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Marvels and Mysteries of Mathematics

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LECTURE 6

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Outline

Outline

Meaning and Content of Mathematics

The word Mathematics comes from Greek $\mu\alpha\theta\eta\mu\alpha$ (máthéma), meaning "knowledge" or "lesson" or "learning".

It is the study of topics such as

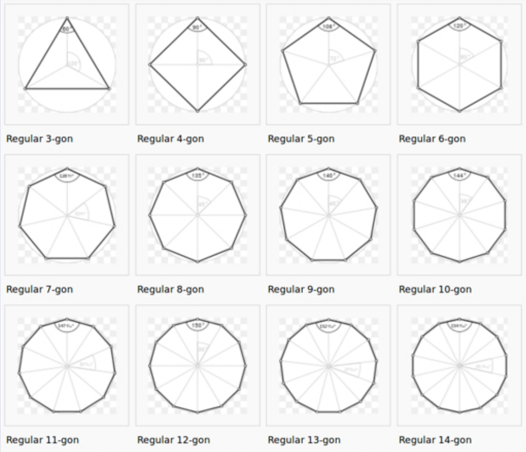
- Quantity: [Numbers. Arithmetic]
- Structure: [Patterns. Algebra]
- Space: [Geometry. Topology]
- Change: [Analysis. Calculus]

Outline

Euler's polyhedron formula.

Carving up the globe.

Regular Polygons



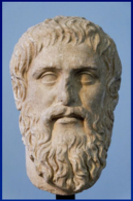
The Platonic Solids (polyhedra)

Tetrahedron (four faces)	Cube or hexahedron (six faces)	Octahedron (eight faces)	Dodecahedron (twelve faces)	Icosahedron (twenty faces)
				

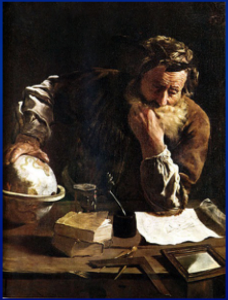
These five regular polyhedra were discovered in ancient Greece, perhaps by Pythagoras.

Plato used them as models of the universe.

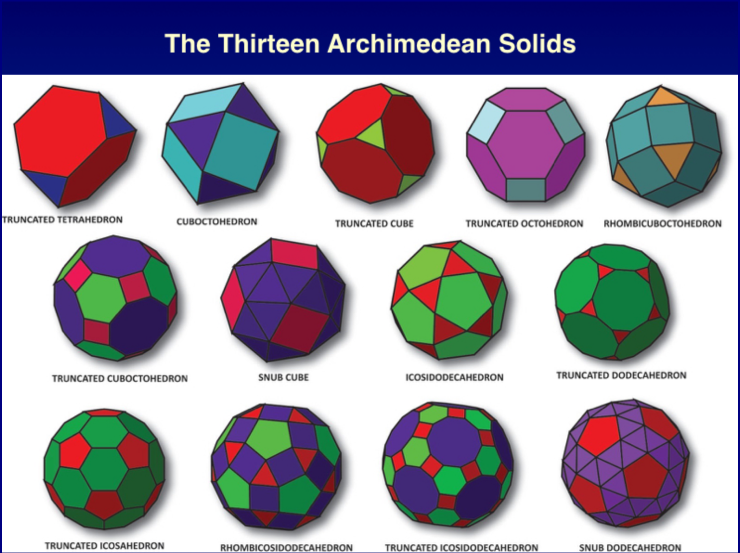
They are analysed in Book XIII of Euclid's Elements.



There are only five **Platonic** solids.



But **Archimedes** found, using different types of polygons, that he could construct 13 new solids.



Check V - E + F for the Truncated Cube

Euler's Polyhedron Formula

The great Swiss mathematician, **Leonard Euler**, noticed that, for all (convex) polyhedra,

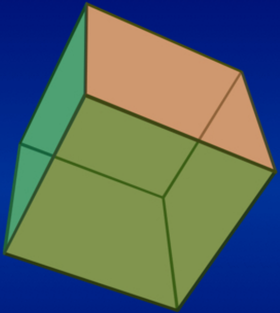
$$V - E + F = 2$$

where

- V = Number of vertices
- E = Number of edges
- F = Number of faces

Mnemonic: *Very Easy Formula*

For example, a Cube

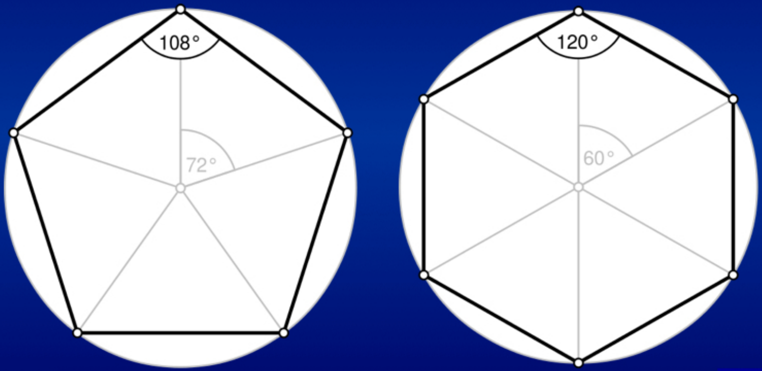


Number of vertices: V = 8
Number of edges: E = 12
Number of faces: F = 6

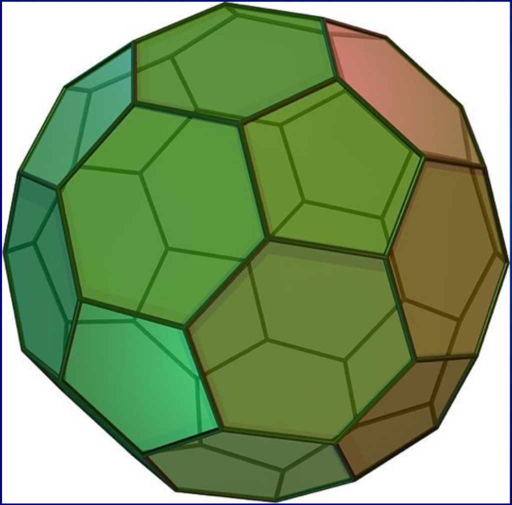
$$(V - E + F) = (8 - 12 + 6) = 2$$

Mnemonic: ***V**ery **E**asy **F**ormula*

Pentagons and Hexagons



The Truncated Icosahedron

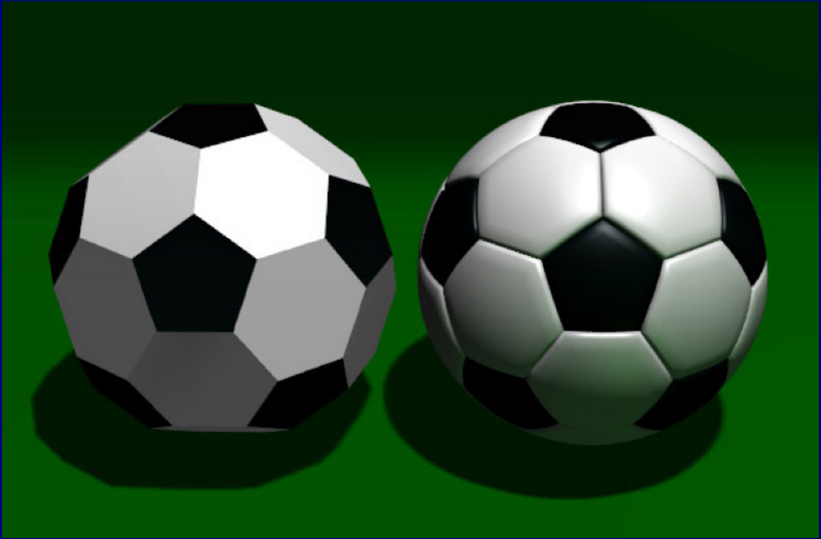


An Archimedean solid with pentagonal and hexagonal faces.

The Truncated Icosahedron

Whare have you seen this before?

The Truncated Icosahedron



The "Buckyball", introduced at the 1970 World Cup Finals in Mexico.

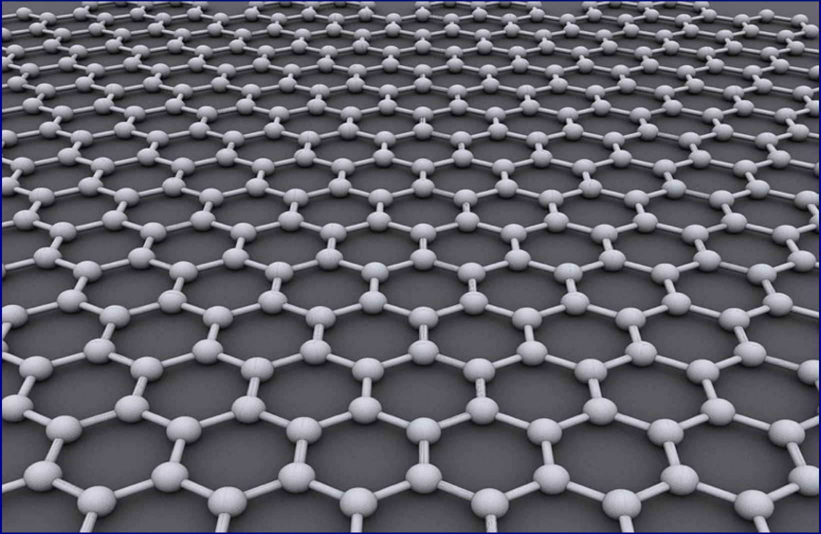
It has 32 panels: 20 hexagons and 12 pentagons.

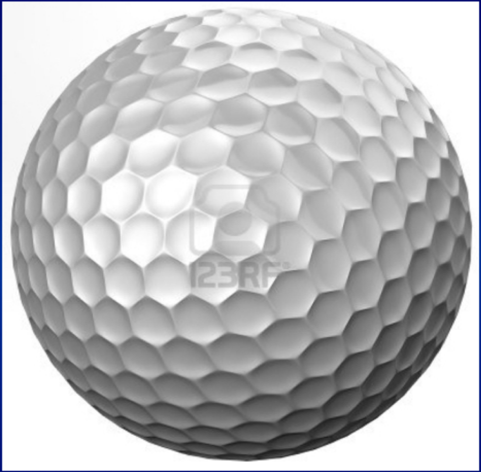

A Geodesic Dome designed by the American architect
Richard Buckminster "Bucky" Fuller.

Buckminsterfullerene is a molecule with formula C_{60}

It was first synthesized in 1985.

Graphene
A hexagonal pattern of carbon one atom thick

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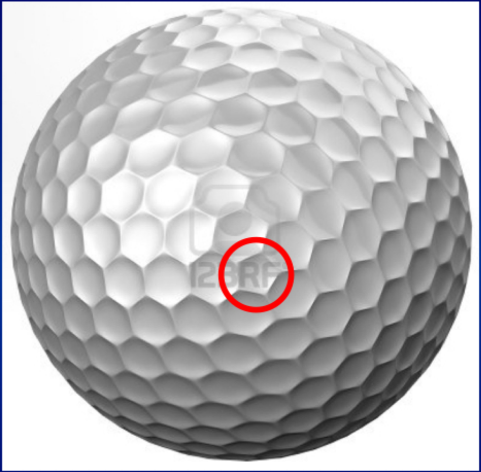
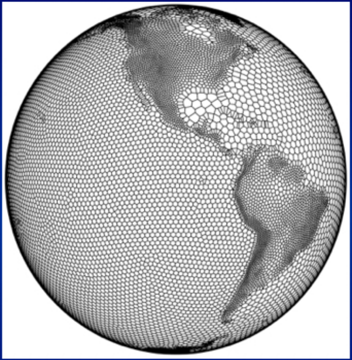
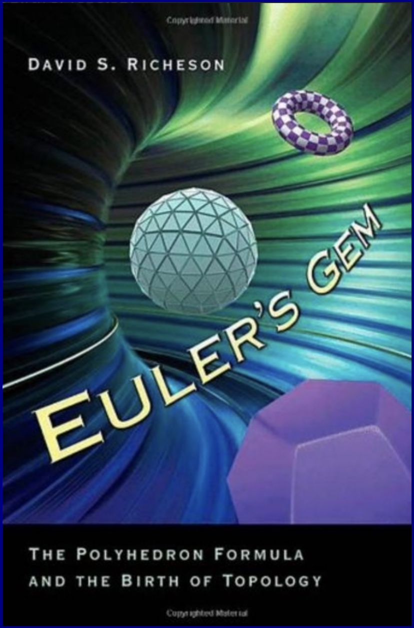


Euler's Polyhedron Formula

$V - E + F = 2$

still holds.

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EULER'S GEM
THE POLYHEDRON FORMULA
AND THE BIRTH OF TOPOLOGY
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Outline

Introduction

Euler's Gem

Distraction 7: Plus Magazine

Cantor's Theorem

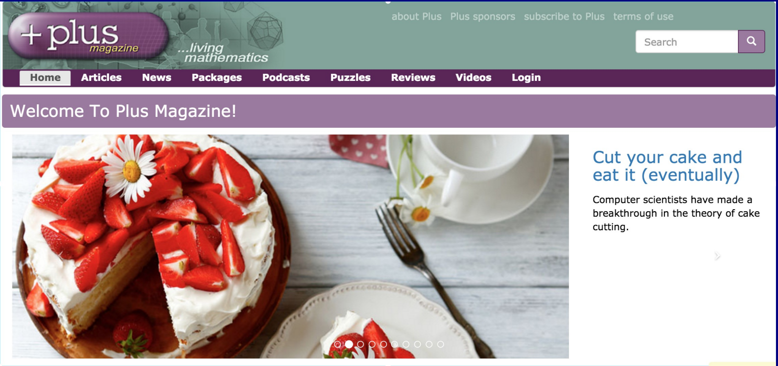
Distraction 6A: Slicing a Pizza (Again)

Astronomy II

Parity of the Rational Numbers

Distraction 8: Sum by Inspection

Distraction 7: Plus Magazine



PLUS: The Mathematics e-zine

`https://plus.maths.org/`

Outline

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Bijections

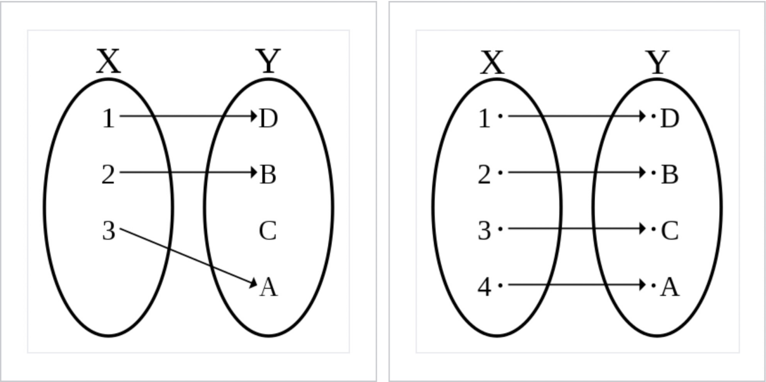
Mathematicians call a 1:1 correspondence
from one set onto another a **bijection.**

Cantor used this approach to compare infinite sets:

> If there is a bijection between two sets
> they are said to be **the same size.**

Cantor built an entire theory of infinity on this idea.

Left: An Injection. Right: A Bijection.



An **injective** non-surjective function (injection, not a bijection)

An **injective** surjective function (bijection)

An injection: A $1:1$ mapping from A into B.

A surjection: Any mapping from A onto B.

A bijection: A map that is an injection and surjection.

A bijection is $1:1$ with an inverse that is also $1:1$.

Cantor's theorem states that, for any A, there is no surjection from A to its power set $\mathcal{P}(A)$.

Therefore, $\mathrm{card}\, A < \mathrm{card}\, \mathcal{P}(A)$.

Colloquially, Therefore, A is smaller than $\mathcal{P}(A)$.

The Power Set

Let's start with a simple example:

$$A = \{1, 2, 3\}$$

The Power set of A is the set of all subsets of A:

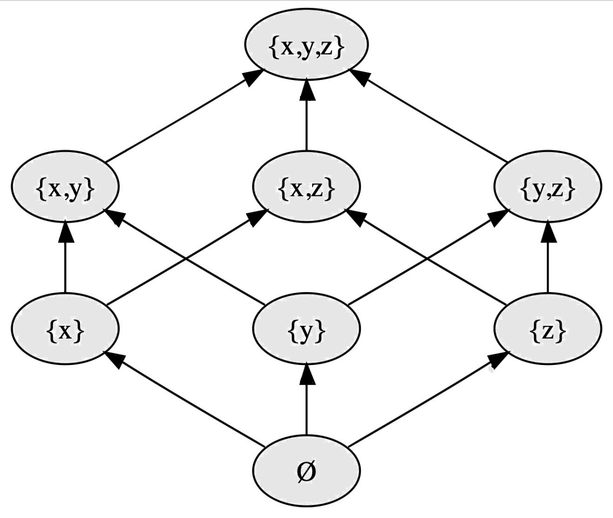
$$\mathcal{P}(A) = \Big\{ \varnothing, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\} \Big\}$$

Let us consider a 1–1 map from A into $\mathcal{P}(A)$:

$$1 \to \{1\} \qquad 2 \to \{2\} \qquad 3 \to \{3\}$$

It does not cover the full power set.

Clearly, this is not surprising, since there are 3 elements in A and $8 = 2^3$ elements in $\mathcal{P}(A)$.



The power set of the set $\{x, y, z\}$.

== Topological structure of a cube ($\#(A) = 2^3$) ==

Cantor's Theorem

Obviously, it is the same for any finite set:

If $\#A = n$ **then** $\#\mathcal{P}(A) = 2^n > n$

so there cannot be a bijection between them (a map that is one-to-one in both directions).

Cantor's genius was to show that this is also true for infinite sets.

In 1891, Georg Cantor published a seminal paper,

Über eine elementare Frage
der Mannigfaltigkeitslehren
(On an elementary question
of the theory of manifolds)

in which his “diagonal argument” first appeared.

He proved that the real numbers are uncountable:

$$\text{card}\,\mathbb{R} > \text{card}\,\mathbb{N} \equiv \aleph_0 \,.$$

His theorem is much more general: it means
there is no greatest order of infinity.

The proof is quite simple, but subtle and clever.

For finite sets, it is obvious:
A set with n elements has 2^n subsets.

Thus, every finite set is smaller than its power set.

Cantor's argument is applicable to all sets,
finite, countable and uncountable.

The theorem states that there is never a bijection between a set A and its power set $\mathcal{P}(A)$.

$$\operatorname{card} A < \operatorname{card} \mathcal{P}(A) ,$$

where $\operatorname{card} A$ represents the cardinality of a set A.

Repeating the power set operation, we have

$$\operatorname{card} \mathcal{P}(A) < \operatorname{card} \mathcal{P}(\mathcal{P}(A)) .$$

This process can be iterated indefinitely.

There is no limit to this process and we can generate an infinite sequence of ever-greater infinities.

The Diagonal Argument: $r \in (0, 1)$

$\mathbb{N}$	$\leftrightarrow$	*reals in* (0,1)
1	$\leftrightarrow$	.835987...
2	$\leftrightarrow$	.250000...
3	$\leftrightarrow$	.559423...
4	$\leftrightarrow$	.500000...
5	$\leftrightarrow$	.728532...
6	$\leftrightarrow$	.845312...
$\vdots$		$\vdots$

Choose a number

$$r = 0.960143\ldots$$

It differs from the first number in the first digit.

It differs from the second number in the second.

And so on . . . .

So, r is not in the list.

The Real Numbers

The case of the real numbers $\mathbb{R}$ is of central interest.

Cantor defined $$\text{card}\,\mathbb{N} = \aleph_0\,.$$

Every real number can be expressed as an infinite sequence of natural numbers (e.g. $\pi = 3.1415\ldots$).

So, the real numbers can be mapped $1:1$ onto the power set of the natural numbers.

$$\text{card}\,\mathbb{R} = 2^{\aleph_0}\,.$$

Cantor's theorem then implies that

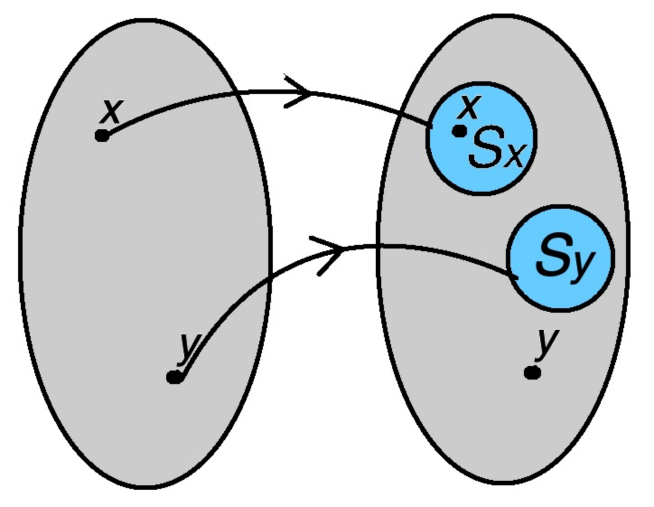
$$\text{card}\,\mathbb{R} > \aleph_0\,.$$

so the real numbers are uncountable.

Proof (*Wikipedia: Cantor's Theorem*)

Theorem (Cantor). Let f be a map from set A to its power set $\mathcal{P}(A)$. Then $f : A \rightarrow \mathcal{P}(A)$ is not surjective. As a consequence, $\operatorname{card}(A) < \operatorname{card}(\mathcal{P}(A))$ holds for any set A.

Proof: Consider the set $B = \{x \in A \,|\, x \notin f(x)\}$. Suppose to the contrary that f is surjective. Then there exists $\xi \in A$ such that $f(\xi) = B$. But by construction, $\xi \in B \iff \xi \notin f(\xi) = B$. This is a contradiction. Thus, f cannot be surjective. On the other hand, $g : A \rightarrow \mathcal{P}(A)$ defined by $x \mapsto \{x\}$ is an injective map. Consequently, we must have $\operatorname{card}(A) < \operatorname{card}(\mathcal{P}(A))$. ■



A map from a set to its power set.

The Unending Hierarchy

If A has cardinality $\mathfrak{N}$ then $\mathcal{P}(A)$ has cardinality $2^{\mathfrak{N}}$.

Cantor introduced the beth-numbers:

$$\beth_0 = \mathrm{card}\ \mathbb{N}\,, \quad \beth_1 = \mathrm{card}\ \mathcal{P}(\mathbb{N})\,, \quad \beth_2 = \mathrm{card}\ \mathcal{P}(\mathcal{P}(\mathbb{N}))\,, \quad \dots$$

These numbers can also be expressed as follows:

$$\beth_0 = \aleph_0\,, \quad \beth_1 = 2^{\beth_0}\,, \quad \beth_2 = 2^{\beth_1}\,, \quad \dots$$

The relationship between the aleph and beth numbers involves the continuum hypothesis.
See post, "Aleph, Beth, Continuum" on `thatsmaths`

Outline

Distraction 6A: Slicing a Pizza (Again)

Cut the pizza using only straight cuts.

There should be exactly one piece of pepperoni on each slice of pizza.

Minimum number of cuts?

Abstract Formulation

A Previous Problem:

Plane cut by n lines. How many regions are formed?

Abstract Formulation

A Previous Problem:

Plane cut by *n* lines. How many regions are formed?

Cuts	Segments (1D)	Regions (2D)	Solids (3D)
0	1	1	1
1	2	2	2
2	3	4	4
3	4	7	8
4	5	11	15
5	6	16	26
6	7	22	42

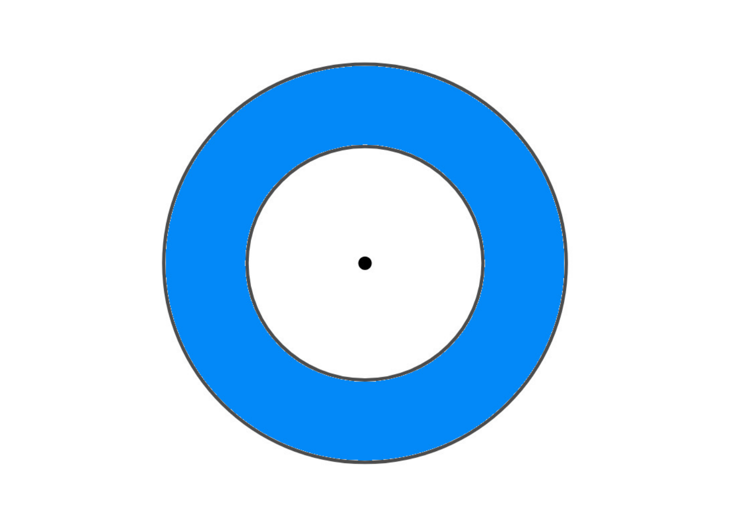
What is the pattern here?

Cutting Lines, Planes and Spaces

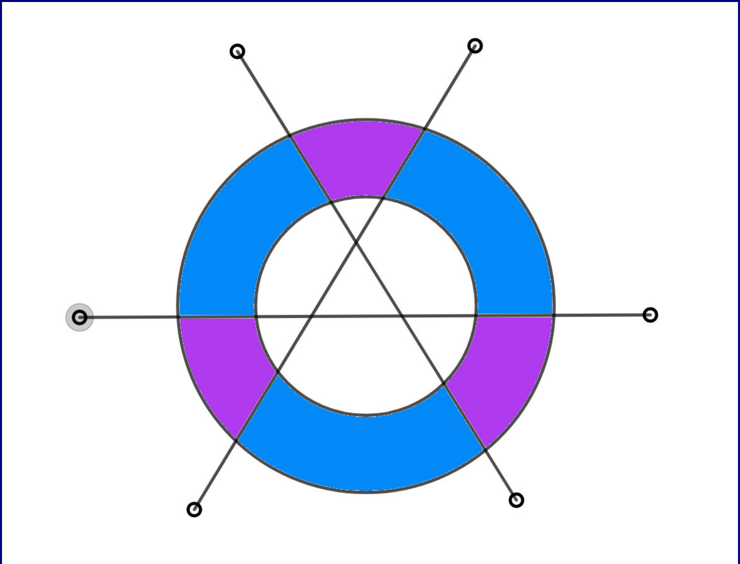
Cuts	Segments (1D)	Regions (2D)	Solids (3D)
0	1	1	1
1	2	2	2
2	3	4	4
3	4	7	8
4	5	11	15
5	6	16	26
6	7	22	42

There is a pattern here.
It is reminiscent of Pascal's Triangle.

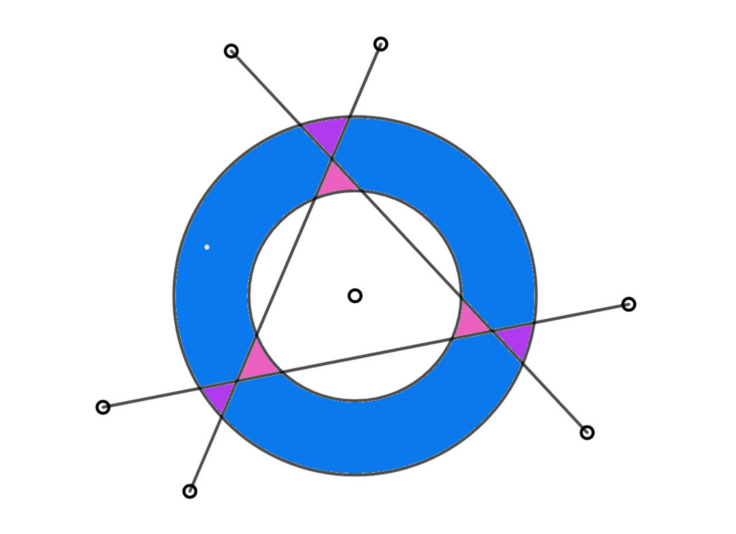
Distraction 6A: Slicing a (Flat) Doughnut



Distraction 6A: Slicing a (Flat) Doughnut



Distraction 6A: Slicing a (Flat) Doughnut



Outline

The Scientific Revolution

INTRODUCTION

This week, we will look at developments in the sixteenth and seventeenth centuries.

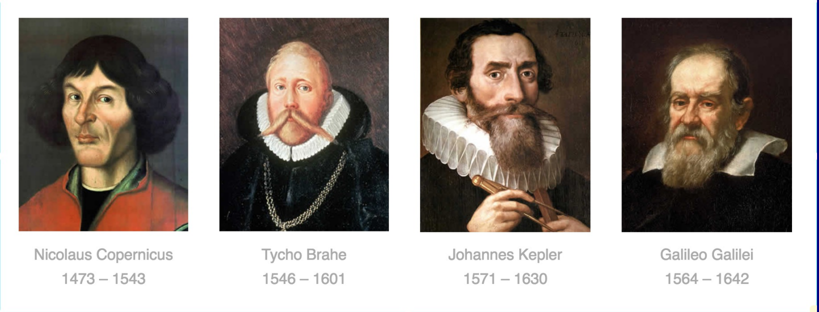


Figure from mathigon.org

The Heliocentric Model

In 1543, Nicolaus Copernicus (1473–1543) published *"On the Revolutions of the Celestial Spheres"*.

He explained that the Sun is at the centre of the universe and that the Earth and planets move around it in circular orbits.

The Heliocentric Model

In 1543, Nicolaus Copernicus (1473–1543) published *"On the Revolutions of the Celestial Spheres"*.

He explained that the Sun is at the centre of the universe and that the Earth and planets move around it in circular orbits.

Danish astronomer Tycho Brahe (1546–1601) made very accurate observations of the movements of the planets, and developed his own model of the solar system.

Johannes Kepler (1571–1630)

Johannes Kepler (1571–1630) succeeded Brahe as imperial mathematician.

After many years of struggling, Kepler succeeded in formulating his three Laws of Planetary Motion.

Kepler's Laws describe the solar system much as we know it to be true today.

Kepler's Laws

- **The planets move on elliptical orbits, with the Sun at one of the two foci.**
  This explains why the Sun appears larger at some times of the year and smaller at others.
- **A line joining the planet and the Sun sweeps out equal areas in equal times.**
  This means that a planet moves faster when close to the Sun, and slower when further away.
- **The square of the orbital period is proportional to the cube of the mean radius of the orbit.**
  This law allows us to find the size of the orbit of a planet if we know the orbital time. Or vice versa.

Jovian Year from Kepler's Third Law

- Distance from Sun to Earth: 1.0 AU
- Distance from Sun to Jupiter: 5.2 AU
-
- Rotation Period of Earth: 1 Year
- Rotation Period of Jupiter: To Be Found

Jovian Year from Kepler's Third Law

- **Distance from Sun to Earth: 1.0 AU**
- **Distance from Sun to Jupiter: 5.2 AU**
-
- **Rotation Period of Earth: 1 Year**
- **Rotation Period of Jupiter: To Be Found**

$$\frac{{P_J}^2}{{P_E}^2} = \frac{{R_J}^3}{{R_E}^3}$$

$$P_J{}^2 = R_J{}^3$$

$$P_J = R_J{}^{\frac{3}{2}} \qquad P_J = (5.2)^{\frac{3}{2}} \approx 12 \textbf{ Years}$$

Jovian Year from Kepler's Third Law

- **Distance from Sun to Earth: 1.0 AU**
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-
- **Rotation Period of Earth: 1 Year**
- **Rotation Period of Jupiter: To Be Found**

$$\frac{{P_J}^2}{{P_E}^2} = \frac{{R_J}^3}{{R_E}^3}$$

$${P_J}^2 = {R_J}^3$$

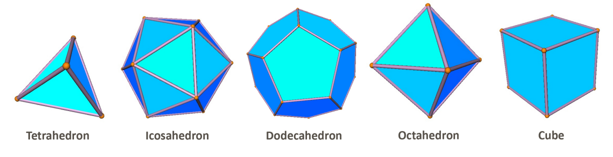
$$P_J = {R_J}^{\frac{3}{2}} \qquad P_J = (5.2)^{\frac{3}{2}} \approx 12 \textbf{ Years}$$

Do this in reverse: get distance from period.

The *Mysterium Cosmographicum*

There were six known planets in Kepler's time:
Mercury, Venus, Earth, Mars, Jupiter, Saturn.

There are precisely five platonic solids:



This gave Kepler an extraordinary idea!

https://thatsmaths.com/2016/10/13/
\keplers-magnificent-mysterium-cosmographicum/

Galileo Galelii (1564–1630)

Galileo introduced the telescope to astronomy, and made some dramatic discoveries.

He observed the four large moons of Jupiter revolving around that planet.

He established the laws of inertia that underlie Newton's dynamical laws.

Four Remarkable Scientists

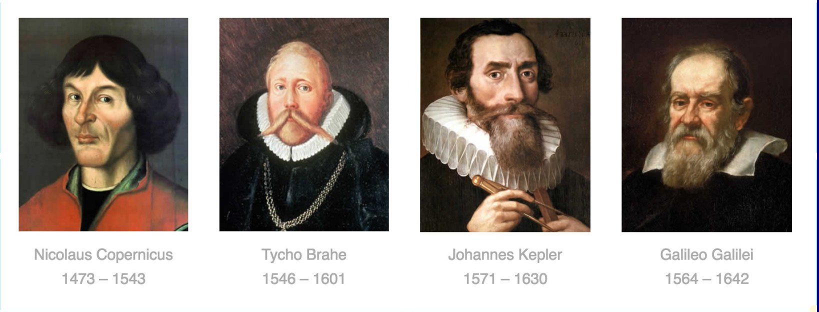


Figure from mathigon.org

Isaac Newton (1642–1727)

In 1687, Isaac Newton published the Principia Mathematica. He established the mathematical foundations of dynamics.

Between any two masses there is a force:

$$F = \frac{GMm}{r^2}$$

This is the force of gravity and gravity is what makes the planets move around the Sun.

Newton's Laws imply and explain Kepler's laws.

Outline

Partitioning the Rational Numbers

The natural numbers $\mathbb{N}$ split nicely into two subsets:

$$\begin{aligned} \mathbf{N}_{\mathrm{O}} &= \{1, 3, 5, 7, \dots\} \\ \mathbf{N}_{\mathrm{E}} &= \{2, 4, 6, 8, \dots\}. \end{aligned}$$

The odd and even numbers are ‘equinumerous’.

A similar split applies to the integers $\mathbb{Z}$:

$$\begin{aligned} \mathbf{Z}_{\mathrm{O}} &= \{\dots -3, -1, +1, +3, +5, \dots\} \\ \mathbf{Z}_{\mathrm{E}} &= \{\dots -4, -2, 0, +2, +4, \dots\}. \end{aligned}$$

The integers form an abelian group $(\mathbb{Z}, +)$.
$\mathbf{Z}_{\mathrm{E}}$ is an additive subgroup of $(\mathbb{Z}, +)$.
It is of index 2, with cosets $\mathbf{Z}_{\mathrm{E}}$ and $\mathbf{Z}_{\mathrm{E}} + 1$.

Parity

The distinction between odd and even is called parity. Parity is defined only for the integers (whole numbers).

Can we extend the concept of parity to the rationals?

The usual 'rules' of parity might be required:

1. Sum of even numbers is even; product is even.
2. Sum of odd numbers is even; product is odd.
3. Sum of even and odd is odd; product is even.
4. Odd number plus 1 is even; even plus 1 is odd.

Rules of Parity

Table: Addition (left) and multiplication (right) tables for $\mathbb{Z}$.

$+$	**even**	**odd**
even	*even*	*odd*
odd	*odd*	*even*

$\times$	**even**	**odd**
even	*even*	*even*
odd	*even*	*odd*

Even and Uneven

For $\mathbb{Q}$, we could define a number $q = m/n$ to be even if the numerator m is even and odd if m is odd.

But then $\frac{1}{4} + \frac{1}{4} = \frac{1}{2}$, meaning that two odd rationals might add to yield another odd one.

We distinguish between 'odd' and 'uneven':

$$\text{For } q = m/n\,, \qquad \begin{cases} q \text{ is even} & \text{if } m \text{ is even},\\ q \text{ is uneven} & \text{if } m \text{ is odd}\,. \end{cases}$$

Numerical Evidence

A MATHEMATICA program was written to count the number of even and uneven rationals in $(0, 1)$.

We can list all rationals in $(0, 1)$ in a sequence:

$$\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{6}, \frac{5}{6}, \frac{1}{7}, \frac{2}{7}, \frac{3}{7}, \frac{4}{7}, \frac{5}{7}, \frac{6}{7}, \frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}, \ldots$$

Colloquially, there are:

"twice as many uneven as even rationals".

Numerical Evidence

$$r = \#\{EVEN\}/\#\{UNEVEN\}$$

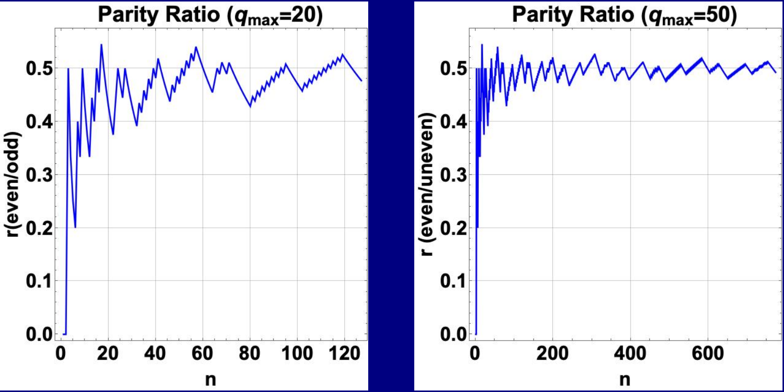


Figure: Parity ratio r for $q_{\max} = 20$ (left) and $q_{\max} = 50$ (right).

A Three-way Split

Is there a natural way of separating the uneven numbers into two subsets? In fact, there is.

For $q = \dfrac{m}{n}$,

$$\begin{cases} q \text{ has even parity if } m \text{ is even,} \\ q \text{ has odd parity if } m \text{ is odd and } n \text{ is odd,} \\ q \text{ has none if } m \text{ is odd and } n \text{ is even.} \end{cases}$$

The term none is an acronym:

none = ‘neither odd nor even’

A Three-way Split

Let e be an even integer and o an odd one.

$$\text{Even: } \frac{e}{o} \qquad \text{Odd: } \frac{o}{o} \qquad \text{None: } \frac{o}{e} .$$

We define three subsets of the rational numbers:

$$\begin{aligned}
\text{Even: } \mathbf{Q}_{\mathrm{E}} &= \{q \in \mathbb{Q} : q = \tfrac{2m}{2n+1} \text{ for some } m, n \in \mathbb{Z}\} \\
\text{Odd: } \mathbf{Q}_{\mathrm{O}} &= \{q \in \mathbb{Q} : q = \tfrac{2m+1}{2n+1} \text{ for some } m, n \in \mathbb{Z}\} \\
\text{None: } \mathbf{Q}_{\mathrm{N}} &= \{q \in \mathbb{Q} : q = \tfrac{2m+1}{2n} \text{ for some } m, n \in \mathbb{Z}\} .
\end{aligned}$$

These three sets are disjoint: $\mathbb{Q} = \mathbf{Q}_{\mathrm{E}} \cup \mathbf{Q}_{\mathrm{O}} \cup \mathbf{Q}_{\mathrm{N}}$.

We may enquire about the relative sizes of the sets.

Addition and multiplication tables for $\mathbb{Q}$.

$+$	even	odd	none
even	*even*	*odd*	*none*
odd	*odd*	*even*	*none*
none	*none*	*none*	*any*

$\times$	even	odd	none
even	*even*	*even*	*any*
odd	*even*	*odd*	*none*
none	*any*	*none*	*none*

Notice that the first two rows and columns are identical to the tables for the integers.

Rules of Parity for the Integers

Table: Addition (left) and multiplication (right) tables for $\mathbb{Z}$.

$+$	*even*	*odd*
even	*even*	*odd*
odd	*odd*	*even*

$\times$	*even*	*odd*
even	*even*	*even*
odd	*even*	*odd*

The Calkin-Wilf Tree

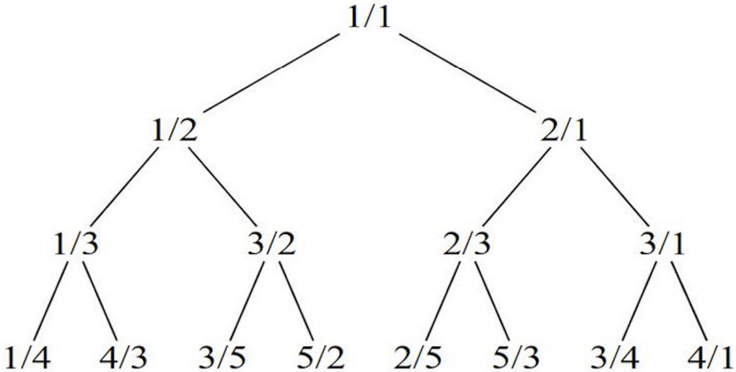


Figure: Another way to organize the rational numbers.

The Calkin-Wilf Tree

The Calkin-Wilf tree is another arrangement of $\mathbb{Q}$.

The Calkin-Wilf tree is complete:

- It includes all the rationals;
- Each positive rational occurs just once.

Everything springs from the root $1/1$.

Each rational has two "children":
for the entry m/n, the children are

$$m/(m+n) \text{ and } (m+n)/n.$$

The Calkin-Wilf Tree

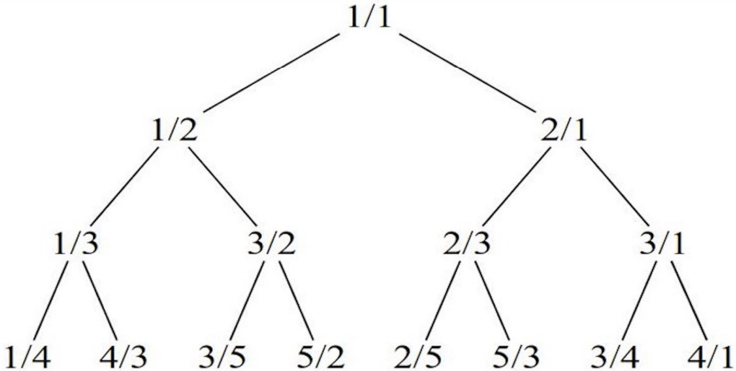
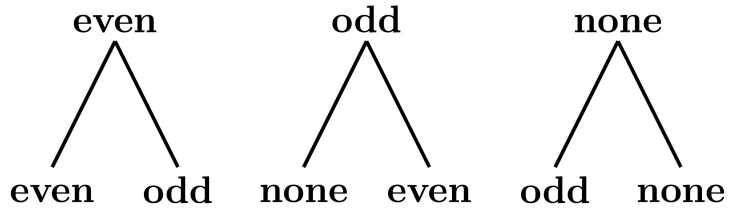


Figure: The first four generations of the Calkin-Wilf tree.

Calkin-Wilf Parity Transfer



If each of the parity classes, even, odd and none, occurs with equal frequency at one generation, then this equality is passed on to the next generation and persists thereafter.

There are equal numbers of rationals whose parity is even, odd and none.

Outline

Distraction 8: Sum by Inspection

Can you guess the sum of this series:

$$\left(\frac{1}{2}\right)^2 + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{8}\right)^2 + \left(\frac{1}{16}\right)^2 + \cdots$$

Distraction 8: Sum by Inspection

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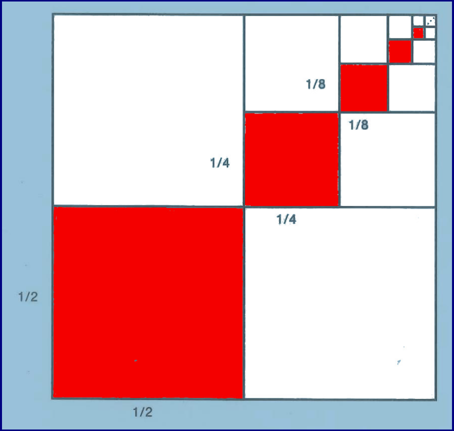
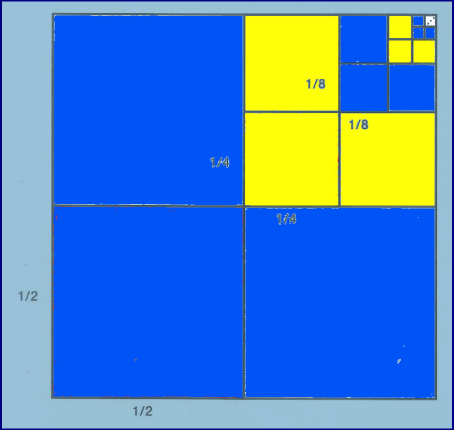
Let's pretend we don't know how to sum a geometric series!

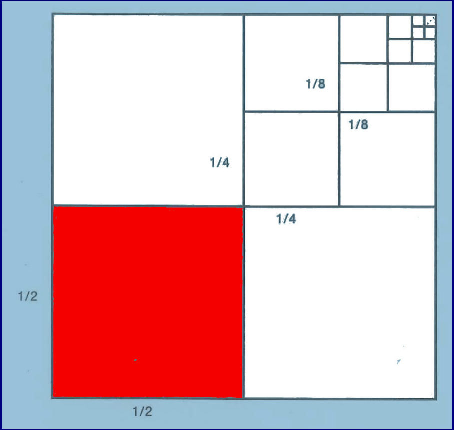
Distraction 8: Sum by Inspection

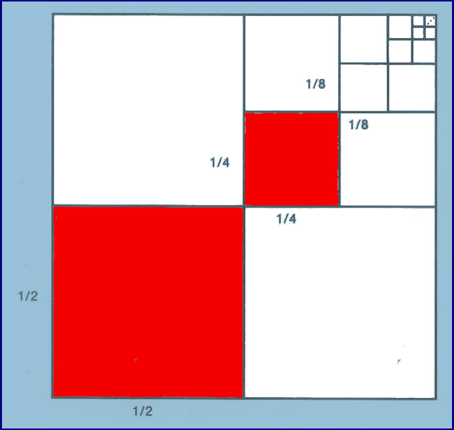


We will find the shaded area without calculation



Subsquares of different scales.


1/8
1/8
1/4
1/4
1/2
1/2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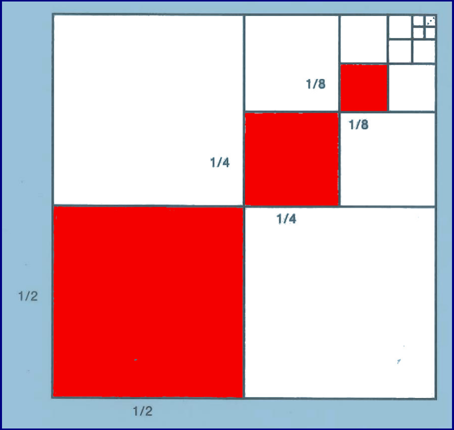
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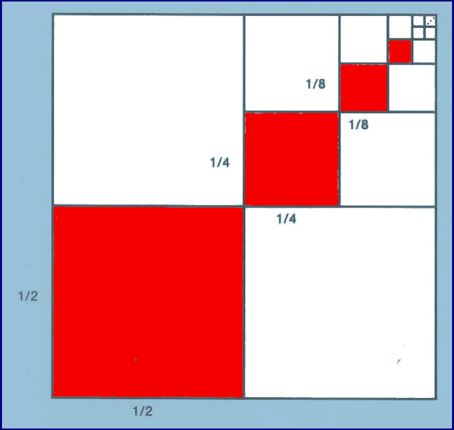
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1/8
1/8
1/4
1/4
1/2
1/2

Proof by Inspection

Look at the figure in two different ways

At each scale, we have three squares the same size, and we keep one of them (red) and omit the others.

So, the area of the shaded squares is $\frac{1}{3}$.

Proof by Inspection

Look at the figure in two different ways

At each scale, we have three squares the same size, and we keep one of them (red) and omit the others.

So, the area of the shaded squares is $\frac{1}{3}$.

However, it is also given by the series

$$\left(\frac{1}{2}\right)^2 + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{8}\right)^2 + \left(\frac{1}{16}\right)^2 + \cdots$$

Therefore we can sum the series:

$$\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \cdots \quad = \quad \frac{1}{3}$$

Thank you