Number Generators

Start with a 20 digit number, the seed:

12345678901234567890

Calculate the square of the number.

Discard the first 10 and last 10 digits,
to get the 20 central digits.

Repeat this process as often as desired.

# Truly Random Number Generators

True random number generators extract randomness from physical phenomena that are completely unpredictable.

Atmospheric noise is the static generated by lightning [globally there are 40 flashes/sec]. It can be detected by an ordinary radio.

# Truly Random Number Generators

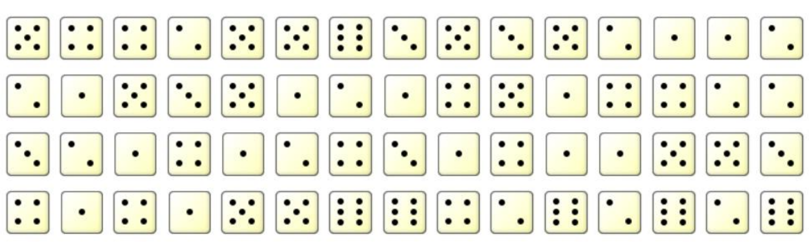
Atmospheric noise passes all the statistical checks for randomness.

Dr Mads Haahr of Trinity College, Dublin uses atmospheric noise to produce random numbers.

Results available on on the website: `random.org`.

# 20 Random Coin Tosses

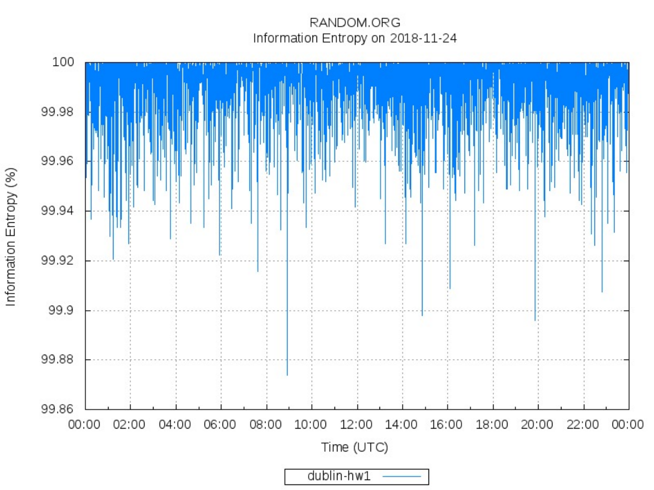
# 60 Dice Rolls



# 100 Random Numbers in [0,99]

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 17 | 60 | 57 | 66 | 4 | 71 | 59 | 36 | 8 | 49 |
| 87 | 64 | 94 | 82 | 6 | 38 | 14 | 87 | 76 | 72 |
| 97 | 38 | 44 | 59 | 56 | 24 | 20 | 6 | 24 | 97 |
| 0 | 40 | 14 | 77 | 18 | 98 | 41 | 39 | 6 | 79 |
| 21 | 59 | 49 | 86 | 91 | 81 | 65 | 64 | 3 | 11 |
| 92 | 17 | 65 | 6 | 37 | 98 | 84 | 17 | 70 | 93 |
| 60 | 52 | 1 | 98 | 20 | 2 | 65 | 9 | 57 | 3 |
| 48 | 86 | 27 | 3 | 71 | 51 | 57 | 56 | 2 | 2 |
| 13 | 14 | 73 | 65 | 11 | 32 | 17 | 7 | 91 | 37 |
| 3 | 8 | 10 | 67 | 0 | 72 | 0 | 42 | 15 | 24 |

# Quality of Random Numbers

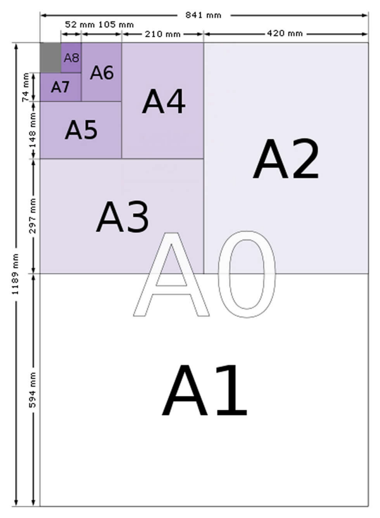


# Pseudo-RNG versus True-RNG

| Characteristic | Pseudo-Random Number Generators | True Random Number Generators |
|---|---|---|
| Efficiency | Excellent | Poor |
| Determinism | Determinstic | Nondeterministic |
| Periodicity | Periodic | Aperiodic |

# Outline

# Standard Paper Sizes




Standard sizes of A-series paper.

The ratio of heights to widths is always $\sqrt{2}$.

# Making a Square

The standard sizes of paper are designed so that each has the same shape (or aspect ratio), and the largest, A0, has an area of one square metre.

**PUZZLE:**
**Is it possible to form a square out of sheets of A4 sized paper (without them overlapping)?**

# Outline

# Topology: a Major Branch of Mathematics

Topology is all about continuity and connectivity.

Here are some of the topics in Topology:

- The Bridges of Königsberg
- Doughnuts and Coffee-cups
- Knots and Links
- Nodes and Edges: Graphs
- The Möbius Band

In this lecture, we look at Knots and Links.

# Pretzel Puzzle

**Look at the two "pretzels" here:**

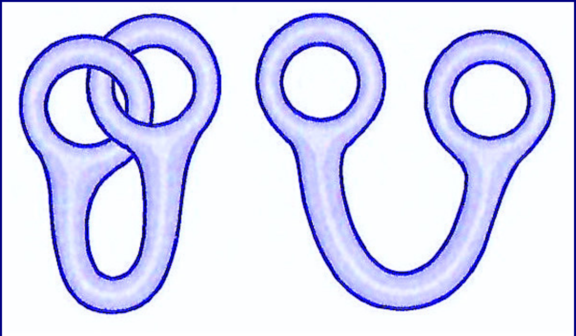


**Figure:** Two "Pretzels". Are they equivalent?

# Knot Theory

A knot is an embedding of the unit circle $S^1$ into three-dimensional space $R^3$.

Two knots are equivalent if one can be distorted into the other without breaking it.

## A knot is a mapping of the unit circle into three-space.

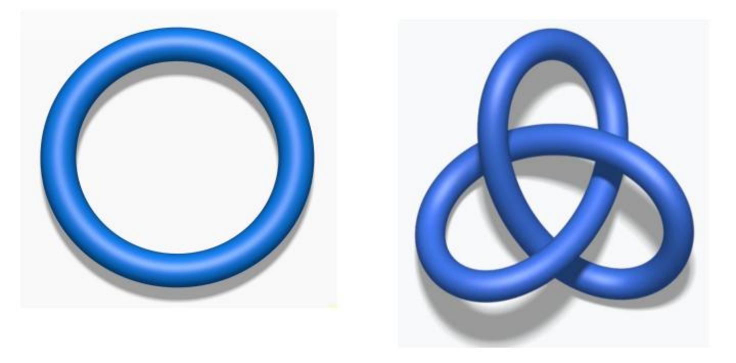


**Figure:** Left: Unknot. Right: Trefoil.

**These two knots aren't equivalent: we can't distort the circle into the trefoil without breaking it.**

# Knots that are Mirror Images

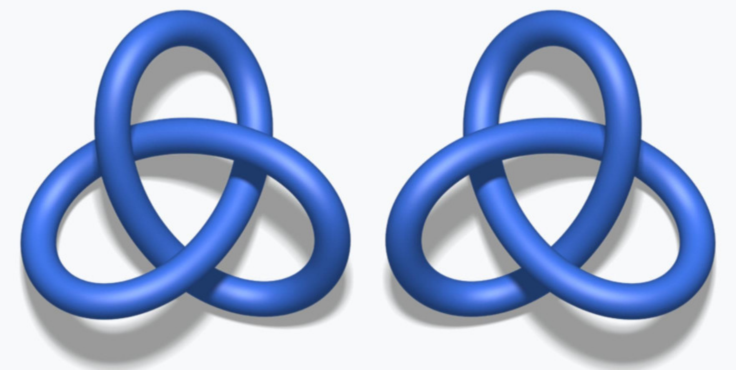


**Figure:** Levo and Dextro Trefoils.

**These knots are not equivalent. We cannot change one to the other without breaking it.**

# The Trefoil is a Chiral Knot

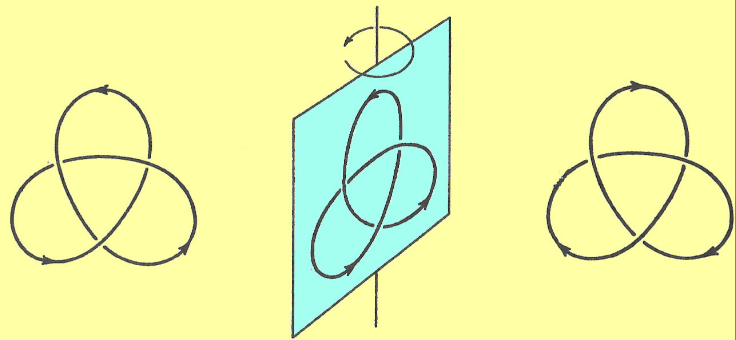


**Figure:** Turn round: no change [direction changes!]

*Eadem mutata resurgo* (Jakob Bernoulli

# Figure-of-Eight: an Achiral Knot

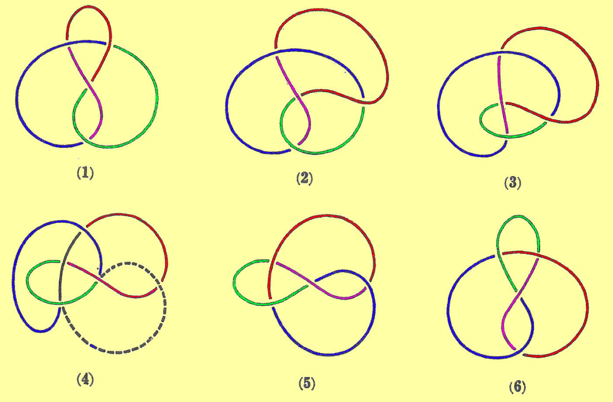



**Figure:** Can be changed to its mirror image.

# Figure-of-Eight: an Achiral Knot

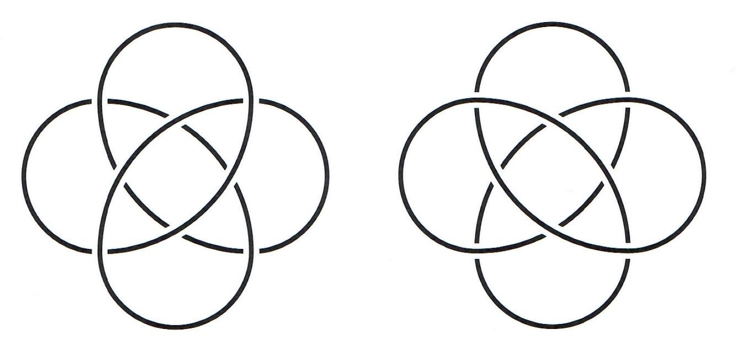


**Figure:** Can be changed to its mirror image.

See p. 15, Cromwell, *Knots and Links*.

# The Simplest Knots and Links

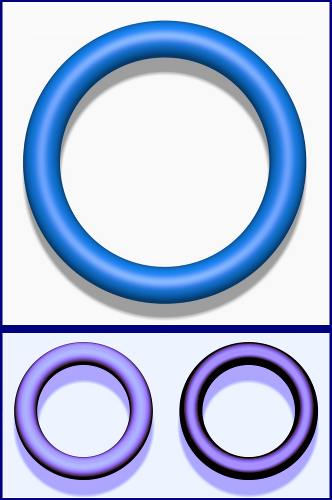


Figure: Top: The Unknot. Bottom: The Unlink.

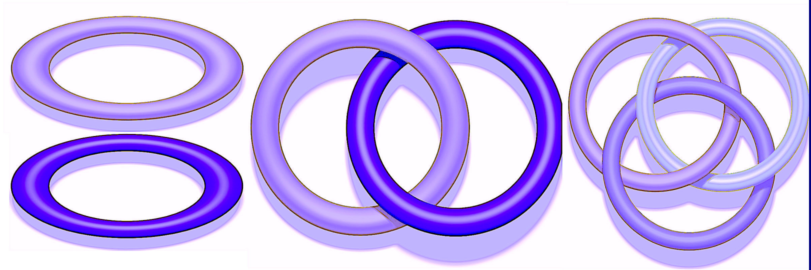


**Figure:** Unlink, Hopf Link and Borromean Rings.

# The Hopf Link

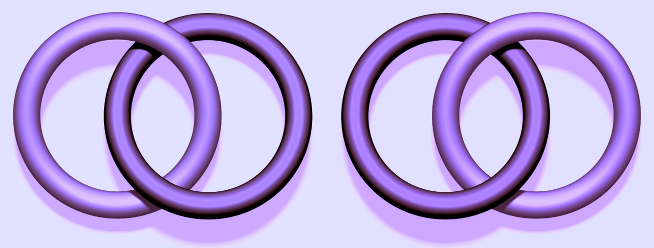


**Figure:** The Hopf Link and its mirror image. Equivalent?

# Rings of Borromeo

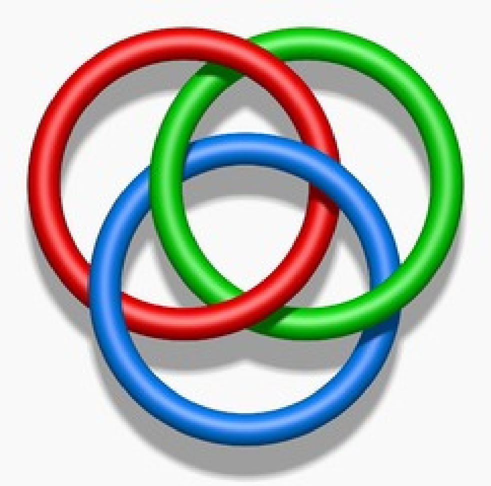


**Figure:** No two rings are linked! Are the three?

# Genus of a Surface

The genus of a topological surface is,
in simple terms, the number of holes in it.

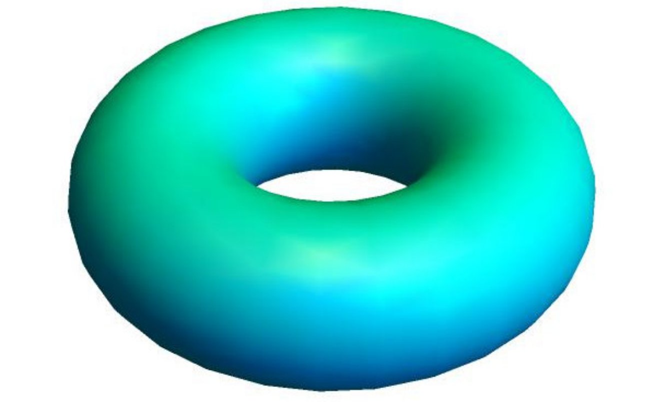
A sphere has no holes, so has genus 0.

A doughnut has one hole, so has genus 1.

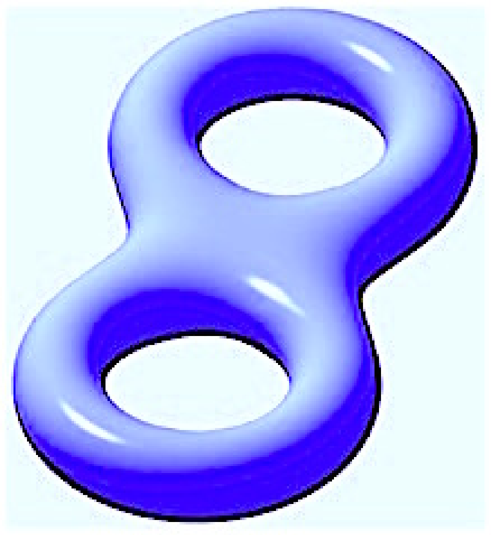
Surfaces can have any number of holes; any genus.

# The Sphere, of Genus 0

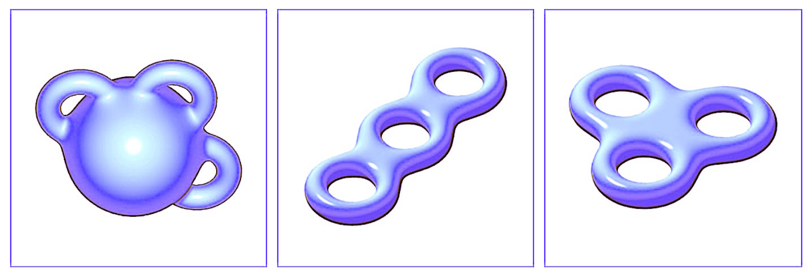
# The Torus, of Genus 1



# The Double Torus, of Genus 2



# Some Surfaces of Genus 3



**Topologists have classified all surfaces in 3-space.**

# Triple Torus

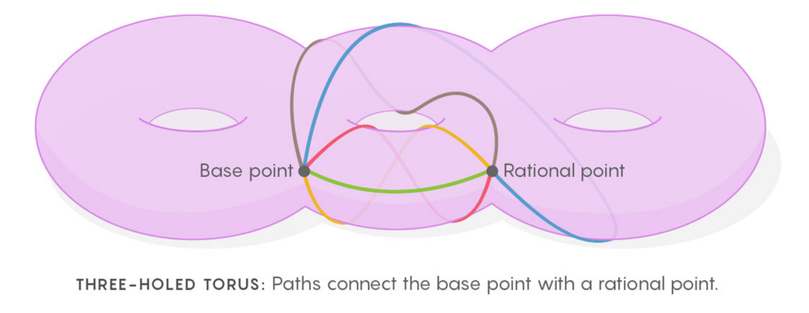
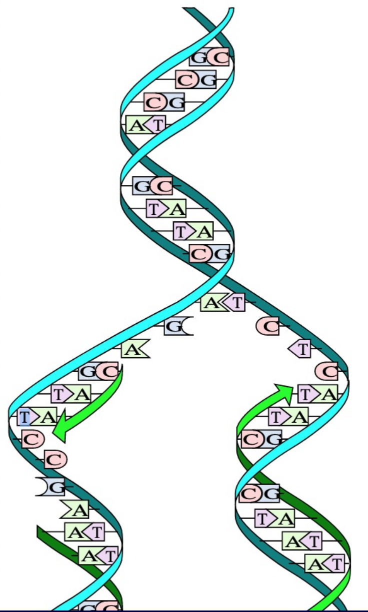



**Figure:** Rational solutions of $x^4 + y^4 = 1$ are on this surface

# DNA Double Helix: Replication



# Topology and DNA Replication

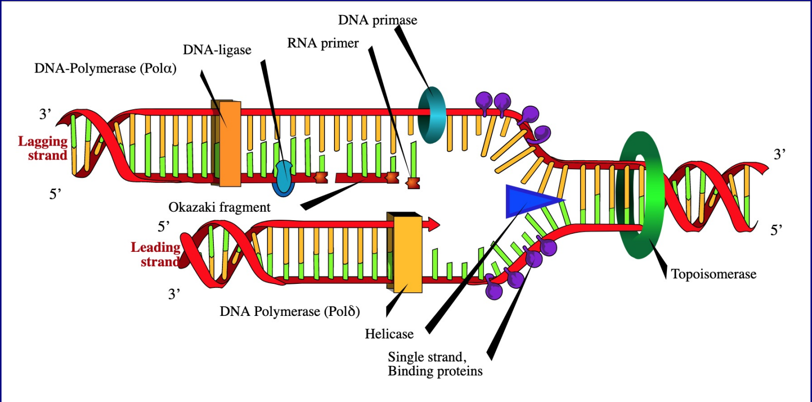



**Figure:** Proteins: Masters of Topology.

# Pretzel Puzzle

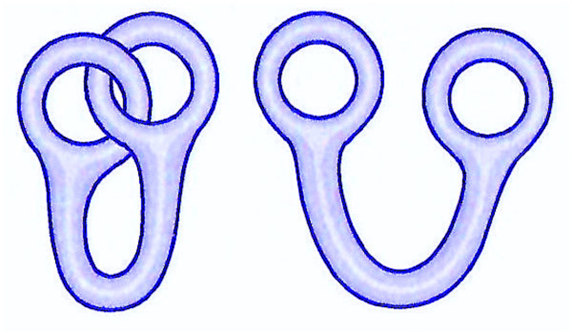


**Figure:** Two "Pretzels". Are they equivalent?

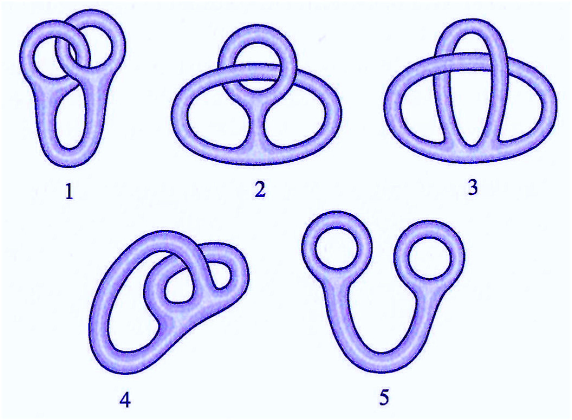



**Figure:** Equivalence!

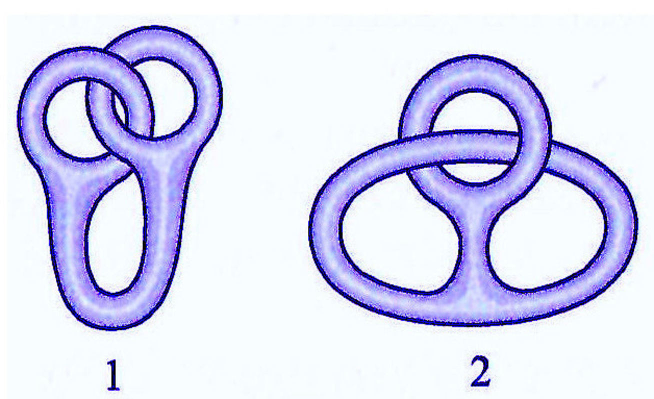



**Figure:** Make the left-hand loop bigger.

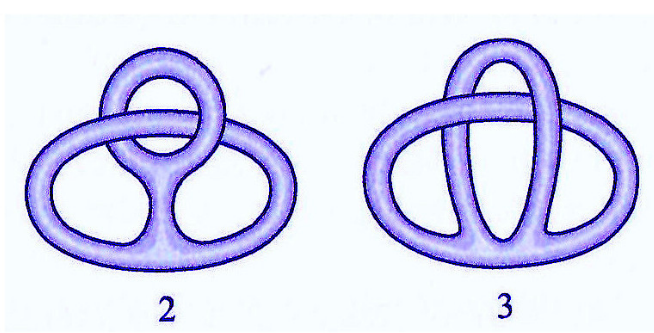



**Figure:** Make the other loop bigger.

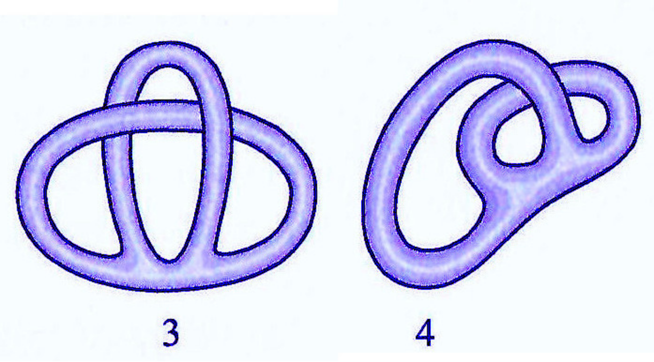



**Figure:** Pull the top loop away to the side.

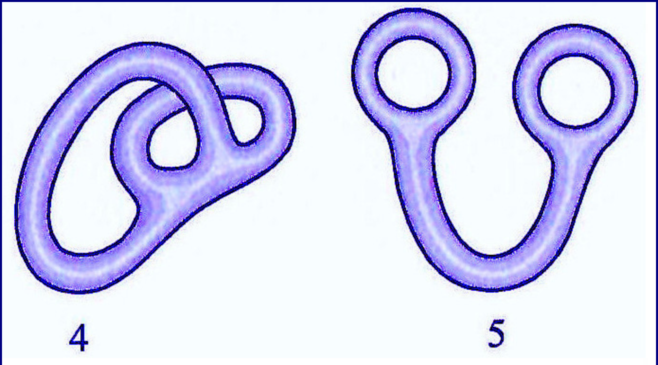



**Figure:** Smoothly distort to the final form.

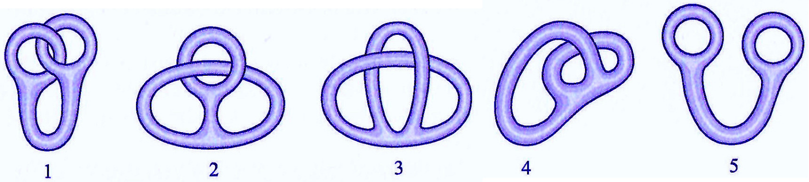



**Figure:** Combining all the distortions. Equivalence!

# Another Surprising Result

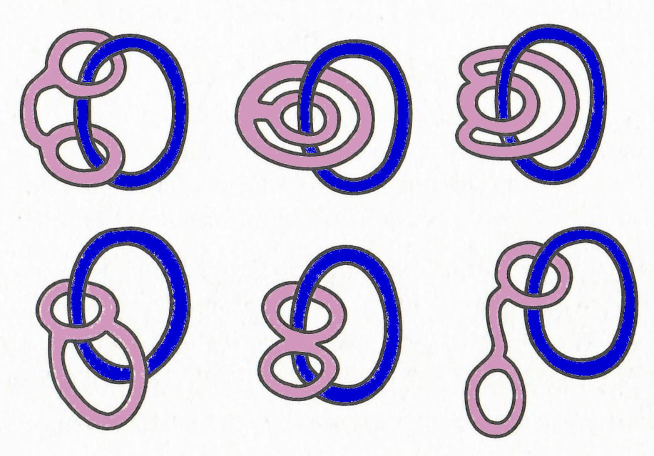


**Figure:** We can unlink one of the hand-cuffs.

Thank you