AweSums

Marvels and Mysteries of Mathematics

•

LECTURE 5

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School of Mathematics & Statistics
University College Dublin

Evening Course, UCD, Autumn 2021

Outline

Outline

Meaning and Content of Mathematics

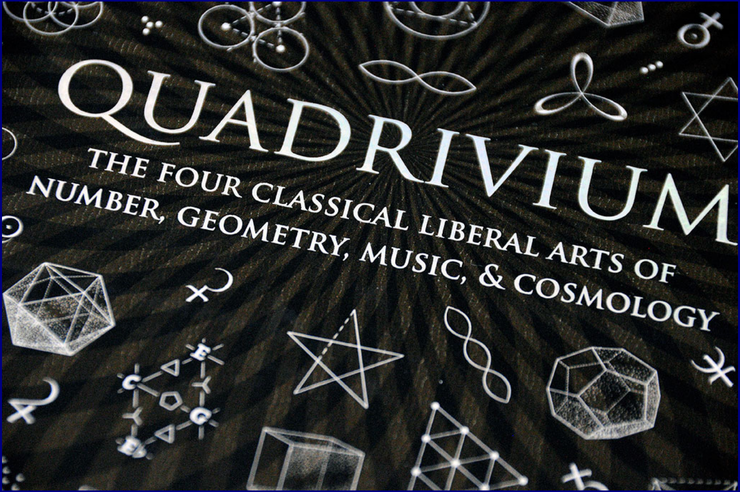
The word Mathematics comes from Greek $\mu\alpha\theta\eta\mu\alpha$ (máthéma), meaning "knowledge" or "lesson" or "learning".

It is the study of topics such as

- Quantity: [Numbers. Arithmetic]
- Structure: [Patterns. Algebra]
- Space: [Geometry. Topology]
- Change: [Analysis. Calculus]

Outline

The Quadrivium



The Quadrivium

The Quadrivium originated with the Pythagoreans around 500 BC.

The Pythagoreans' quest was to find the eternal laws of the Universe, and they organized their studies into the scheme later known as the Quadrivium.

It comprised four disciplines:

- Arithmetic
- Geometry
- Music
- Astronomy

The Quadrivium

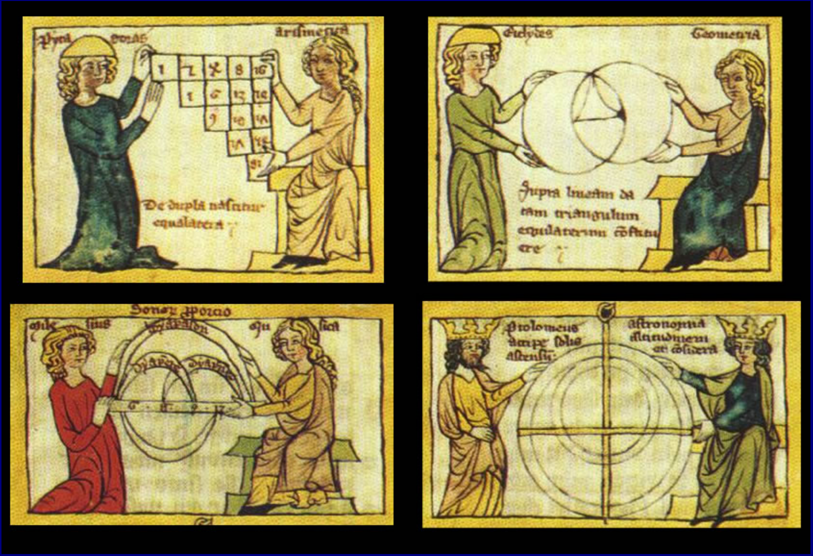
First comes Arithmetic, concerned with the infinite linear array of numbers.

Moving beyond the line to the plane and 3D space, we have Geometry.

The third discipline is Music, which is an application of the science of numbers.

Fourth comes Astronomy, the application of Geometry to the world of space.

The Quadrivium



Static/Dynamic. Pure/Applied

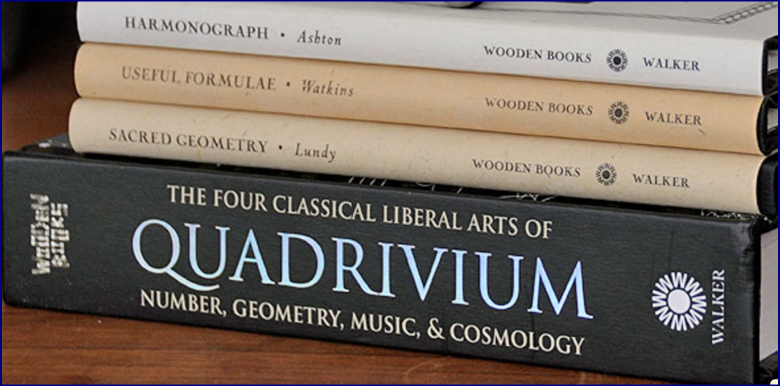
- Arithmetic (static number)
- Music (moving number)
- Geometry (measurement of static Earth)
- Astronomy (measurement of moving Heavens)

Arithmetic represents numbers at rest,
Geometry is magnitudes at rest,

Music is numbers in motion and
Astronomy is geometry in motion.

The first two are pure in nature,
while the last two are applied.

The Quadrivium



For the Greeks, Mathematics embraced all four areas.

The Pythagoreans

Pythagoras distinguished between
quantity and magnitude.

Objects that can be counted
yield quantities or numbers.

Substances that are measured provide magnitudes.

Thus, cattle are counted whereas milk is measured.

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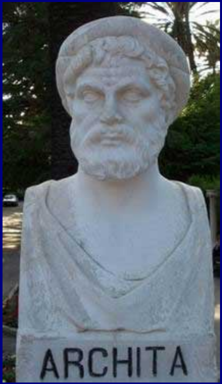
Thus, cattle are counted whereas milk is measured.

Arithmetic studies quantities or numbers and
Music involves the relationship between numbers
and their evolution in time.

Geometry deals with magnitudes, and
Astronomy with their distribution in space.

Archytas (428–350 BC): *ΑΡΧΥΤΑΣ*



Αρχυτας.

Born in Tarentum, son of Hestiaeus.

Mathematician and philosopher.

Pythagorean, student of Philolaus.

Provided a solution for the Delian problem of doubling the cube.

Said to have tutored Plato in mathematics(?)

Archytas (428–350 BC)

Archytas lived in Tarentum (now in Southern Italy).

One of the last scholars of the Pythagorean School and was a good friend of Plato.

The designation of the four disciplines of the Quadrivium was ascribed to Archytas.

His views were to dominate pedagogical thought for over two millennia.

Partly due to Archytas, mathematics has played a prominent role in education ever since.

Plato's Academy

According to Plato, mathematical knowledge was essential for an understanding of the Universe. The curriculum was outlined in Plato's *Republic.*

Inscription over the entrance to Plato's Academy:

"Let None But Geometers Enter Here".

This indicated that the Quadrivium was a prerequisite for the study of philosophy in ancient Greece.

Boethius (AD 480–524)

The Western Roman Empire was in many ways static for centuries.

No new mathematics between the conquest of Greece and the fall of the Roman Empire in AD 476.

Boethius, the 6th century Roman philosopher, was one of the last great scholars of antiquity.

The organization of the Quadrivium was formalized by Boethius.

It was the mainstay of the medieval monastic system of education.

The Quadrivium

Typus Arithmeticae

A woodcut from the book *Margarita Philosophica,* by Gregor Reisch, Freiburg, 1503.

The central figure is Dame Arithmetic, watching a competition between Boethius, using pen and Hindu-Arabic numerals, and Pythagoras, using a counting board or *tabula*.

She looks favourably toward Boethius.

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But how did Boethius know about Hindu-Arabic numerals?

The Liberal Arts

The seven liberal arts comprised
the Trivium and the Quadrivium.

The Trivium was centred on three arts of language:

- Grammar: proper structure of language.
- Logic: for arriving at the truth.
- Rhetoric: the beautiful use of language.

Aim of the Trivium: Goodness, Truth and Beauty.

Aristotle traced the origin of the Trivium back to Zeno.

The Ultimate Goal

The goal of studying the Quadrivium was
an insight into the nature of reality,
an understanding of the Universe.

The Quadrivium offered the seeker of wisdom
an understanding of the integral nature of
the Universe and the role of humankind within it.

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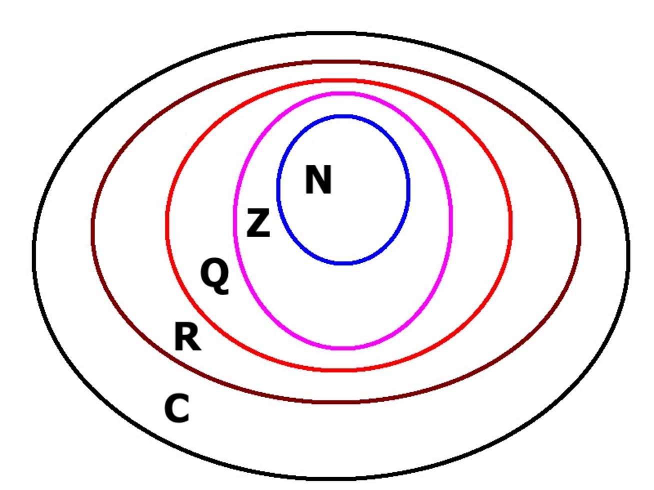
That is our aim in AweSums!

Outline

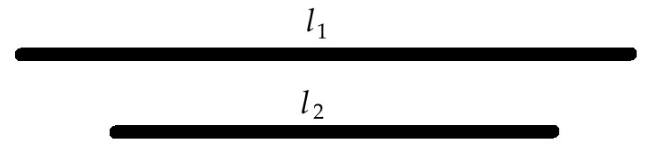
The Hierarchy of Numbers



$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$$

Incommensurability

Suppose we have two line segments



Can we find a unit of measurement such that both lines are a whole number of units?

Can they be co-measured? Are they commensurable?

Are ℓ_1 and ℓ_2 commensurable?

If so, let the unit of measurement be λ.

Then

$$\begin{aligned} \ell_1 &= m\lambda\,, \quad m \in \mathbb{N} \\ \ell_2 &= n\lambda\,, \quad n \in \mathbb{N} \end{aligned}$$

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Therefore

$$\frac{\ell_1}{\ell_2} = \frac{m\lambda}{n\lambda} = \frac{m}{n}$$

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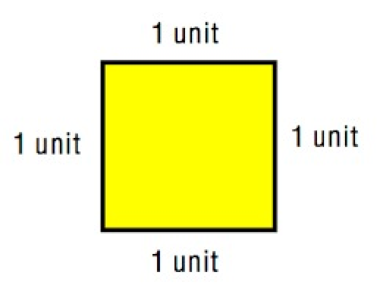
Therefore

$$\frac{\ell_1}{\ell_2} = \frac{m\lambda}{n\lambda} = \frac{m}{n}$$

If not, then ℓ_1 and ℓ_2 are incommensurable.

Irrational Numbers

If the side of a square is of length 1, then the diagonal has length $\sqrt{2}$ (by the Theorem of Pythagoras).



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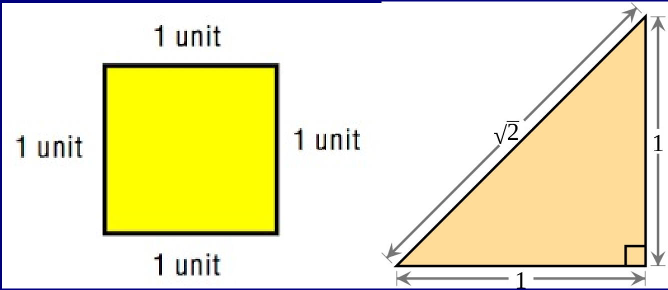

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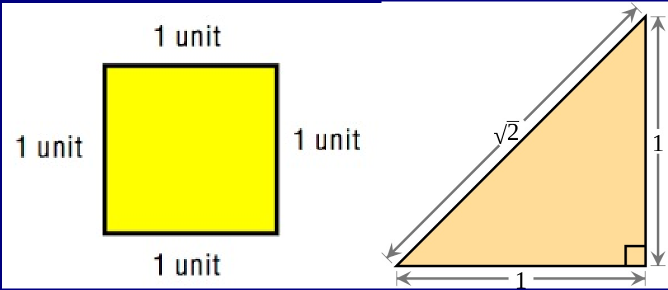

The ratio between the diagonal and the side is:

$$\frac{\text{Diagonal}}{\text{Side Length}} = \sqrt{2}$$

Irrationality of $\sqrt{2}$

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1. Whole numbers
2. Ratios of whole numbers

There were no other numbers.

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Let's suppose that $\sqrt{2}$ is a ratio of whole numbers:

$$\sqrt{2} = \frac{p}{q}$$

We can assume that p and q have no common factors. Otherwise, we just cancel them out.

For example, suppose $p = 42$ and $q = 30$. Then

$$\frac{p}{q} = \frac{42}{30} = \frac{7 \times 6}{5 \times 6} = \frac{7}{5}$$

Remarks on *Reductio ad Absurdum.*

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The Sign of the Four (1890)

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In particular, p and q cannot both be even numbers.

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Now square both sides of the equation $\sqrt{2} = p/q$:

$$2 = \frac{p}{q} \times \frac{p}{q} = \frac{p^2}{q^2} \qquad \text{or} \qquad p^2 = 2q^2$$

This means that p^2 is even. Therefore, p is even.

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Let $p = 2r$ where r is another whole number.

$$\text{Then} \qquad p^2 = (2r)^2 = 4r^2 = 2q^2 \qquad \text{or} \qquad 2r^2 = q^2$$

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Let $p = 2r$ where r is another whole number.

Then $\qquad p^2 = (2r)^2 = 4r^2 = 2q^2 \qquad$ or $\qquad 2r^2 = q^2$

But this means that q^2 is even. So, q is even.

Both p and q are even. This is a contradiction.

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The supposition was that $\sqrt{2}$ is a ratio of two integers that have no common factors:

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By *reductio ad absurdum,* $\sqrt{2}$ is irrational.

It is not a ratio of whole numbers.

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By *reductio ad absurdum,* $\sqrt{2}$ is irrational.

It is not a ratio of whole numbers.

To the Pythagoreans, $\sqrt{2}$ was not a number.

$\kappa\rho\iota\sigma\eta \quad \kappa\alpha\tau\alpha\sigma\tau\rho o\phi\eta$!

$\sqrt{2}$ and the Development of Mathematics

The discovery of irrational quantities had a dramatic effect on the development of mathematics.

Legend has it that the discoveror of this fact was thrown from a ship and drowned.

The result was that focus now fell on geometry, and arithmetic or number theory was neglected.

The problems were not resolved for many centuries.

Outline

MoMath, New York

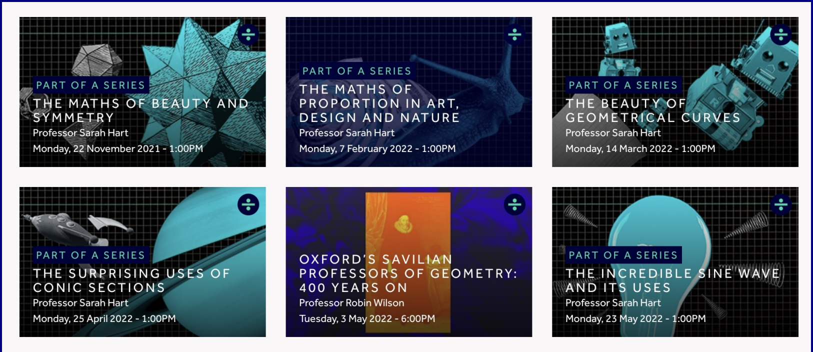
https://momath.org/

Watch the Videos.

Crazy Bike at MoMath

Gresham College London

https://www.gresham.ac.uk/



Search for mathematics lectures

Dublin Applied Maths Meetup

Peter Lynch will speak on

Richardson's Fabulous Forecast Factory

Dublin Applied Maths Meetup
Thursday at 7:30pm.

https://www.meetup.com/Dublin-Applied-Mathematics-Meetup/events/

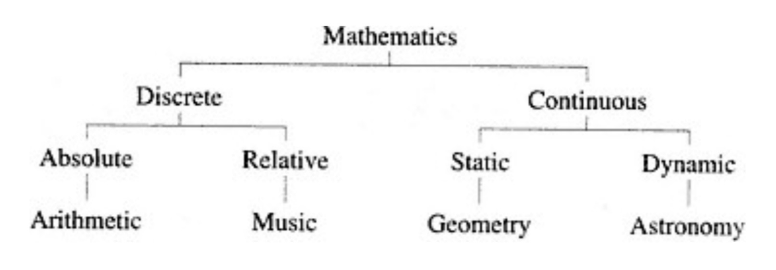
Some News

Any More ???

Not just now!

Outline

The Quadrivium



The Pythagorean model of mathematics

The Ancient Greeks

Mathematics and Astronomy are intimately linked.

Two of the strands of the Quadrivium were Geometry (static) and Cosmology (dynamic space).

Greek astronomer Claudius Ptolemy (c.90–168AD) placed the Earth at the centre of the universe.

The Sun and planets move around the Earth in orbits that are of the most perfect of all shapes: circles.

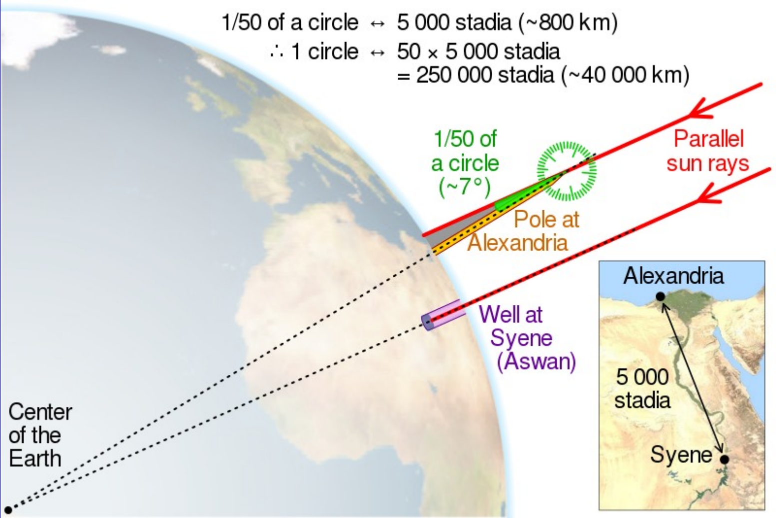
Aristarchus of Samos (c.310–230 BC)

Aristarchus of Samos ('A$\rho\iota\sigma\tau\alpha\rho\chi o\varsigma$), astronomer and mathematician, presented the first model that placed the Sun at the center of the universe.

The original writing of Aristarchus is lost, but Archimedes wrote in his Sand Reckoner:

"His hypotheses are that the fixed stars and the Sun remain unmoved, that the Earth revolves about the Sun on the circumference of a circle, ... "

Eratosthenes (c.276–194 BC)



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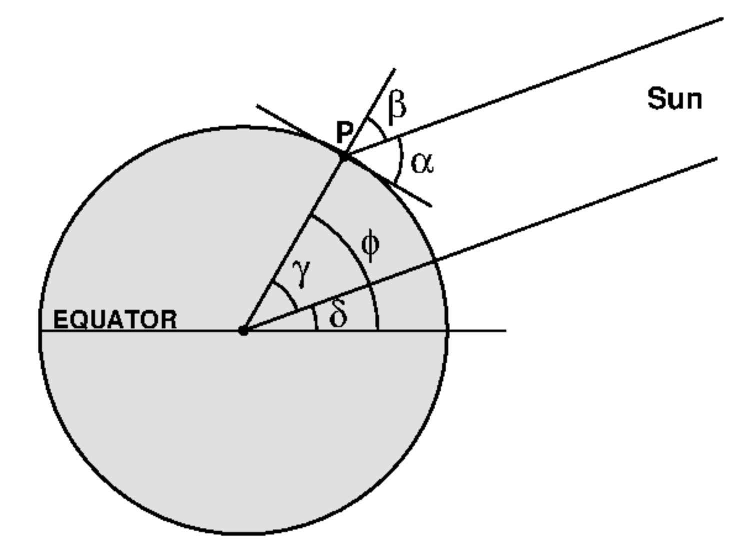


Hipparchus (c.190–120 BC)

Hipparchus of Nicaea ($'I\pi\pi\alpha\rho\chi o\varsigma$) was a Greek astronomer, geographer, and mathematician.

Regarded as the greatest astronomer of antiquity.

Often considered to be the founder of trigonometry.

Famous for

- Precession of the equinoxes
- First comprehensive star catalog
- Invention of the astrolabe
- Invention (perhaps) of the armillary sphere.

Claudius Ptolemy (c.AD 100–170)

Claudius Ptolemy was a Greco-Roman astronomer, mathematician, geographer and astrologer.

He lived in the city of Alexandria.

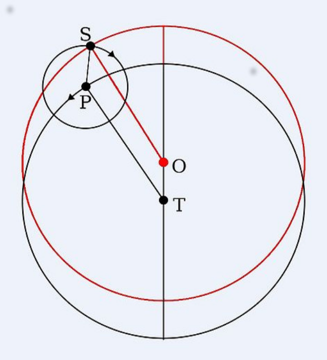
Ptolemy wrote several scientific treatises:

- An astronomical treatise (the Almagest) originally called Mathematical Treatise (Mathematike Syntaxis).
- A book on geography.
- An astrological treatise.

Ptolemy's Almagest is the only surviving comprehensive ancient treatise on astronomy.

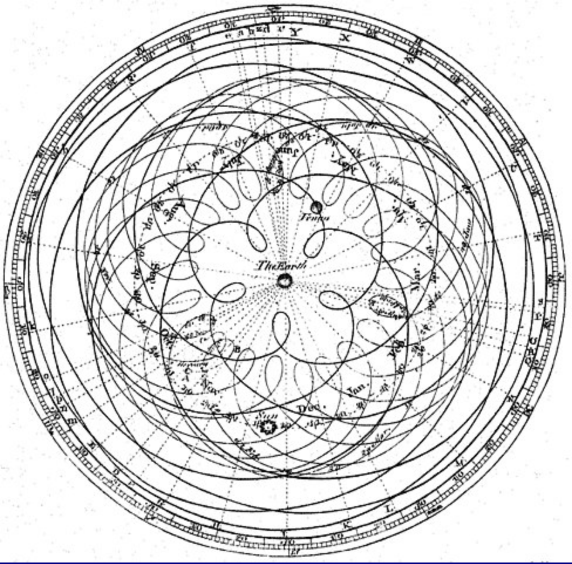
Ptolemy's Model

Ptolemy's model was universally accepted until the appearance of simpler heliocentric models during the scientific revolution.



O is the earth and S the planet.

Ptolemaic Epicycles



“Adding Epicycles”

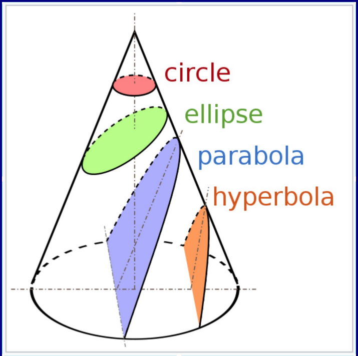
According to **Norwood Russell Hanson** (science historian):

There is no bilaterally symmetrical, nor eccentrically periodic curve used in any branch of astrophysics or observational astronomy which could not be smoothly plotted as the resultant motion of a point turning within a constellation of epicycles, finite in number, revolving around a fixed deferent.

“The Mathematical Power of Epicyclical Astronomy", 1960

Any path — periodic or not, closed or open — can be approximated by a sum of epicycles.

Conic Sections



Circles are special cases of conic sections.

They are formed by a plane cutting a cone at an angle.

Conics were studied by Apollonius of Perga (late 3rd – early 2nd centuries BC).

https://en.wikipedia.org/wiki/Conic_section

The Scientific Revolution

TRAILER

Next week, we will look at developments in the sixteenth and seventeenth centuries.

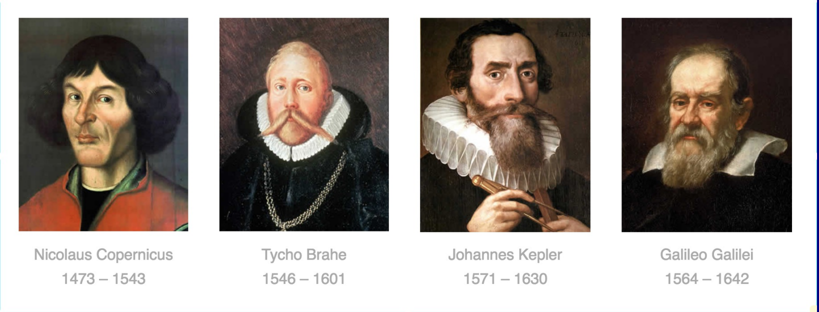


Figure from mathigon.org

Terry Tao's 2020 Hamilton Lecture

`https://www.youtube.com/watch?v=9A0jPchJE0A`

Or go to YouTube and search for "Terry Tao Astronomy"

Note: Tao's talk starts at 7 min 30 sec.

Outline

The Real Numbers

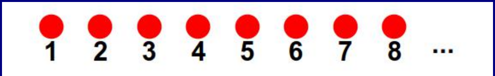
We need to be able to assign a number to a line of any length.

The Pythagoreans found that no number known to them gave the diagonal of a unit square.

It is as if there are gaps in the number system.

We look at the rational numbers and show how to complete them: how to fill in the gaps.

The set $\mathbb{N}$ is infinite, but each element is isolated.



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between any two rationals there is another rational.

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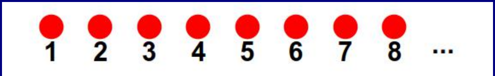

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between any two rationals there is another rational.

PROOF: Let $r_1 = p_1/q_1$ and $r_2 = p_2/q_2$ be rationals.

$$\bar{r} = \tfrac{1}{2}(r_1 + r_2) = \frac{1}{2}\left(\frac{p_1}{q_1} + \frac{p_2}{q_2}\right) = \frac{p_1 q_2 + q_1 p_2}{2 q_1 q_2}$$

is another rational between them: $r_1 < \bar{r} < r_2$.

* * * NEWSFLASH * * *

I am currently trying to devide a partition of the rational numbers into classes, like odd and even. but with three cases.

If I make any progress over the next week or two, I'll tell you about this "hot off the press" maths!

Although $\mathbb{Q}$ is dense, there are gaps.
The line of rationals is discontinuous.

We complete it—filling in the gaps—by defining the limit of any sequence of rationals as a real number.

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WARNING:
We are glossing over a number of fundamental ideas of mathematical analysis:

- What is an infinite sequence?
- What is the limit of a sequence?

To give a particular example, we know that

$$\sqrt{2} = 1.41421356\ldots$$

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$$\{1,\ 1.4,\ 1.41,\ 1.414,\ 1.4142,\ 1.41421, \ldots\}$$

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In terms of fractions, this is the sequence

$$\left\{\frac{1}{1}, \frac{14}{10}, \frac{141}{100}, \frac{1414}{1000}, \frac{14142}{10000}, \frac{141421}{100000}, \ldots\right\}$$

These rational numbers get closer and closer to $\sqrt{2}$.

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EXERCISE:
Construct a sequence in $\mathbb{Q}$ that tends to π.

The Real Number Line

The set of Real Numbers, $\mathbb{R}$, contains all the rational numbers in $\mathbb{Q}$ and also all the limits of sequences of rationals [technically, all ‘Cauchy sequences’].

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We may assume that

- Every point on the number line corresponds to a real number.
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PHYSICS: There are unknown aspects of the microscopic structure of spacetime! These go beyond our 'Universe of Discourse'.

Now we have the chain of sets:

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$$

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$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$$

We can also consider the prime numbers $\mathbb{P}$.
They are subset of the natural numbers, so

$$\mathbb{P} \subset \mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$$

Outline

Pascal's Triangle

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1

Combinatorial Symbol

$$\binom{n}{r}$$ "*n* choose *r*"

This symbol represents the number of combinations of r objects selected from a set of n objects.

$\binom{n}{r}$ are also called **Binomial coefficients.**

Pascal's Triangle: Combinations

$$\binom{0}{0}$$

$$\binom{1}{0} \quad \binom{1}{1}$$

$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$

$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$

$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$

Pascal's Triangle

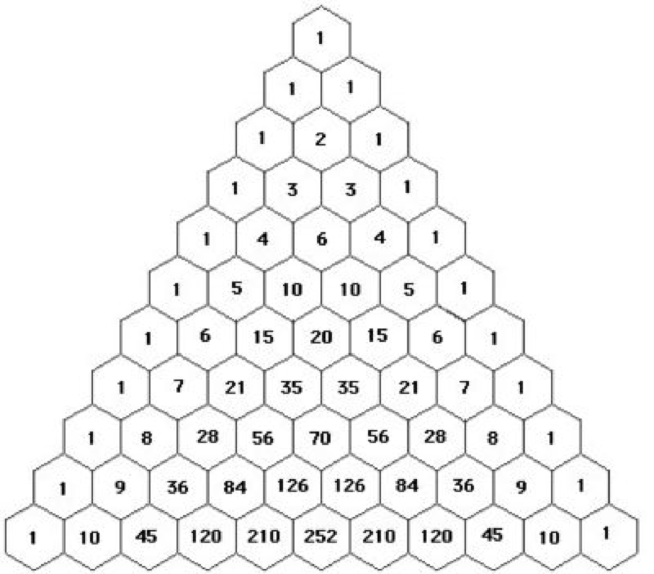
Pascal's triangle is a triangular array of the binomial coefficients.

It is named after French mathematician Blaise Pascal.

It was studied centuries before him in:

- India (Pingala, C2BC)
- Persia (Omar Khayyam, C11AD)
- China (Yang Hui, C13AD).

Pascal's *Traité du triangle arithmétique* (Treatise on Arithmetical Triangle) was published in 1665.


1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1

Pascal's Triangle

The rows of Pascal's triangle are numbered starting with row n = 0 at the top (0-th row).

The entries in each row are numbered from the left beginning with k = 0.

The triangle is easily constructed:

- A single entry 1 in row 0.
- Add numbers above for each new row.

The entry in the nth row and k-th column of Pascal's triangle is denoted $\binom{n}{k}$.

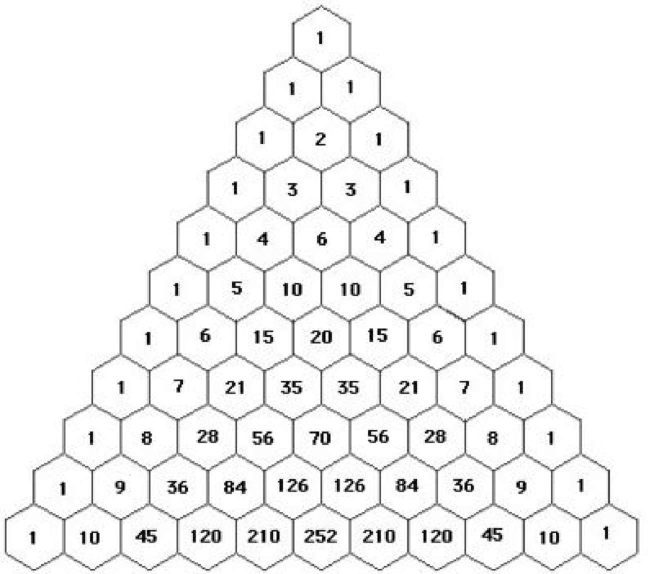
The entry in the topmost row is $\binom{0}{0} = 1$.

Pascal's Identity

The construction of the triangle may be written:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

This relationship is known as Pascal's Identity.


1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1

See Mathigon website

https://mathigon.org/course/sequences/pascals-triangle

Pascal's Triangle & Fibonacci Numbers.

1							
1	1						
1	2	1					
1	3	3	1				
1	4	6	4	1			
1	5	10	10	5	1		
1	6	15	20	15	6	1	
1	7	21	35	35	21	7	1

Figure: Pascal's Triangle and Fibonacci Numbers

Where are the Fibonacci Numbers hiding here?

Sierpinski's Gasket


Sierpinski Gasket: usual construction by trisection.

Sierpinski's Gasket is constructed by starting with an equilateral triangle, and successively removing the central triangle at each scale.

Sierpinski's Gasket at Stage 6

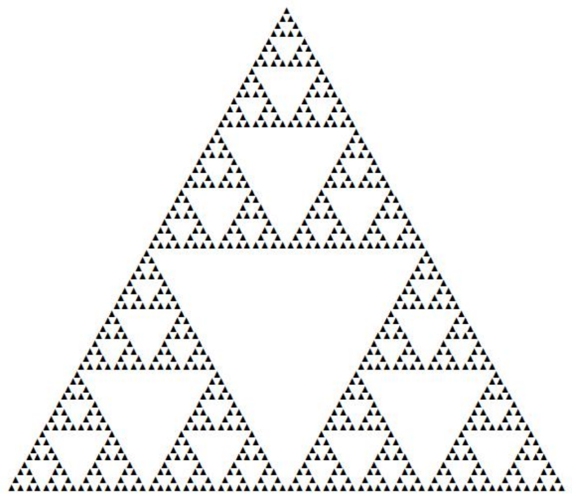


Figure: Result after 6 subdivisions

Sierpinski's Gasket in Pascal's Triangle

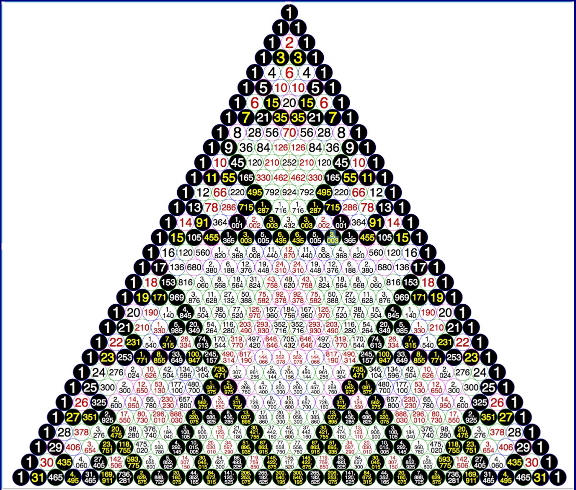


Figure: Odd numbers are in black

Remember Walking in Manhattan?

♖	1	1	1
1	2	3	4
1	3	6	10
1	4	10	20

Figure: Number of routes for a rook in chess.

Geometric Numbers in Pascal's Triangle