AweSums

Marvels and Mysteries of Mathematics

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LECTURE 4

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Outline

Outline

Meaning and Content of Mathematics

The word **Mathematics** comes from Greek $\mu\alpha\theta\eta\mu\alpha$ (máthéma), meaning "knowledge" or "lesson" or "learning".

It is the study of topics such as

- Quantity: [Numbers. Arithmetic]
- Structure: [Patterns. Algebra]
- Space: [Geometry. Topology]
- Change: [Analysis. Calculus]

Outline

Distraction 13: Conway's Puzzle

Find a 10-digit number ABCDEFGHIJ such that:

1. A is divisible by 1
2. AB is divisible by 2
3. ABC is divisible by 3
4. ABCD is divisible by 4
5. ABCDE is divisible by 5
6. ABCDEF is divisible by 6
7. ABCDEFG is divisible by 7
8. ABCDEFGH is divisible by 8
9. ABCDEFGHI is divisible by 9
10. ABCDEFGHIJ is divisible by 10

Each letter is a digit (1,2,3,4,5,6,7,8,9,0).

Distraction 13. Solution

(1): Try every possible permutation:

$$10! = 3{,}628{,}800$$

(2) Use division rules to reduce this number.

Distraction 13. Solution: X XXX XXX XXX

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(2) Use division rules to reduce this number.

(3) Go to this page in *The Guardian:* https://www.theguardian.com/science/2020/apr/20/can-you-solve-it-john-horton-conway-playful-maths-genius
(or Google for Conway's number puzzle Bellos)

(4) Go to this page in *Quanta Magazine:* https://www.quantamagazine.org/three-math-puzzles-inspired-by-john-horton-conway-20201015/

Outline

Enigmas of Infinity

Zeno of Elea devised several paradoxes involving infinity.

He argued that one cannot travel from A to B: to do so, one must first travel half the distance, then half of the remaining half, then half the remainder, and so on.

He concluded that motion is logically impossible.

Zeno was misled by his belief that the sum of finite quantities must grow without limit as more are added.

Enigmas of Infinity

Systematic mathematical study of infinite sets began around 1875 when Georg Cantor developed a theory of transfinite numbers.

He reasoned that the method of comparing the sizes of finite sets could be carried over to infinite ones.

If two finite sets, for example the cards in a deck and the weeks in a year, can be matched up one to one they must have the same number of elements.

Bijections

Mathematicians call a 1:1 correspondence a **bijection.**

Cantor used this approach to compare infinite sets: if there is a bijection between them, two sets are said to be **the same size.**

Cantor built an entire theory of infinity on this idea.

Hilbert's Hotel

We will look at a fantasy devised by David Hilbert.

We could call it a Gedankenexperiment

It was introduced by Hilbert in 1924 in a lecture

Über das Unendliche.

Hilbert's Hotel

Hilbert's Grand Hotel

Hilbert's Hotel (an amusing metaphor) illustrates some of the surprising and counter-intuitive properties of infinity.

He imagined a hotel with an infinite number of rooms.

Even with the hotel full, there is always room to accommodate an extra guest.

Hilbert's Grand Hotel

Hilbert's Hotel (an amusing metaphor)
illustrates some of the surprising and
counter-intuitive properties of infinity.

He imagined a hotel with an infinite number of rooms.

Even with the hotel full, there is always
room to accommodate an extra guest.

Simply move guest 1 to room 2, guest 2 to room 3
and so on, thereby vacating the first room.

Indeed, an infinite number of new arrivals could be accommodated: for all rooms n, move the guest in room n to room $2n$, and magically all the odd-numbered room become vacant.

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Indeed, a countably infinite number of busses, each with a countably infinite number of passengers, can be accommodated.

Video:
https://www.youtube.com/watch?v=Uj3_KqkI9Zo&t=191s

https://mathigon.org/world/Infinity

Outline

William Rowan Hamilton

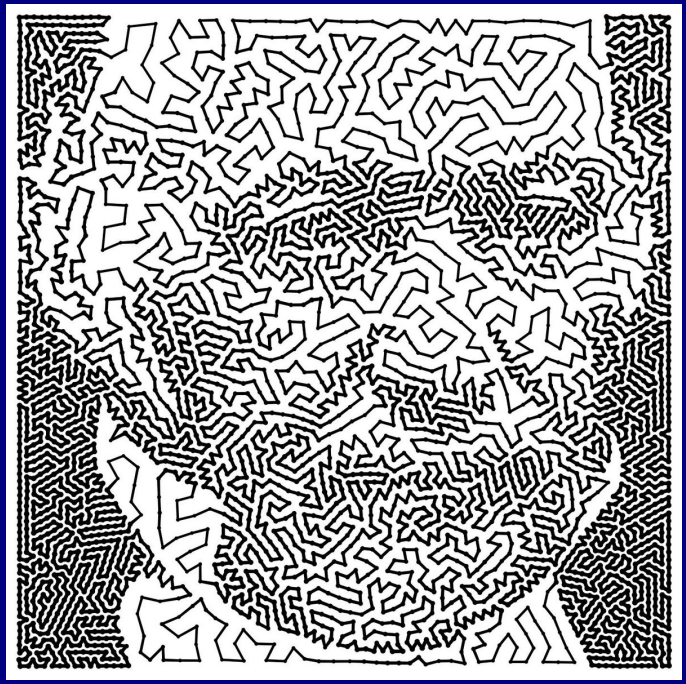
William Rowan Hamilton was Ireland's most renowned mathematician.

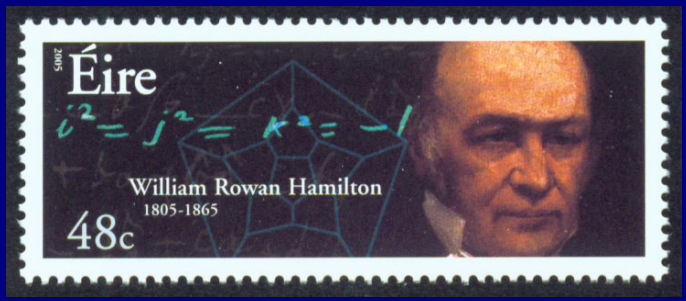
He made fundamental contributions to mathematics and physics.

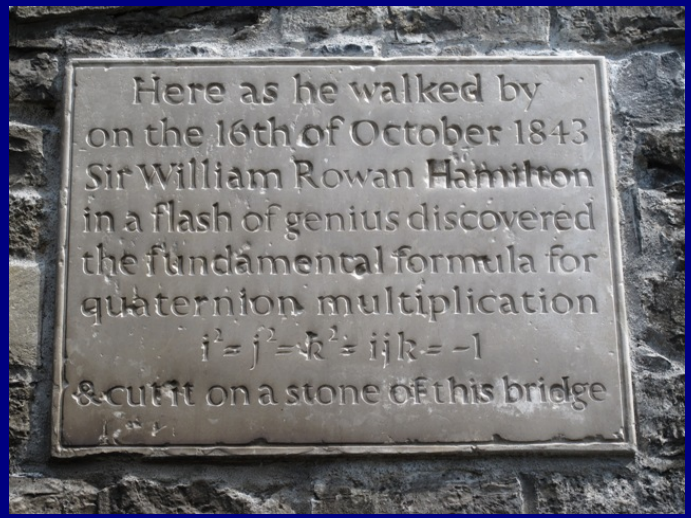
His discoveries include

- Least Action Principle
- Canonical equations of dynamics
- Quaternions

Conical refraction / Bridge / Stamps




Éire
2005
$i^2 = j^2 = k^2 = -1$
William Rowan Hamilton
1805-1865
48c


Here as he walked by
on the 16th of October 1843
Sir William Rowan Hamilton
in a flash of genius discovered
the fundamental formula for
quaternion multiplication
$i^2 = j^2 = k^2 = ijk = -1$
& cut it on a stone of this bridge

Hamilton's Icosian Game

Hamilton invented a game, called The Icosian Game.

It involves finding a path along a graph that visits every vertex and returns to the starting point.

Such a solution is called a Hamiltonian Cycle.

Dodecahedron

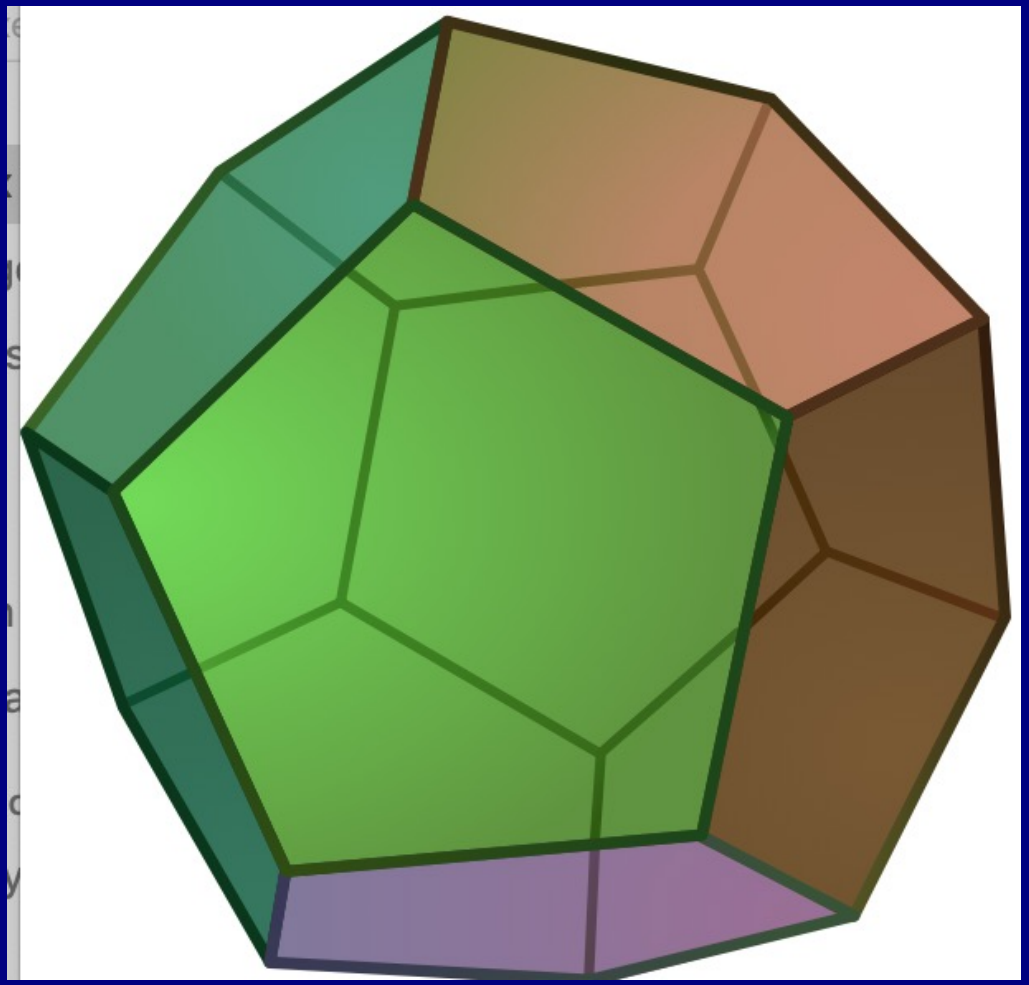


Figure: See Wikipedia page Dodecahedron.

Icosian Grid with Irish Locations

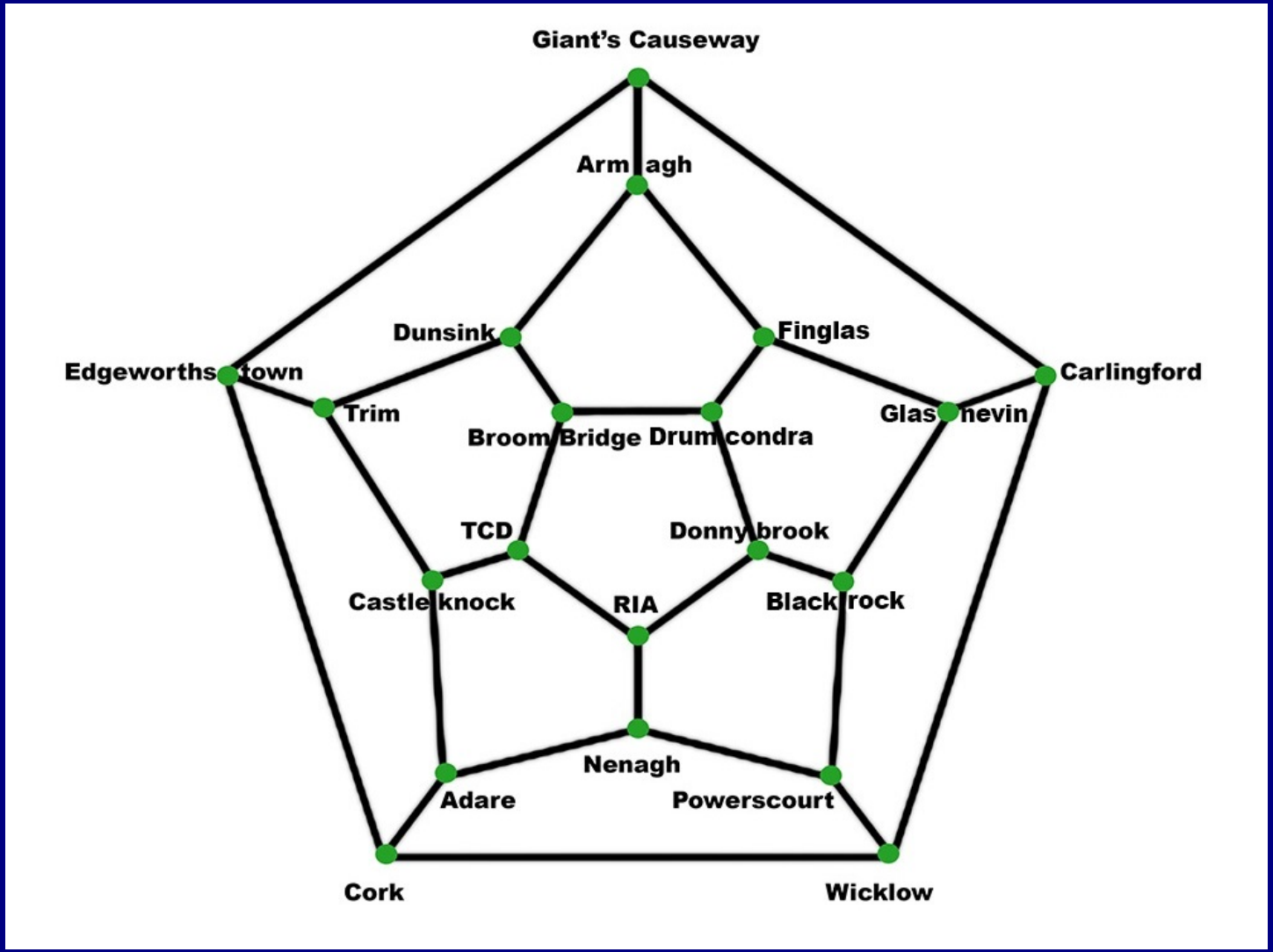


Figure: Credit Colm Mulcahy

Outline

Topology: a Major Branch of Mathematics

Topology is all about continuity and connectivity.

We will look at a few examples of connectivity.

- A Circle
- A Square
- A Triangle

What makes them different?
What makes them the same?

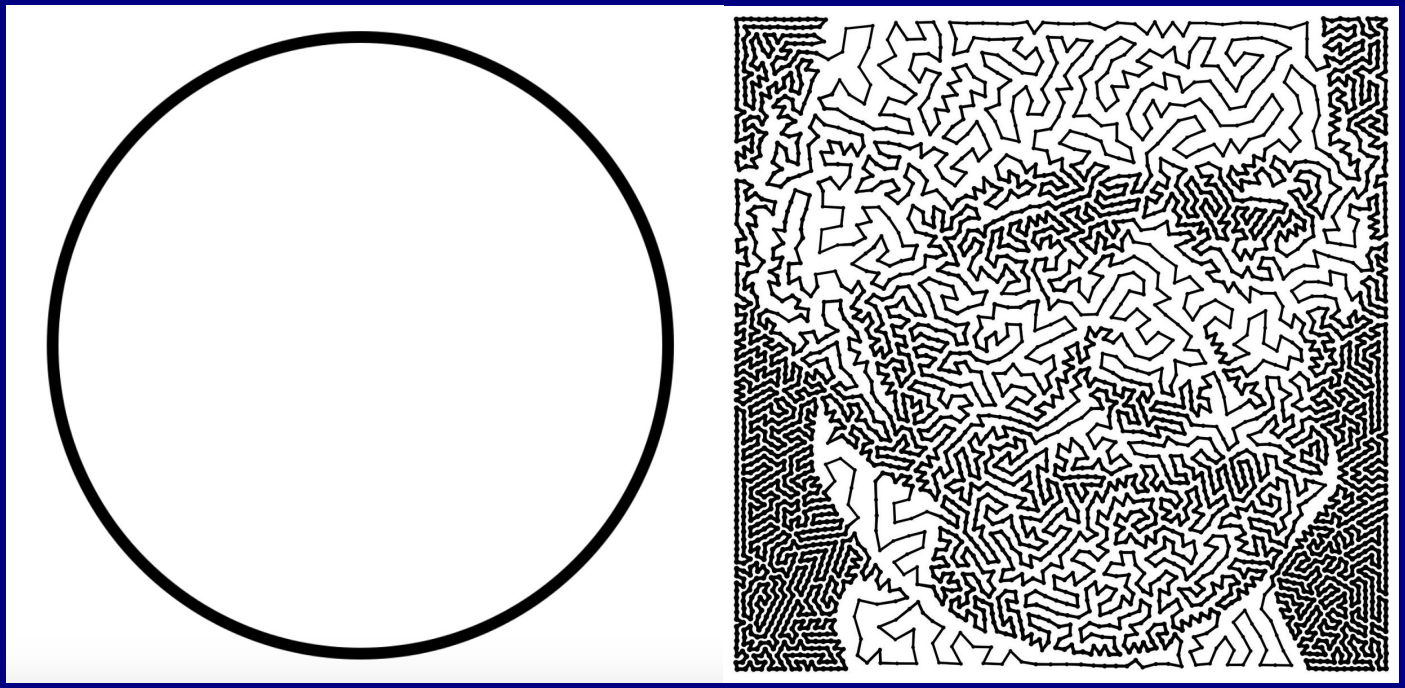


Figure: Topologically equivalent curves in the plane

Jordan Curves are topologically equivalent to a circle.

They are also called simple closed curves and are important for the Travelling Salesman Problem.

Topology: a Major Branch of Mathematics

Topology is all about continuity and connectivity.

We will look at a few aspects of Topology.

- The Bridges of Königsberg
- Doughnuts and Coffee-cups
- Knots and Links
- Nodes and Edges: Graphs
- The Möbius Band

Let us start with the London Underground Map.

The London Underground Map

Figure: Topographical map of the Underground

The London Underground Map

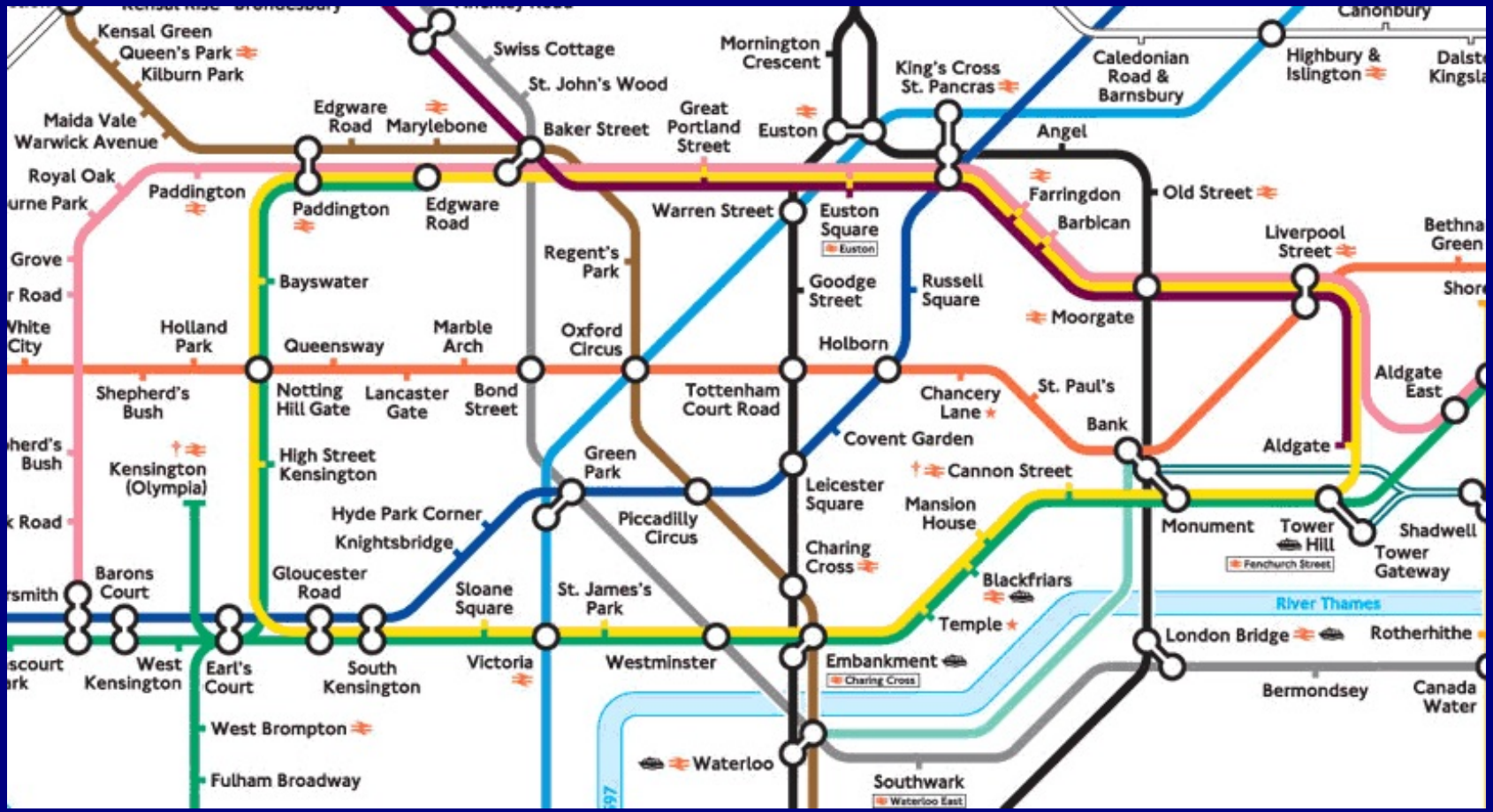


Figure: Topological map of the Underground

The London Underground Map

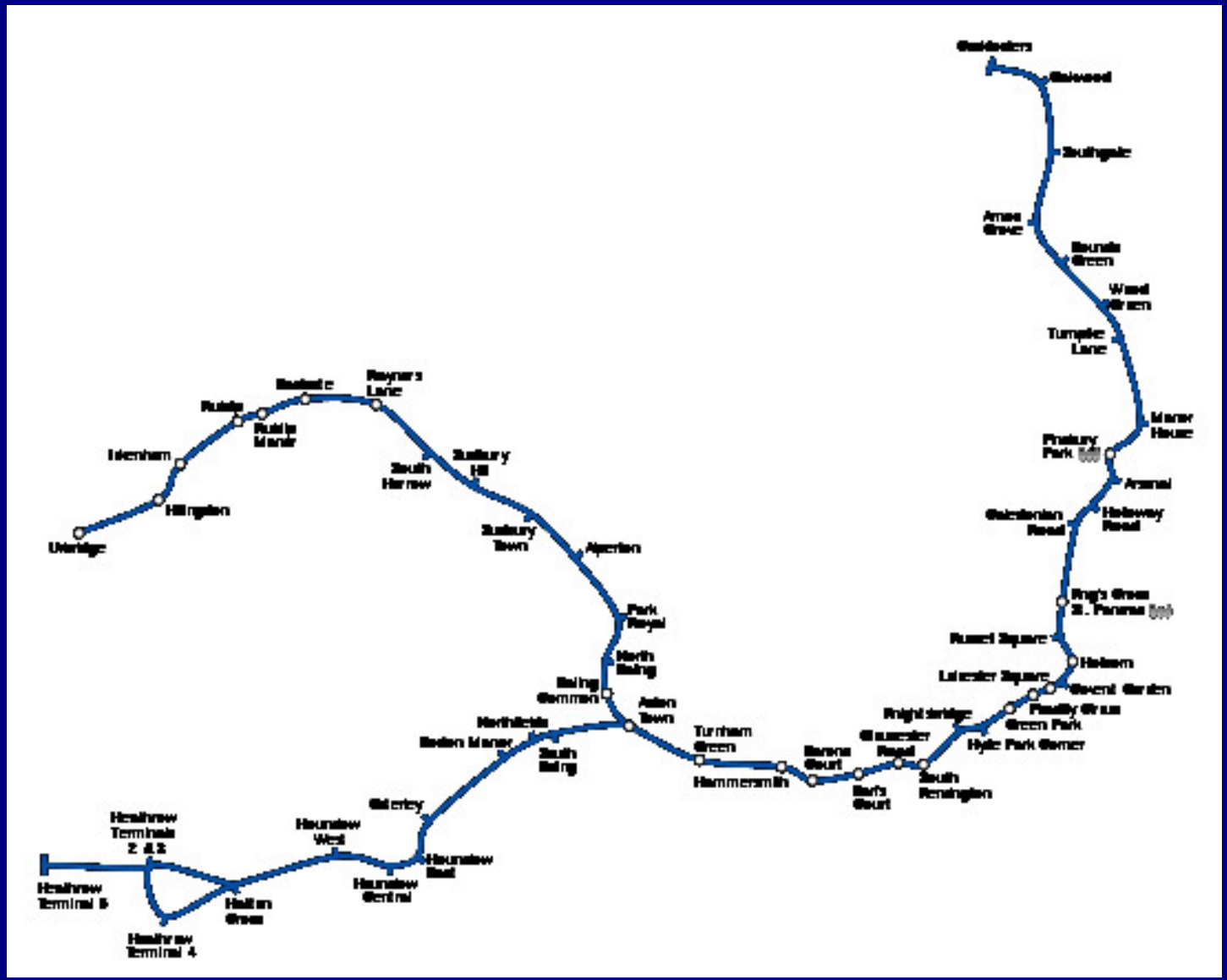
Properties of a simple closed loop:

- No branches. No travel options.
- Start anywhere: end up there again.
- Definite direction (CW or CCW).
- An Inside and an Outside.

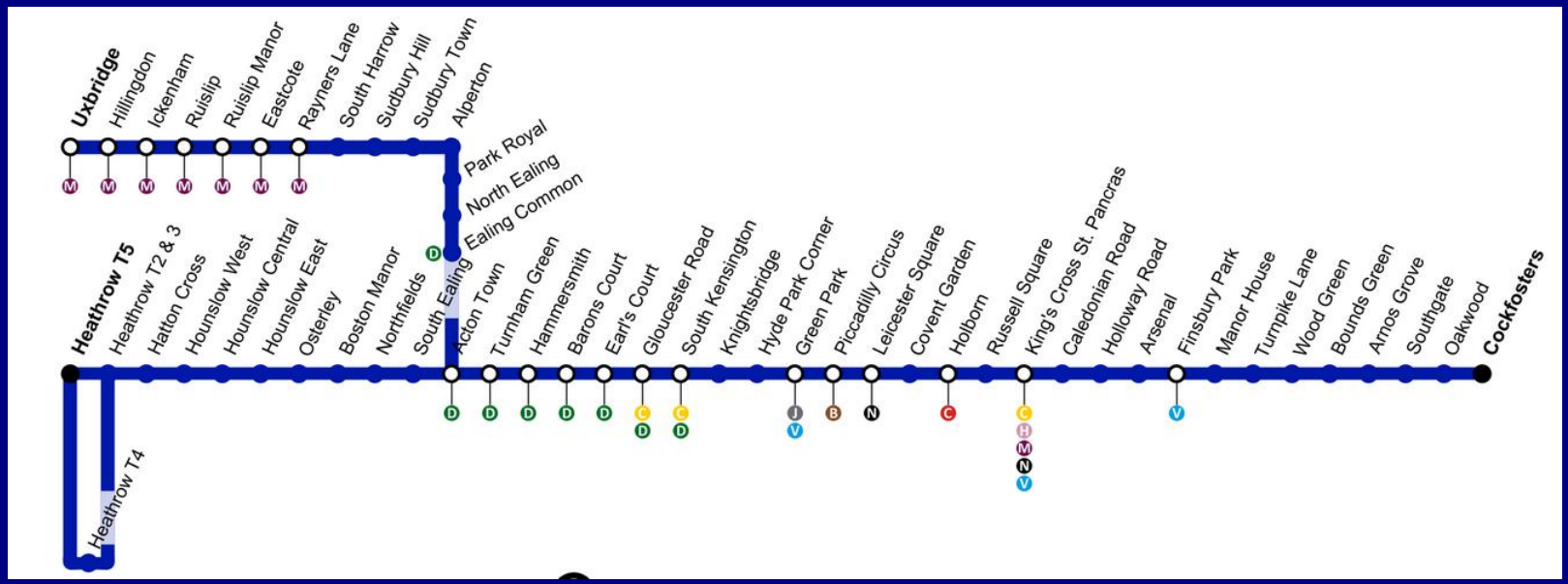
It is topologically equivalent to a circle.

Draw a (complicated) simple loop.

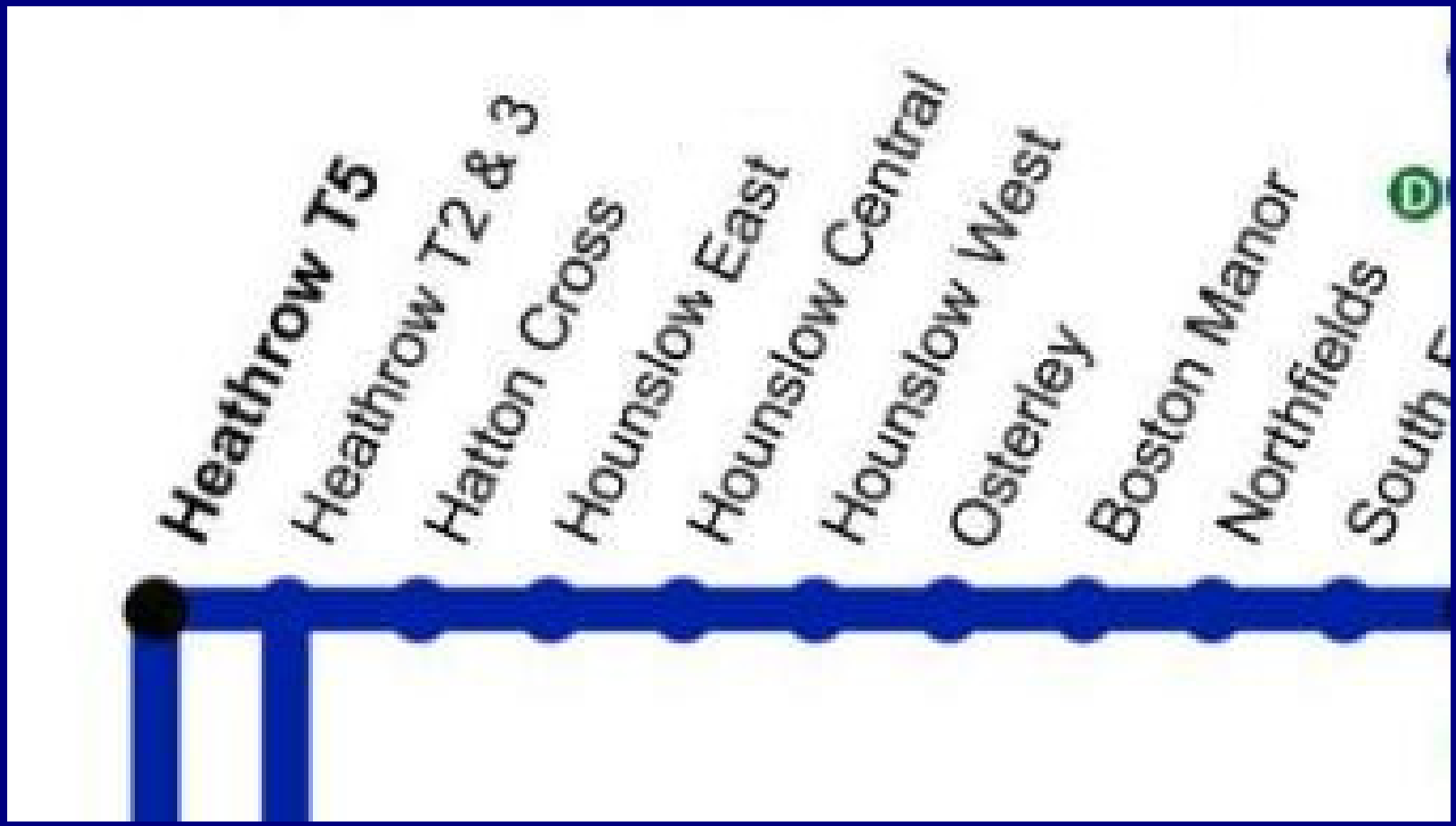
Piccadilly Line, Topographic



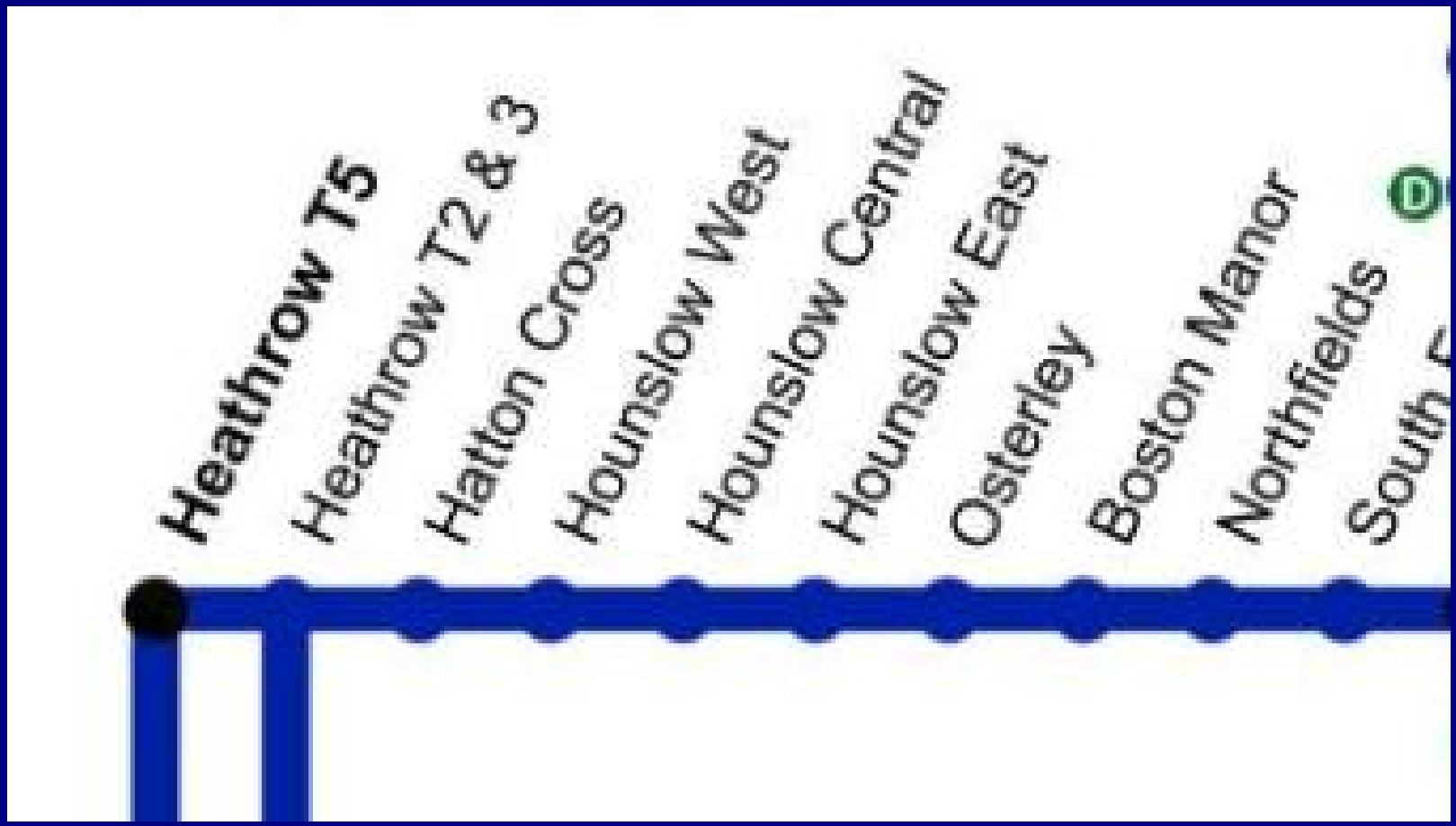
Piccadilly Line, Topological



Piccadilly Line, Detail



Piccadilly Line, Detail



Spaghetti Junction on M50

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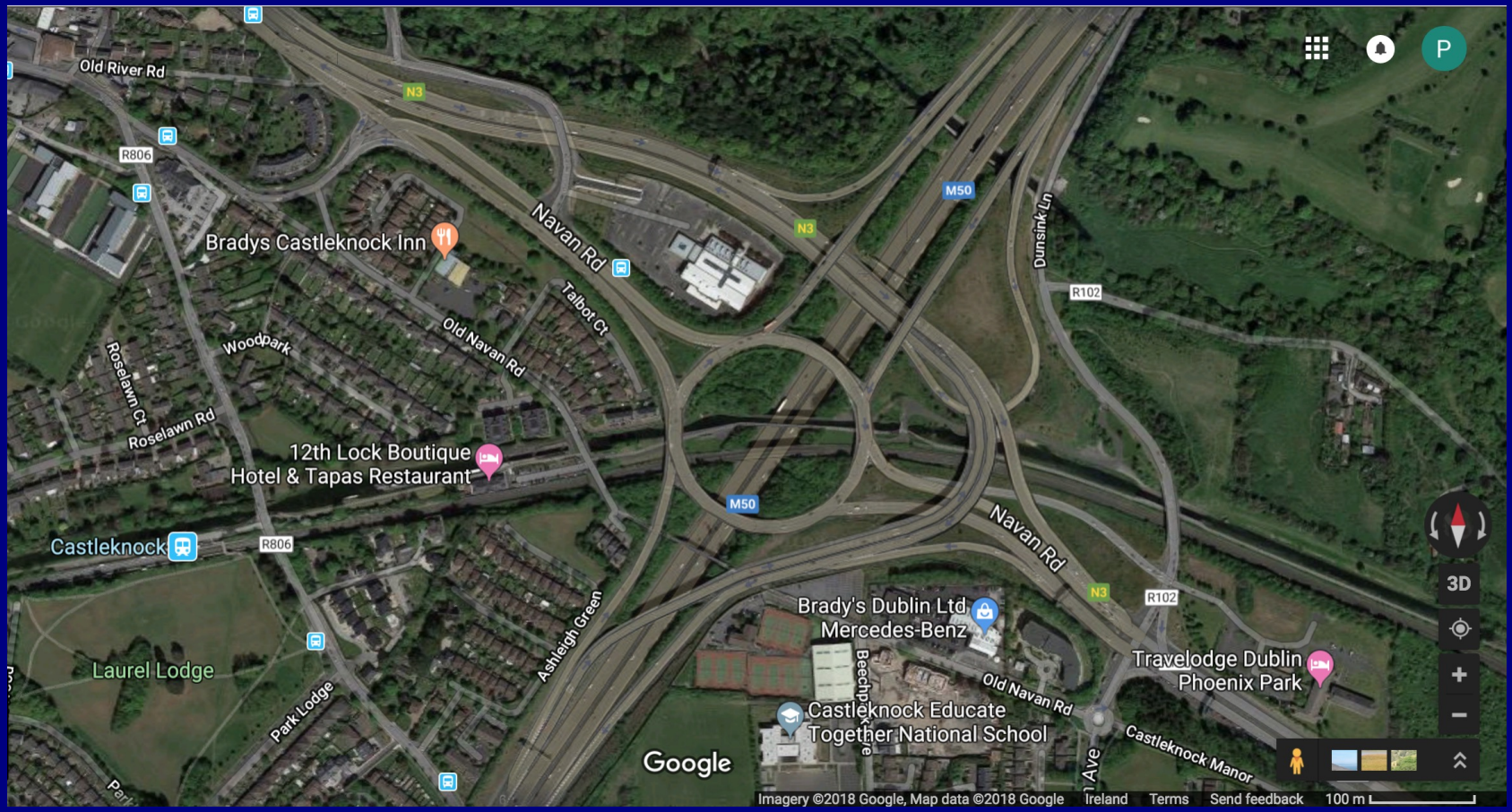
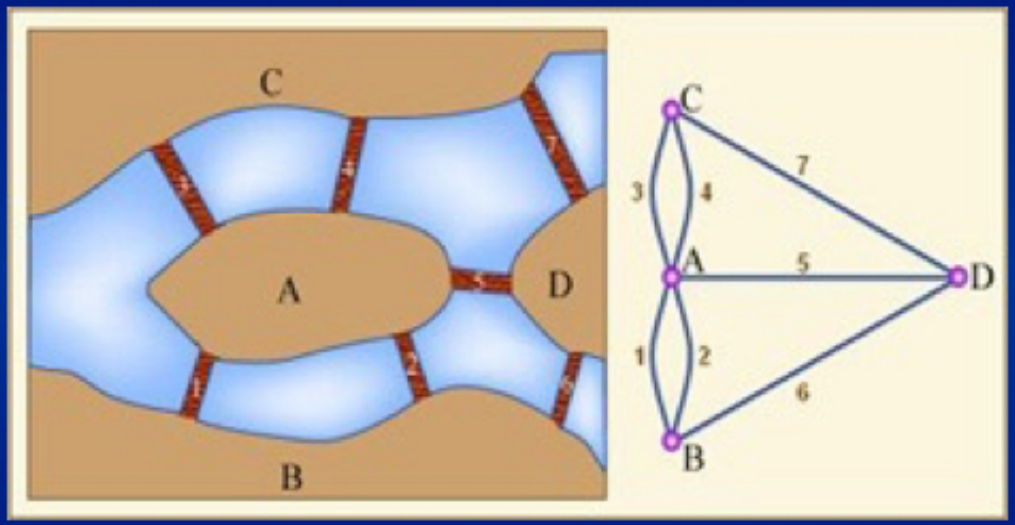
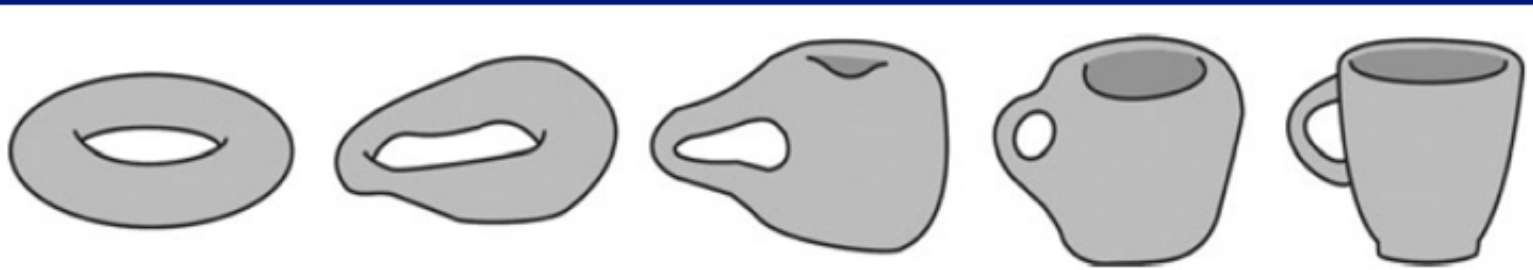


Topology
is often called
Rubber Sheet Geometry





Definition of a Topologist

Continuous distortion without tearing or glueing.

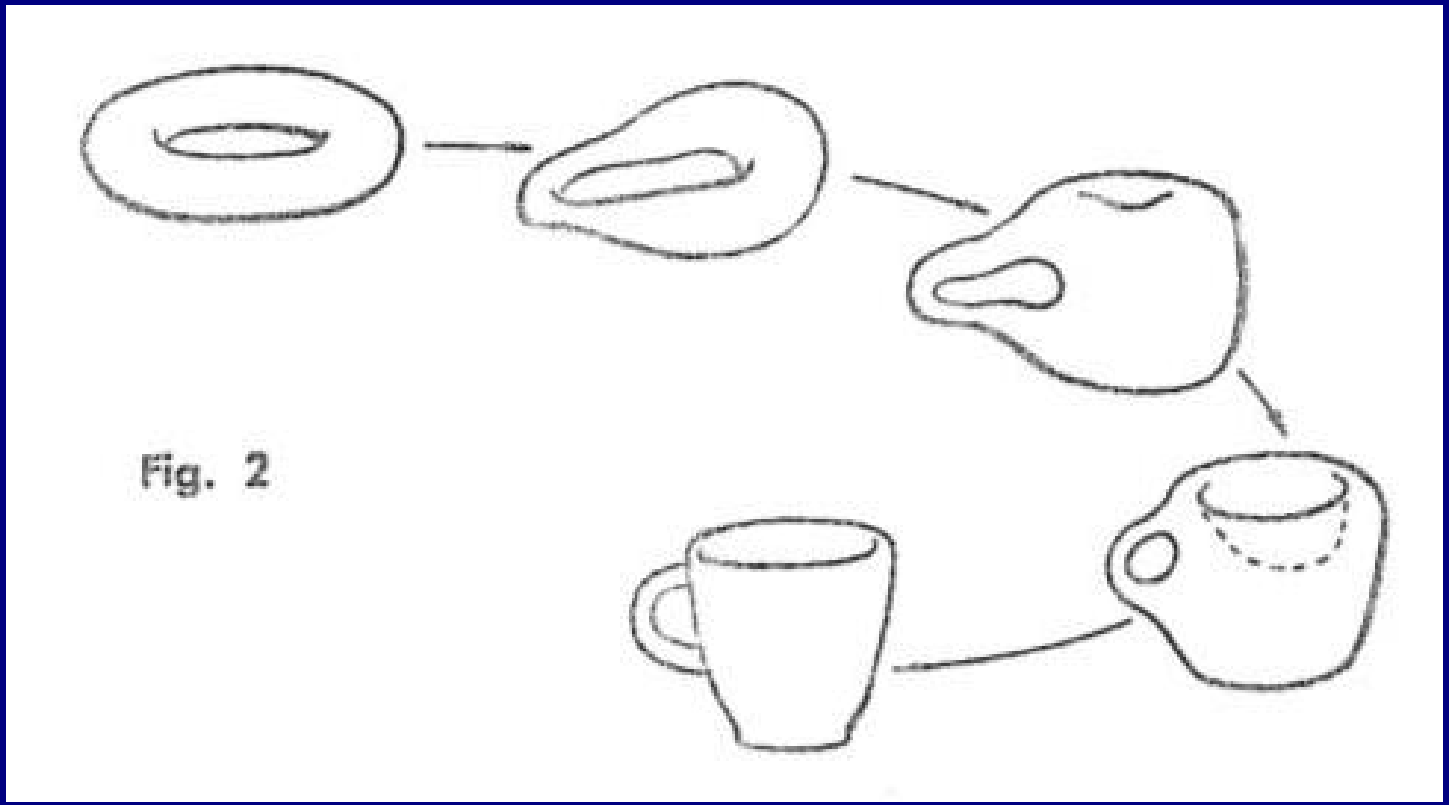


Figure: "A topologist is someone who doesn't know the difference between a doughnut and a coffee-cup." [Joke!]

Topological Invariance

Topology is about Continuity and connectedness.

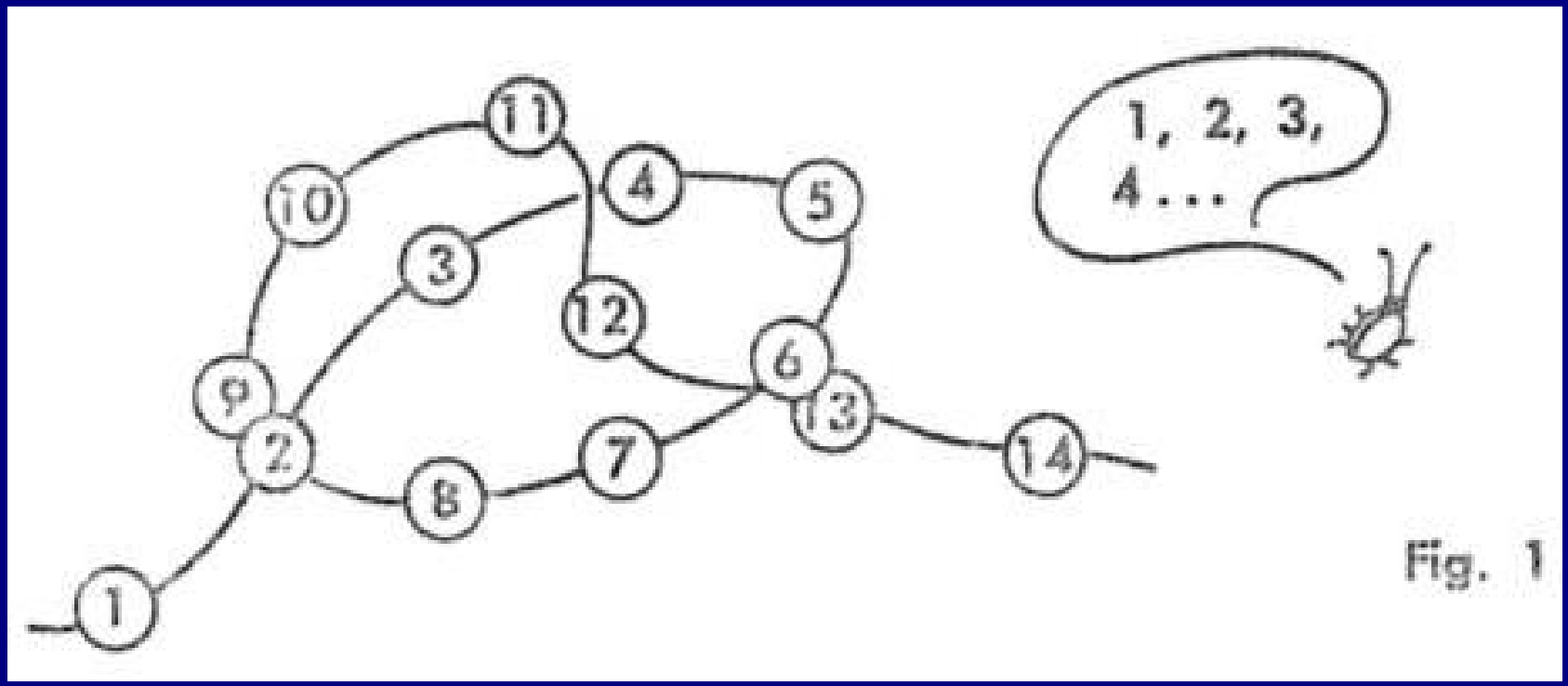
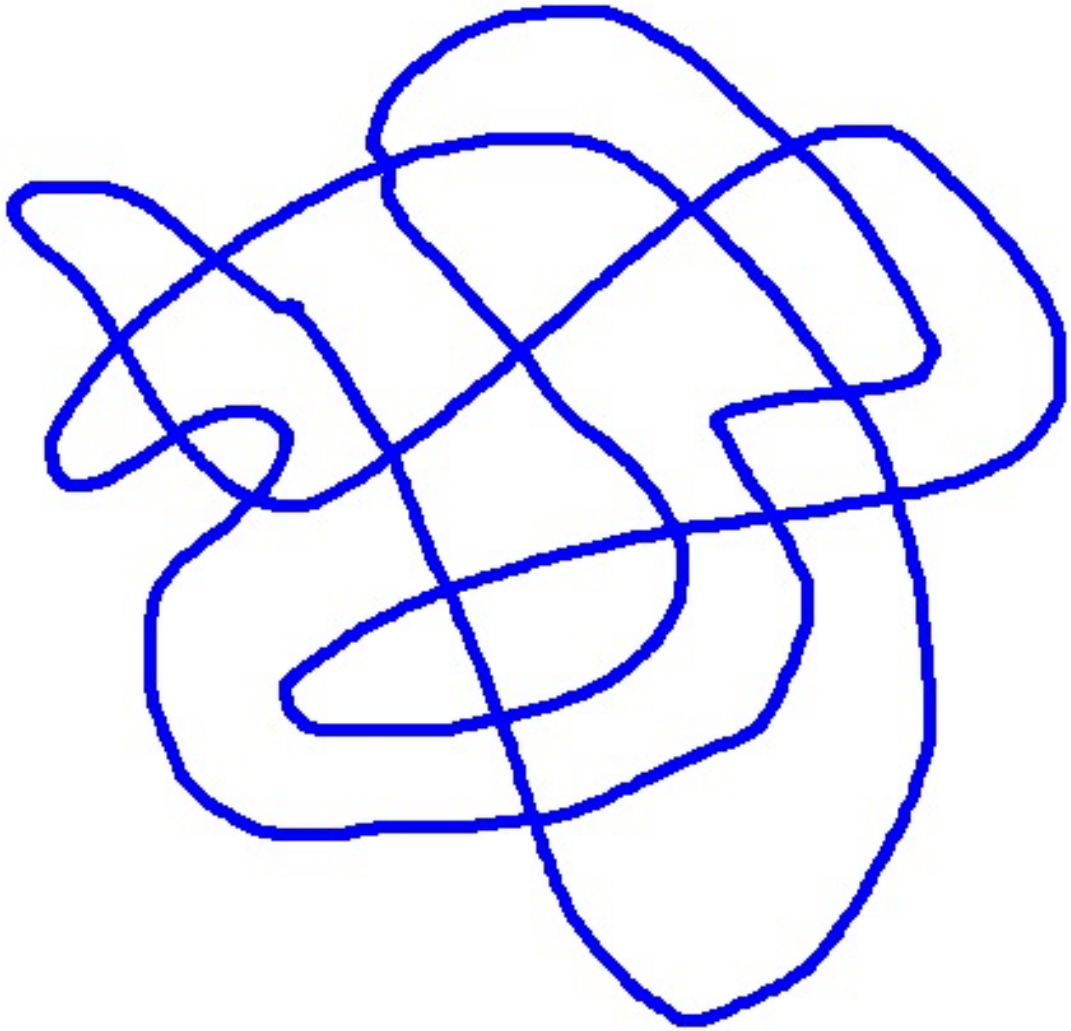


Figure: Order of points unchanged under distortion. A bug sees only the order of the points, not the shape of the curve.

The Doodle Theorem



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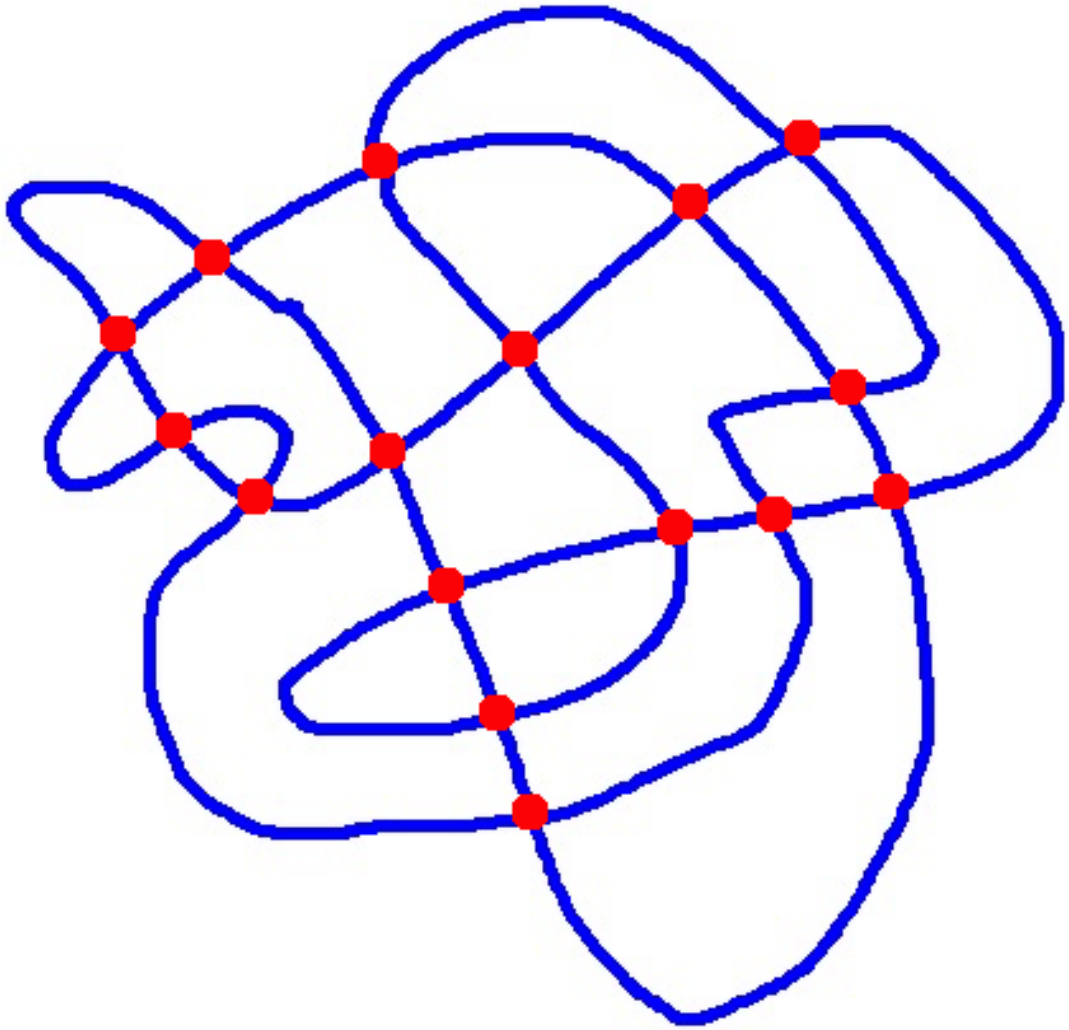

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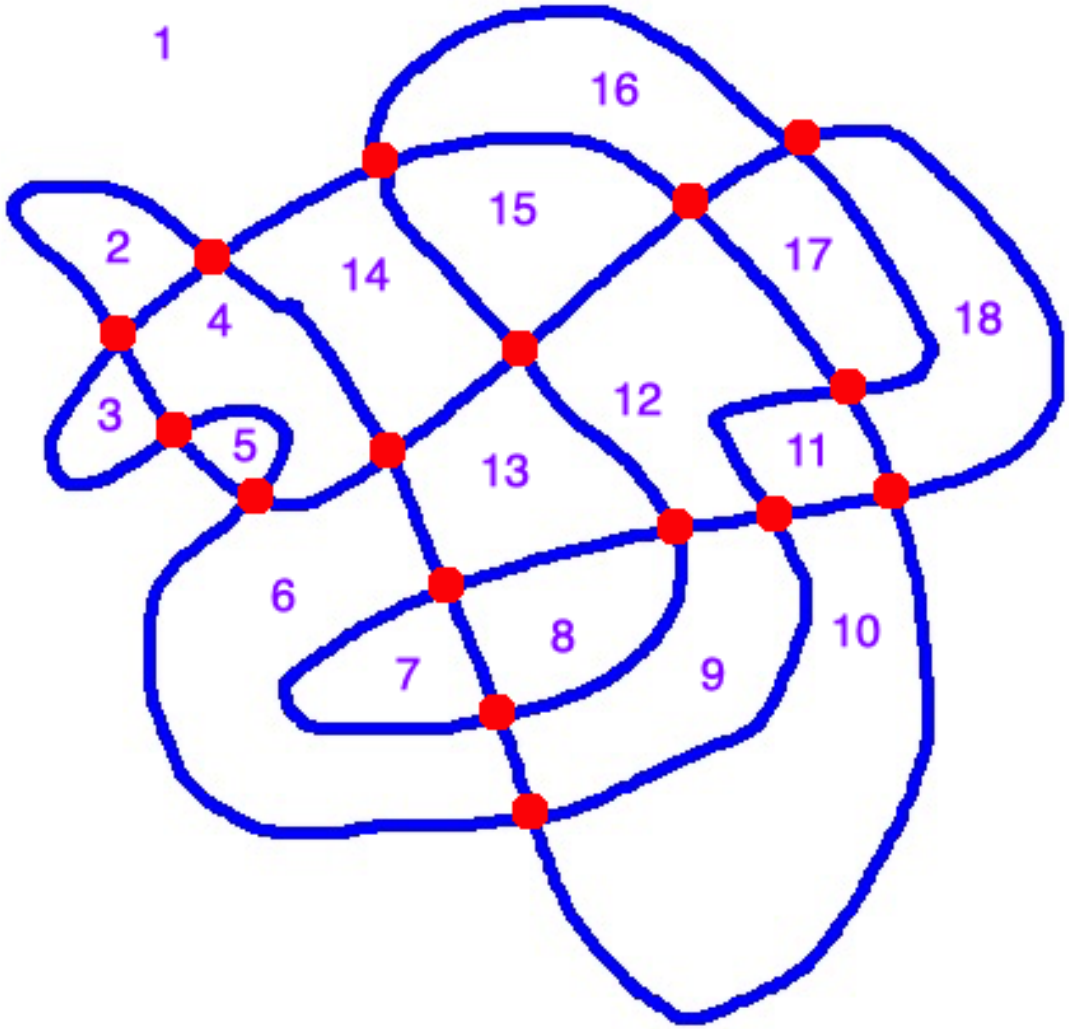

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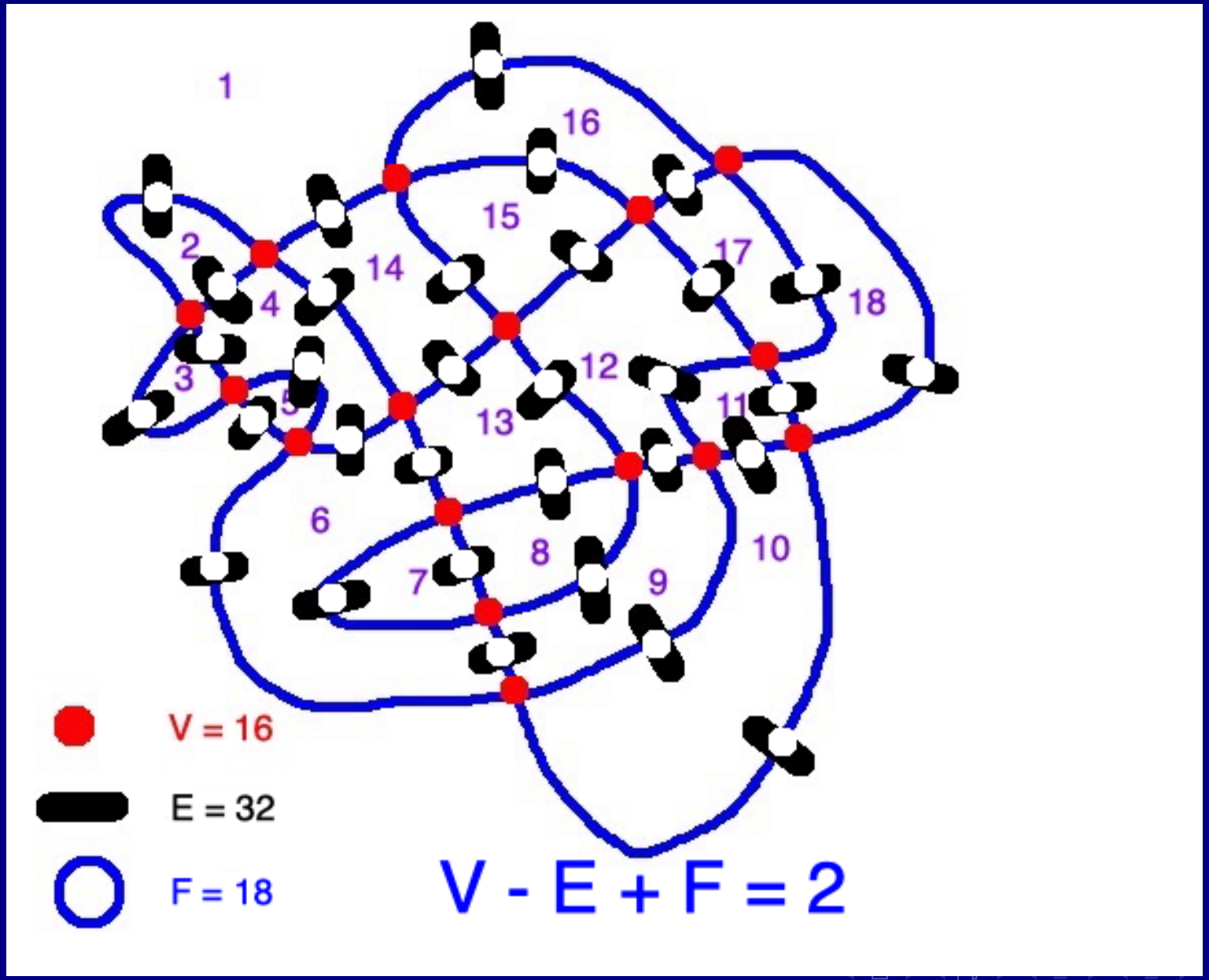
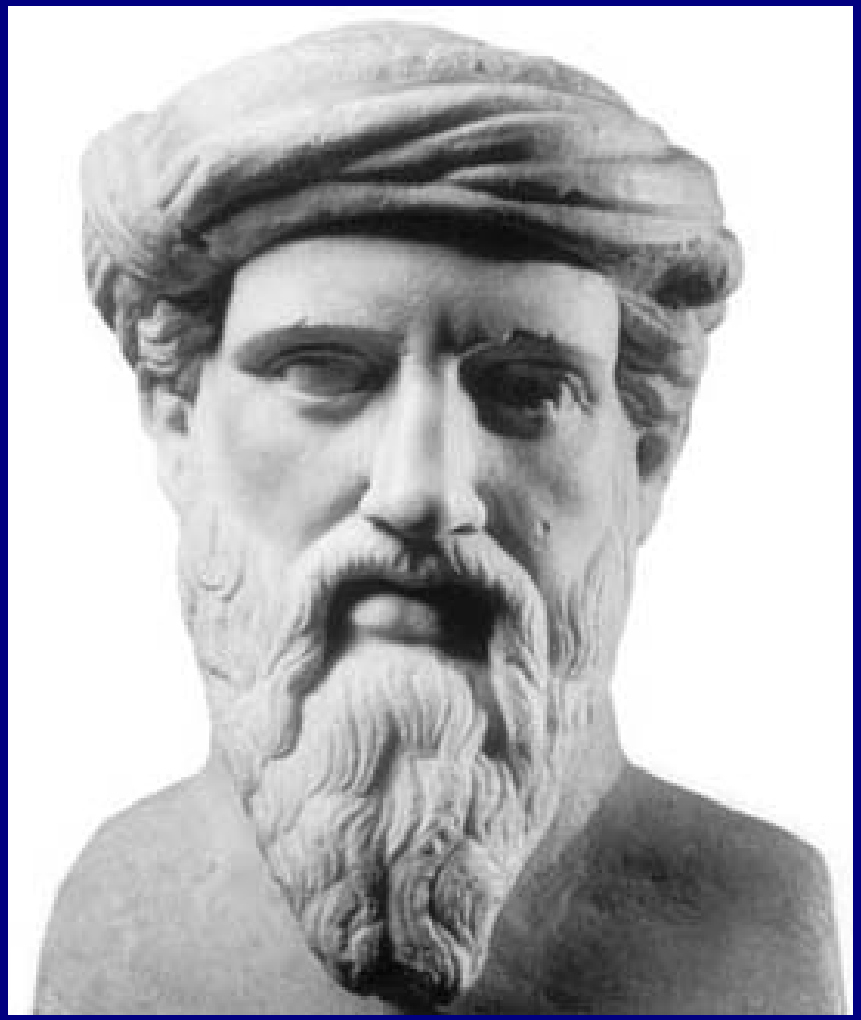


Outline



The Thallasic Age

The period from 800 BC to AD 800.

$\Theta\alpha\lambda\alpha\sigma\sigma\alpha$ **— the Sea.**

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- **The first Olympic Games in 776 BC**
- **Homer and Hesiod lived around 700 BC**
- **Greek mathematics began to thrive**
- **First two major figures: Thales and Pythagoras.**

Pythagoras (c. 570–495 BC)

Pythagoras was

- Born on the island of Samos (off Turkey).
- Philosopher, mystic, prophet and religious leader.
- Contemporary with Confucius and Lao-Tzu.

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Words philosophy (love of learning) and mathematics (that which is learned) attributed to Pythagoras.

May have been first person to imagine that natural phenomena can be understood through mathematics.

Pythagoras (c. 570–495 BC)

- No contemporary documents
- Myth, legend and tradition
- Second or third hand accounts
  often written centuries later
- Aristotle's biography no longer extant.

Hardly any statement about Pythagoras uncontested.

Difficult to separate history from myth and legend.

Pythagoras (c. 570–495 BC)

- Travelled to Egypt, Babylon and perhaps India
- Mathematics, astronomy and religious lore
- Theorem on right-angled triangles
- Result known to Babylonians 1000 years earlier
- No record of a proof by Pythagoras survives.

The Pythagoreans

Around 530 BC Pythagoras moved to Croton in Magna Graecia (now Southern Italy).

He established an organization or school (philosophical / religious / political).

Both men and women were members of "The Pythagoreans"

Adherents were very secretive:
Bound by an oath of allegiance

Led lives of temperance; observed strict moral codes.

Pythagorean Women

"Women were given equal opportunity to study as Pythagoreans, and learned practical domestic skills in addition to philosophy.

"Women were held to be different from men, sometimes in positive ways.

"The priestess, philosopher and mathematician Themistoclea is regarded as Pythagoras' teacher; Theano, Damo and Melissa as female disciples."

From the Wikipedia article: The Pythagoreans.

Pythagorean Quotes

- “I was Euphorbus at the siege of Troy.”
- “In anger, refrain from both speech and action.”
- “Educate the children and it won’t be necessary to punish the men.”
- “Abstain from beans!”

Pythagorean Quotes

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- “There is geometry in the humming of the strings,
There is music in the spacing of the spheres.”
- “Number rules the universe.”

Harmony & Discord

By tradition, Pythagoras discovered
the principles of musical harmony.

Stringed instruments produce harmonious sounds
when string lengths are ratios of small numbers.

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Stringed instruments produce harmonious sounds when string lengths are ratios of small numbers.

Extended this idea to the heavens: planets emit sounds according to their speed of movement

Concept of the "harmony of the spheres".

Johannes Kepler: Harmonices Mundi

All is Number

The motto of the Pythagoreans: *All is Number.*

All natural phenomena in the universe can be expressed using whole numbers or ratios of them.

For the Pythagoreans, numbers were
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Modern physics holds that, at its deepest level, the universe is mathematical in nature.

This view is a topic of current serious discussion (***The Mathematical Universe***, by Max Tegmark).

Thank you