AweSums

Marvels and Mysteries of Mathematics

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LECTURE 3

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Evening Course, UCD, Autumn 2021

Outline

Introduction

The Nippur Tablet

Cutting the Plane

Set Theory II

Greek Alphabet

Counting Infinite Sets

Distraction 2B: Books

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Meaning and Content of Mathematics

The word Mathematics comes from Greek $\mu\alpha\theta\eta\mu\alpha$ (máthéma), meaning "knowledge" or "lesson" or "learning".

It is the study of topics such as

- Quantity (numbers)
- Structure (patterns)
- Space (geometry)
- Change (analysis).

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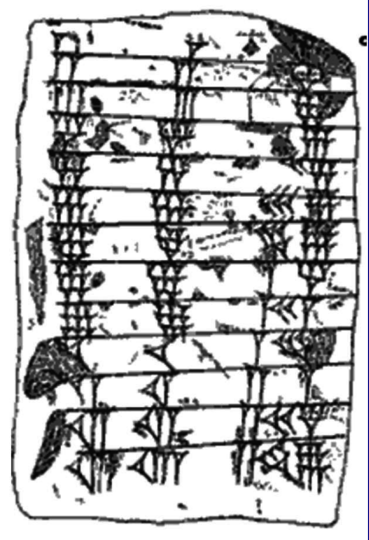
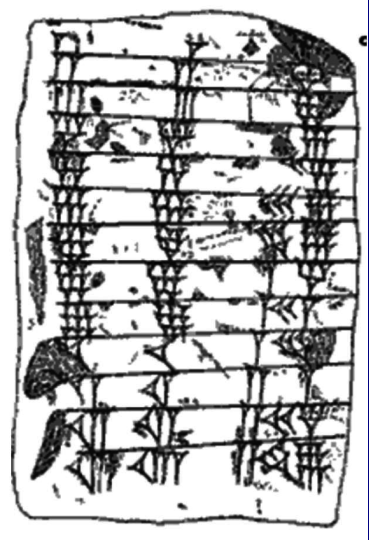
The Nippur Tablet



What is the last line?

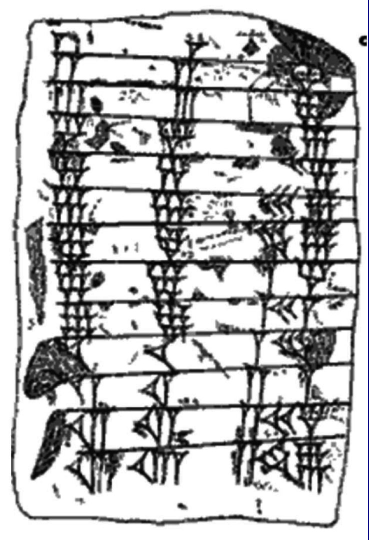
The Nippur Tablet



What is the last line?
The last line states that

$$13 \times 13 = 2 \times 60 + 49 = 169$$

The Nippur Tablet



What is the last line?
The last line states that

$$13 \times 13 = 2 \times 60 + 49 = 169$$

But it could be

$$13 \times 13 = 2 \times 60^2 + 40 \times 60 + 9$$

which comes to 9609. Babylonian numeration is ambiguous.

There is no zero!

The Nippur Tablet

What purpose could the Nippur Tablet have had?

What use could there be for a list of squares?

The Nippur Tablet

What purpose could the Nippur Tablet have had?

What use could there be for a list of squares?

Perhaps it was used for multiplication!

After a brief refresher on school maths,
we show how this can be done.

Refresher: Some School Maths

How do we do multiplication of binomials

$$(a + b) \times (c + d) \ ?$$

Refresher: Some School Maths

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$$(a + b) \times (c + d) \ ?$$

This can be evaluated by expanding twice:

$$a \cdot (c + d) + b \cdot (c + d) = a \cdot c + a \cdot d + b \cdot c + b \cdot d$$

Refresher: Some School Maths

How do we do multiplication of binomials

$$(a+b)\times(c+d)\ ?$$

This can be evaluated by expanding twice:

$$a\cdot(c+d)+b\cdot(c+d)=a\cdot c+a\cdot d+b\cdot c+b\cdot d$$

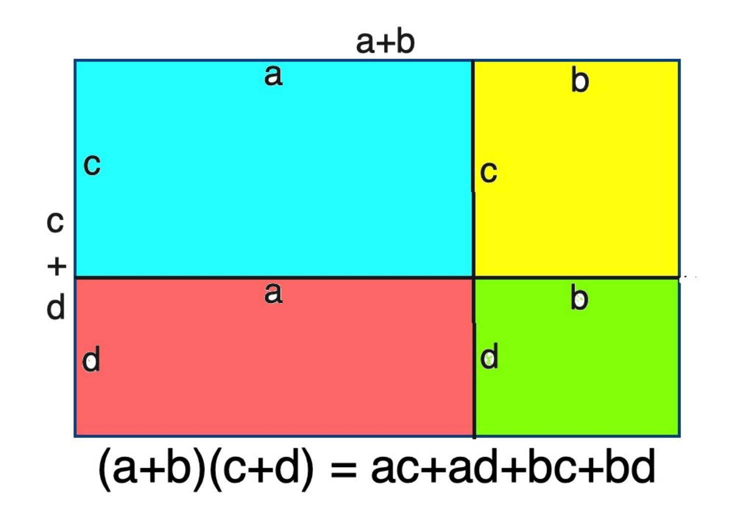
A special case where the two factors are equal:

$$(a+b)\cdot(a+b) \quad = \quad a\cdot a+a\cdot b+b\cdot a+b\cdot b$$

so that

$$(a+b)^2 \quad = \quad a^2+2ab+b^2$$

Geometric Reasoning



Multiplication by Squaring

Let a and b be any two numbers:

$$(a+b)^2 = a^2 + 2ab + b^2$$
$$(a-b)^2 = a^2 - 2ab + b^2$$

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Subtracting, we get

$$(a+b)^2 - (a-b)^2 = 4ab$$

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Subtracting, we get

$$(a+b)^2 - (a-b)^2 = 4ab$$

Thus, we can find the product using squares:

$$ab = \frac{1}{4}\left[(a+b)^2 - (a-b)^2\right] .$$

Every product is the difference of two squares ($\div 4$).

Multiplication by Squaring

$$\frac{1}{4}\left[(a+b)^2 - (a-b)^2\right] = ab$$

Let us take a particular example: $37 \times 13 = ?$

$$a = 37 \quad b = 13 \quad a+b = 50 \quad a-b = 24\,.$$

Multiplication by Squaring

$$\frac{1}{4}\left[(a+b)^2-(a-b)^2\right] = ab$$

Let us take a particular example: $37 \times 13 = ?$

$$a = 37 \quad b = 13 \quad a+b = 50 \quad a-b = 24\,.$$

$$\begin{aligned}\tfrac{1}{4}[50^2 - 24^2] &= \tfrac{1}{4}[2500 - 576] \\ &= \tfrac{1}{4}[1924] \\ &= 481 \\ &= 37 \times 13\,.\end{aligned}$$

Perhaps this was the function of the Nippur tablet.

Practicalities in Babylon

$$ab = \frac{1}{4}\left[(a+b)^2 - (a-b)^2\right] .$$

Suppose it was important to be able to multiply numbers up to, say, 100.

A full multiplication table would have 10,000 entries. With 20 products on each tablet, this would mean 500 clay tablets!

A table of squares up to 200 would require only 10 clay tablets.

Refresher: Some School Maths

How do we calculate

$$a^2 - b^2 \, ?$$

Refresher: Some School Maths

How do we calculate

$$a^2 - b^2 \, ?$$

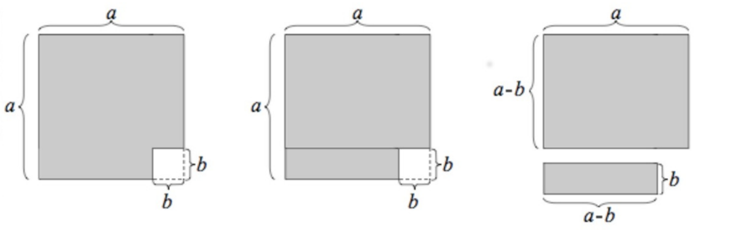
In school we may learn that

$$a^2 - b^2 = (a+b) * (a-b)$$

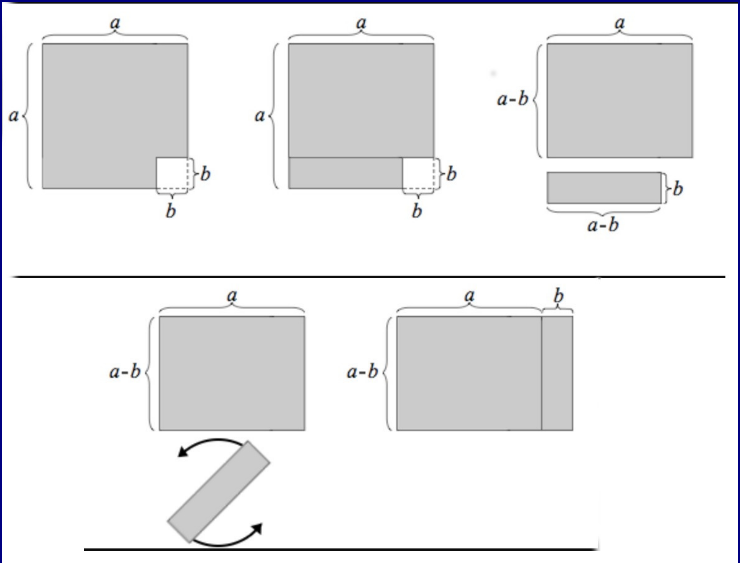
But can we make this understandable?

Yes: using pictures.

A Pictorial Proof ($a > b$)



A Pictorial Proof



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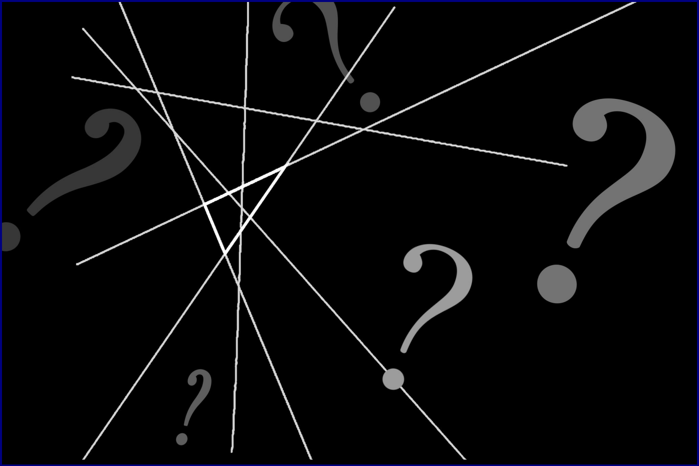
Digression: A Simple Puzzle



Six random lines. How many regions?

A Problem of Jakob Steiner

Jakob Steiner (1796-1863)

What is the <u>maximum</u> number of parts into which a plane can be divided by *n* straight lines?

Solving a Simple Puzzle

George Pólya, a famous Hungarian mathematician, wrote a book called

How to Solve It.

He gave many helpful tips for solving problems.

One of the key rules was:

If you cannot solve a problem,
Try to solve a simpler problem.

Let's do some Experimental Mathematics.

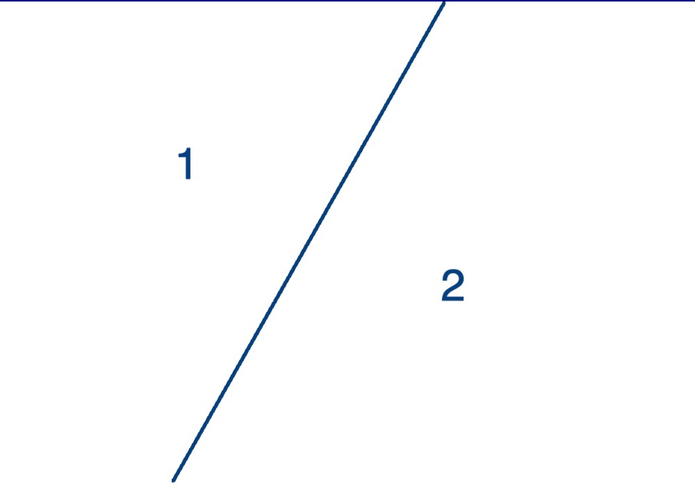
Technical Restrictions

The lines must be in a generic configuration:

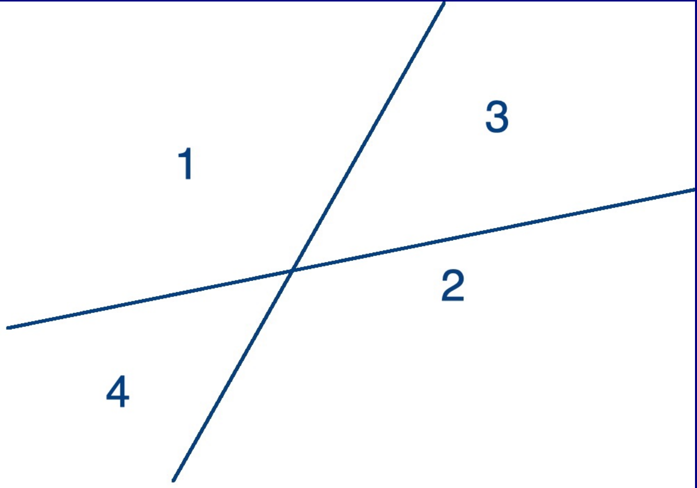
- All the lines are distinct.
- No two lines can be parallel.
- No point is on more than two lines.

If these restrictions are violated, a minute perturbation will be sufficient to remove the problem so that they are satisfied.

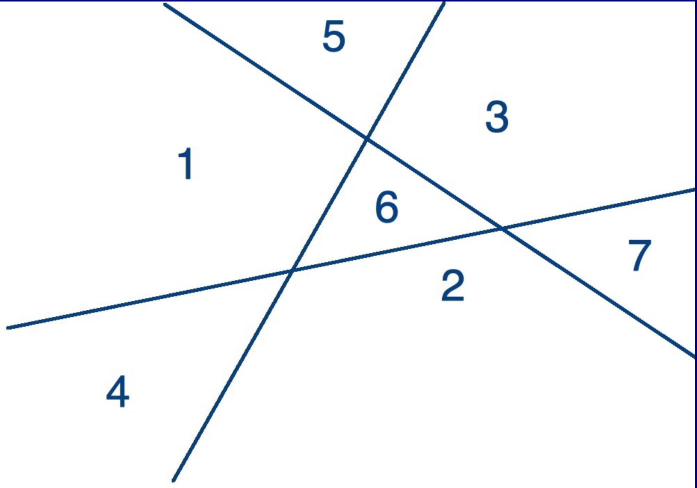
No Lines: 1 Region.



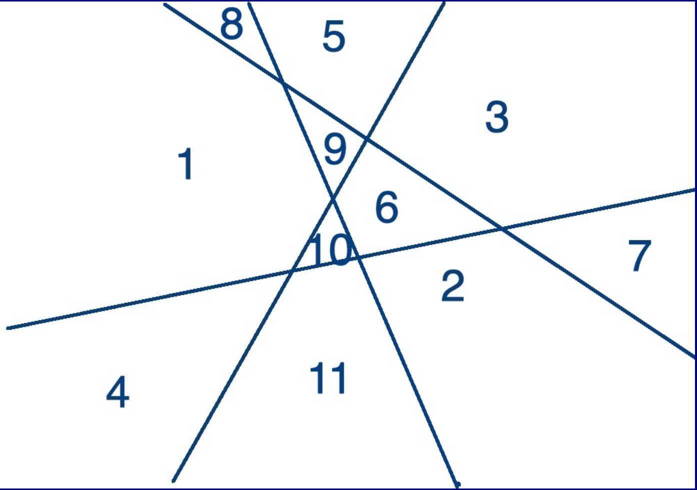
1 Line: 2 Regions.



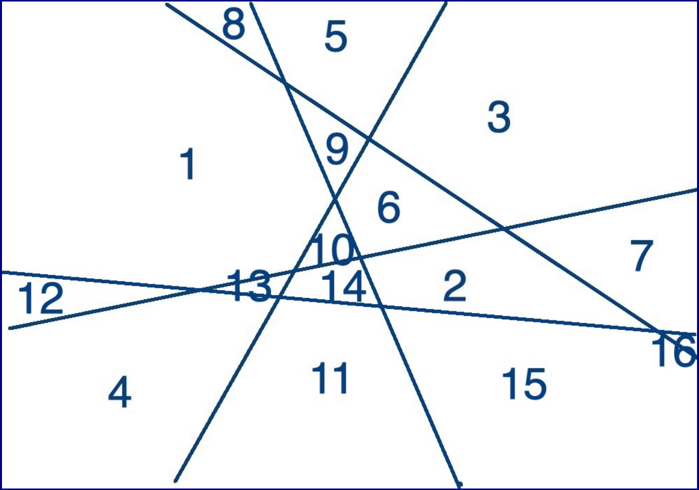
2 Lines: 4 Regions.



3 Lines: 7 Regions.



4 Lines: 11 Regions.



5 Lines: 16 Regions.

Finding a Pattern

Lines	Regions
0	1
1	2
2	4
3	7
4	11
5	16

Finding a Pattern

Lines	Regions
0	$1 = 1$
1	$2 = 1 + 1$
2	$4 = 1 + 1 + 2$
3	$7 = 1 + 1 + 2 + 3$
4	$11 = 1 + 1 + 2 + 3 + 4$
5	$16 = 1 + 1 + 2 + 3 + 4 + 5$

Finding a Pattern

Lines	Regions
0	$1 = 1$
1	$2 = 1 + 1$
2	$4 = 1 + 1 + 2$
3	$7 = 1 + 1 + 2 + 3$
4	$11 = 1 + 1 + 2 + 3 + 4$
5	$16 = 1 + 1 + 2 + 3 + 4 + 5$

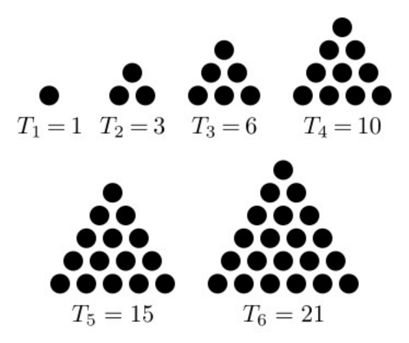
$$R_n = R_{n-1} + n \qquad \textbf{or} \qquad R_n = 1 + \frac{n(n+1)}{2} .$$

Triangular Numbers

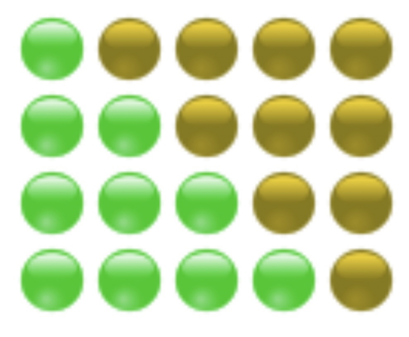
Numbers of the form

$$T_n = 1 + 2 + 3 + \cdots + n$$

are called triangular numbers:

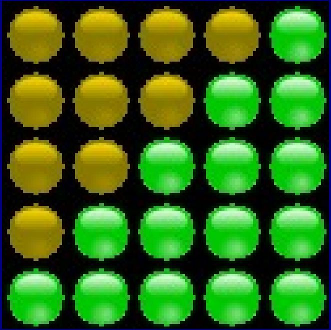


Triangular Numbers



$$2T_4 = 4(4+1) \quad \textbf{(green+yellow)} \quad T_4 = \frac{4(4+1)}{2} = 10\,.$$

Triangular Numbers



$$T_4 + T_5 = 5 \times 5 \quad \textbf{(yellow+green)}\,.$$

$$T_{n-1} + T_n = n^2\,.$$

Proving the Pattern: Heuristic Argument

We have found a pattern for the number of regions:

$$R_n = \frac{n(n+1)}{2} + 1 = \left(\frac{n^2+n+2}{2}\right) .$$

But we have not proved it mathematically.
Perhaps it breaks down for larger n.

We will not give a formal proof, but just an argument that suggests the formula is correct.

Suppose we have $n-1$ lines. The n-th line has to cross each one of the other lines.
It also has to extend in both directions.

So n new regions are created.

Consequences of the Pattern

We have found a pattern for the number of regions:

$$R_n = R_{n-1} + n \, .$$

Is this of any practical importance?

Perhaps not. But you might consider the following problem to be of interest:

How many pieces of cake can you get by making n straight slices?

This leads us to the **Lazy Caterer's Sequence.**

The Lazy Caterer's Sequence

Also known as the central polygonal numbers.

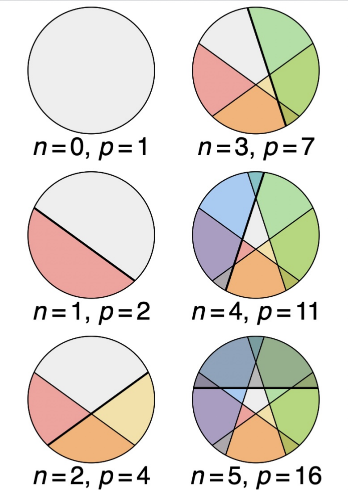
The maximum number of pieces of a disk (cake, pancake or pizza) that can be made with a given number of straight cuts.

Three cuts produce six pieces if the cuts all meet at a common point, but up to seven if they don't.

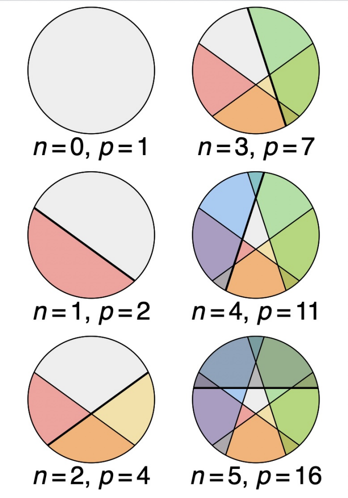
$$1, 2, 4, 7, 11, 16, 22, 29, 37, 46, 56, 67, 79, 92, 106, \ldots$$

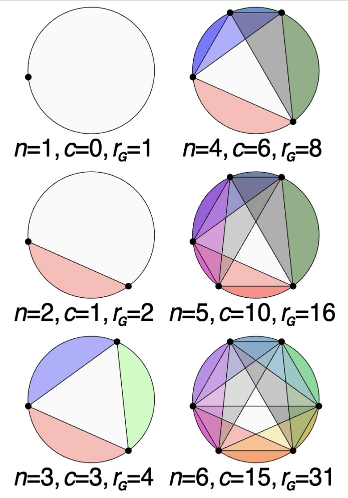
See `oeis.org`

The Pitfalls of Generalizing



The Pitfalls of Generalizing

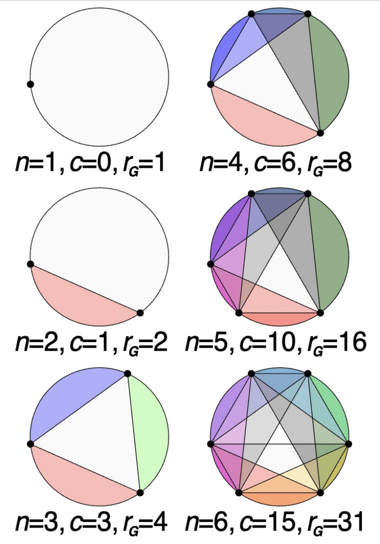




Lines **Chords**

Circle Division by Lines and Chords

The Pitfalls of Generalizing



Circle Division by Chords.

The sequence begins

$$1, 2, 4, 8, 16, \dots$$

It is tempting to assume the number of regions is

$$R_n = n^2$$

for all values of n.

But this formula breaks down for $n = 6$.

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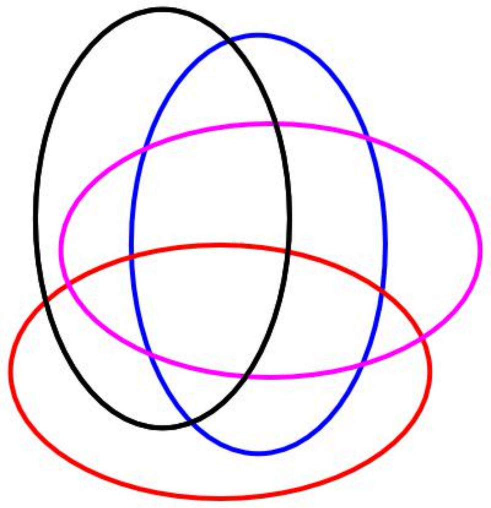
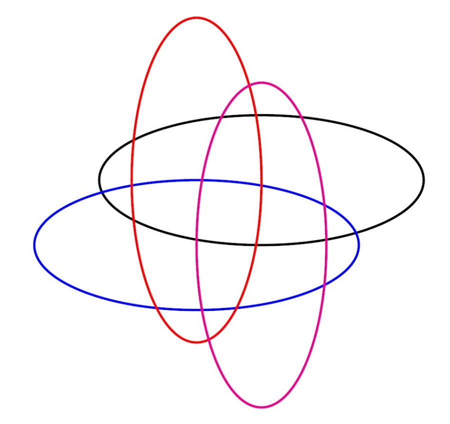
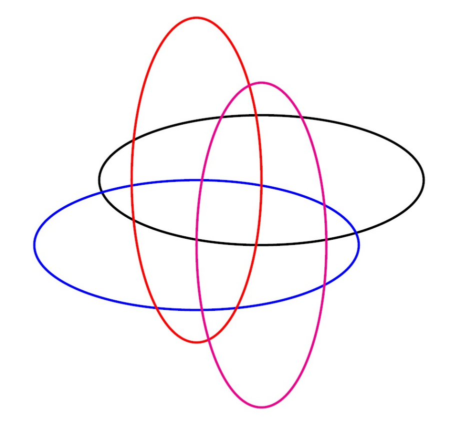
Venn Diagram for 4 Sets



Venn-4 Diagram: Symmetric

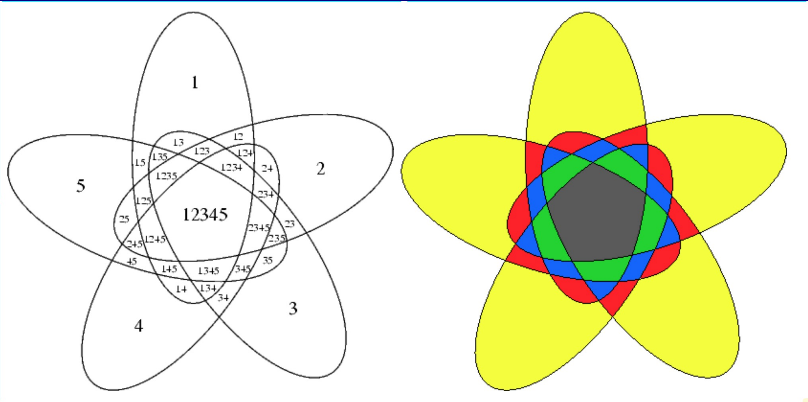


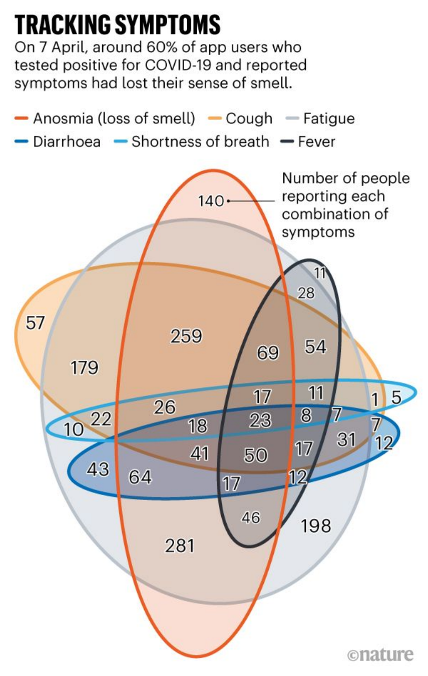
Venn-4 Diagram: Symmetric



Challenge: Construct a symmetric Venn-4 diagram.

Venn Diagram for 5 Sets



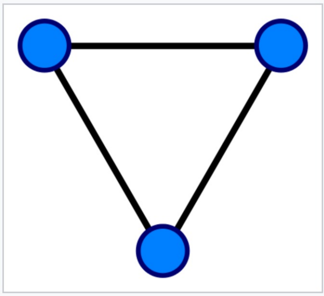


From Science journal *Nature.*

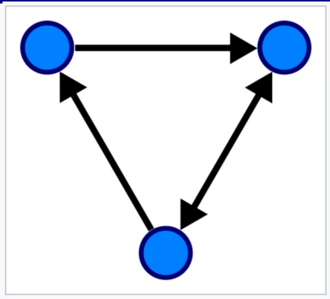
A diagram that is very poorly designed and difficult to understand.

Mathematatial Graphs: Joining the Dots

A *graph* is a set of vertices joined by edges.

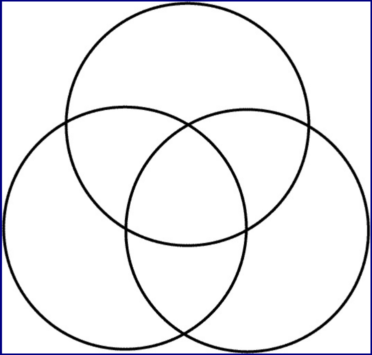


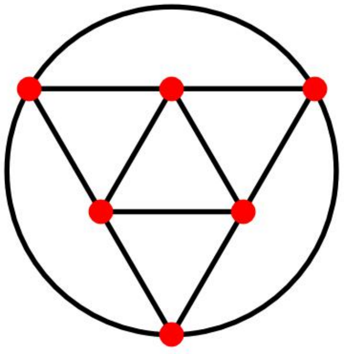
Undirected Graph



Directed Graph

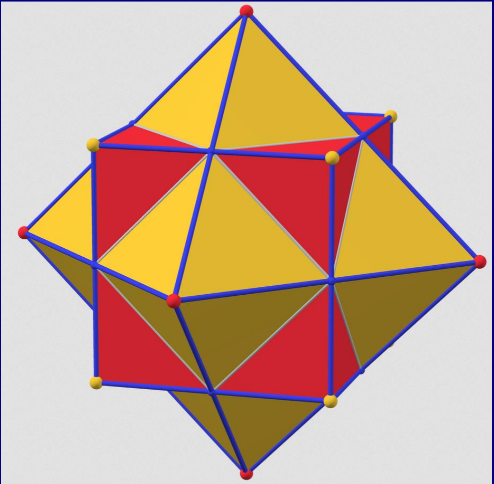
Venn Diagram as a Graph



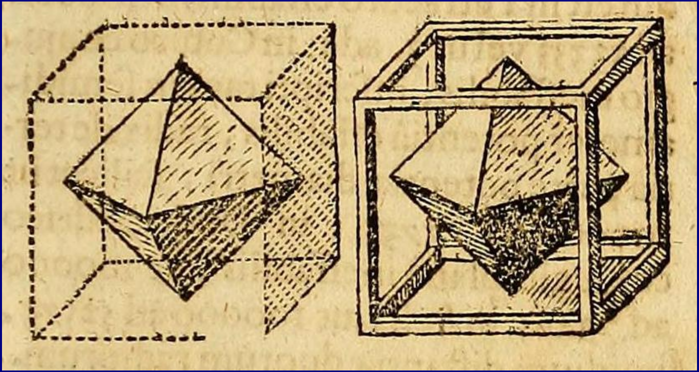


Graph is equivalent to an octahedron

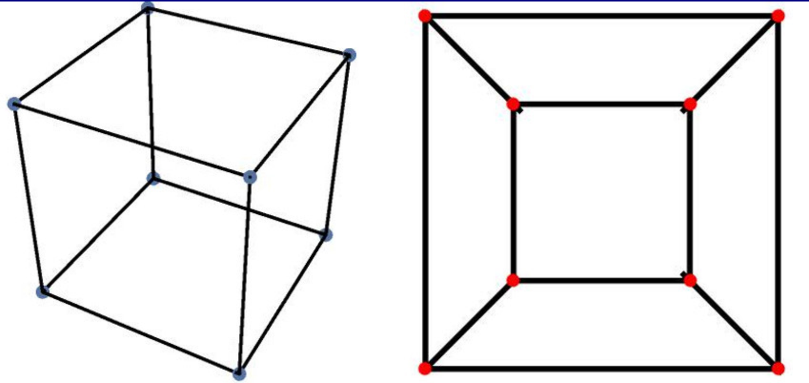
Cube and Octahedron are Duals



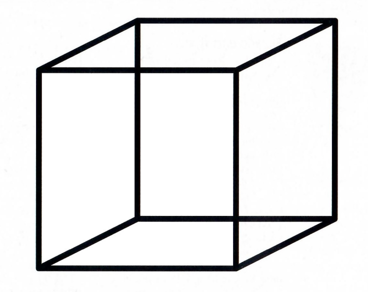
From Kepler's *Harmonices Mundi*



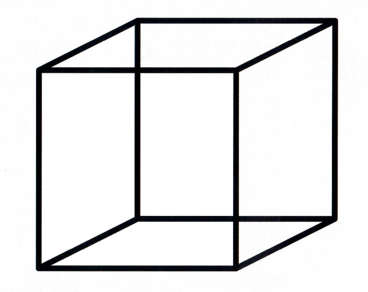
Venn3 Dual as a Cube

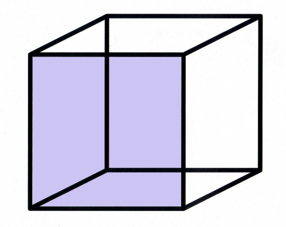


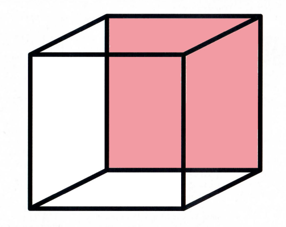
The Necker Cube



The Necker Cube







See blog post

Venn Again's Awake

on my mathematical blog thatsmaths.com

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The Greek Alphabet

Ελληνικό αλφάβητο

The Greek Alphabet

Ελληνικό αλφάβητο

Some Motivation

- Greek letters are used extensively in maths.
- Greek alphabet is the basis of the Roman one.
- Also the basis of the Cyrillic and others.

The Greek Alphabet

Ελληνικό αλφάβητο

Some Motivation

- Greek letters are used extensively in maths.
- Greek alphabet is the basis of the Roman one.
- Also the basis of the Cyrillic and others.
- A great advantage for touring in Greece.
- You already know several of the letters.
- It is simple to learn in small sections.

Ursa Major

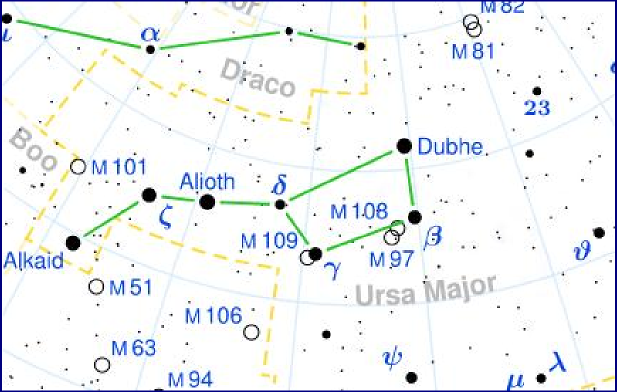

Figure: The Great Bear: Dubhe is α-Ursae Majoris.

Letter	Name	Sound	
		Ancient[5]	Modern[6]
Α α	alpha, άλφα	[a] [aː]	[a]
Β β	beta, βήτα	[b]	[v]
Γ γ	gamma, γάμμα	[g], [ŋ][7]	[ɣ] ~ [ʝ], [ŋ][8] ~ [ɲ][9]
Δ δ	delta, δέλτα	[d]	[ð]
Ε ε	epsilon, έψιλον	[e]	[e]
Ζ ζ	zeta, ζήτα	[zd][A]	[z]
Η η	eta, ήτα	[ɛː]	[i]
Θ θ	theta, θήτα	[tʰ]	[θ]
Ι ι	iota, ιώτα	[i] [iː]	[i], [ʝ],[10] [ɲ][11]
Κ κ	kappa, κάππα	[k]	[k] ~ [c]
Λ λ	lambda, λάμδα	[l]	[l]
Μ μ	mu, μυ	[m]	[m]

Letter	Name	Sound	
		Ancient[5]	Modern[6]
Ν ν	nu, νυ	[n]	[n]
Ξ ξ	xi, ξι	[ks]	[ks]
Ο ο	omicron, όμικρον	[o]	[o]
Π π	pi, πι	[p]	[p]
Ρ ρ	rho, ρώ	[r]	[r]
Σ σ/ς[13]	sigma, σίγμα	[s]	[s]
Τ τ	tau, ταυ	[t]	[t]
Υ υ	upsilon, ύψιλον	[y] [yː]	[i]
Φ φ	phi, φι	[pʰ]	[f]
Χ χ	chi, χι	[kʰ]	[x] ~ [ç]
Ψ ψ	psi, ψι	[ps]	[ps]
Ω ω	omega, ωμέγα	[ɔː]	[o]

Figure: The Greek Alphabet (from Wikipedia)

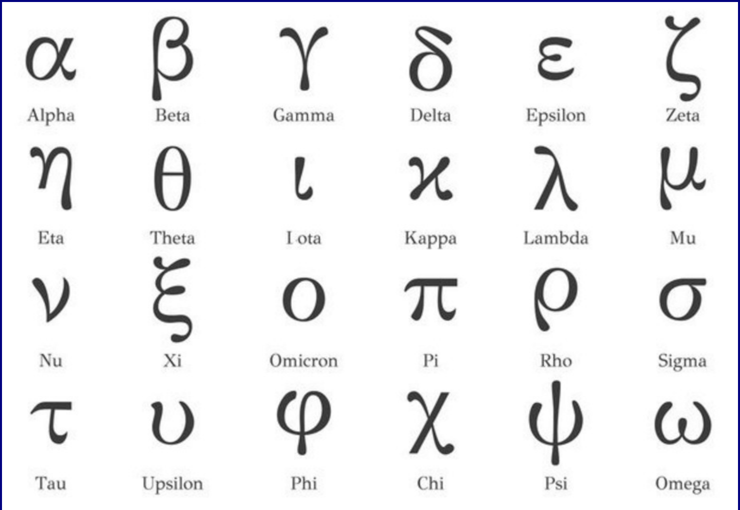

Figure: 24 beautiful letters

The First Six Letters

The first group of six letters.

α	β	γ	δ	ϵ	ζ
A	B	Γ	Δ	E	Z

The Next Six Letters

The second group of six letters.

$$\eta \quad \theta \quad \iota \quad \kappa \quad \lambda \quad \mu$$

$$\mathrm{H} \quad \Theta \quad \mathrm{I} \quad \mathrm{K} \quad \Lambda \quad \mathrm{M}$$

The Next Six Letters

The third group of six letters.

$$\nu \quad \xi \quad o \quad \pi \quad \rho \quad \sigma$$

$$\mathrm{N} \quad \Xi \quad \mathrm{O} \quad \Pi \quad \mathrm{P} \quad \Sigma$$

The Last Six Letters

The final group of six letters.

τ	υ	ϕ	χ	ψ	ω
T	Υ	Φ	X	Ψ	Ω


Αεροδρόμιο
Airport
Ξενοφώντος
Ομόνοια
Μητροπόλεως
Ερμού
Πειραιώς
Β. Σοφίας


Κέντρο
Center
Λιμάνι
Port


Μουσειο Ακρόπολης
Acropolis Museum


Αθήνα
Δελφοί
400M


ΤΑΞΙ

A Few Greek Words (for practice)

$\kappa\lambda\iota\mu\alpha\xi$
$\delta\rho\alpha\mu\alpha$
$\nu\epsilon\kappa\tau\alpha\rho$
$\kappa\omega\lambda o\nu$

$\kappa o\sigma\mu o\varsigma$
$\mu\alpha\theta\eta\mu\alpha$
$\beta\iota\beta\lambda\iota o$
$\iota\delta\epsilon\alpha$

A Few Greek Words (for practice)

$\kappa\lambda\iota\mu\alpha\xi$
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Climax: $\kappa\lambda\iota\mu\alpha\xi$
Drama: $\delta\rho\alpha\mu\alpha$
Nectar: $\nu\epsilon\kappa\tau\alpha\rho$
Colon: $\kappa\omega\lambda o\nu$

Cosmos: $\kappa o\sigma\mu o\varsigma$
Maths: $\mu\alpha\theta\eta\mu\alpha$
Book: $\beta\iota\beta\lambda\iota o$
Idea: $\iota\delta\epsilon\alpha$

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$\kappa\omega\mu\alpha$
$\psi\upsilon\kappa\eta$
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$\alpha\nu\alpha\theta\epsilon\mu\alpha$
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$\kappa\alpha\tau\alpha\sigma\tau\rho o\phi\eta$

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Coma: $\kappa\omega\mu\alpha$
Psyche: $\psi\upsilon\kappa\eta$
Crisis: $\kappa\rho\iota\sigma\iota\varsigma$

Anathema: $\alpha\nu\alpha\theta\epsilon\mu\alpha$
Ambrosia: $\alpha\mu\beta\rho o\sigma\iota\alpha$
Catastrophe: $\kappa\alpha\tau\alpha\sigma\tau\rho o\phi\eta$

Outline

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Distraction 2B: Books

There is no Largest Number

Children often express bemusement at the idea that there is no largest number.

Given any number, 1 can be added to it to give a larger number.

But the implication that there is no limit to this process is perplexing.

The concept of infinity has exercised the greatest minds throughout the history of human thought.

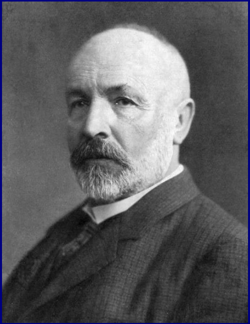
Degrees of Infinity

In the late 19th century, Georg Cantor showed that there are different degrees of infinity.

In fact, there is an infinite hierarchy of infinities.

Cantor brought into prominence several paradoxical results that had a profound impact on the development of logic and of mathematics.

Georg Cantor (1845–1918)



Cantor discovered many remarkable properties of infinite sets.

Cardinality

Finite Sets have a finite number of elements.

Example: The Counties of Ireland form a finite set.

Counties $=$ {Antrim, Armagh, . . . , Wexford, Wicklow}

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For a finite set A, the **cardinality** of A is:

The number of elements in A

One-to-one Correspondence

A particular number, say 5, is associated with all the sets having five elements.

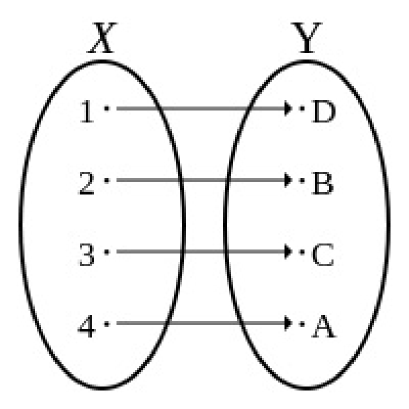
For any two of these sets, we can find a 1-to-1 correspondence between the elements of the two sets.

The number 5 is called the cardinality of these sets.

Generalizing this:

Any two sets are the same size (or cardinality) if there is a 1-to-1 correspondence between them.

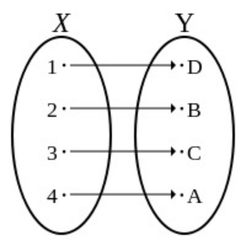
One-to-one Correspondence



Equality of Set Size: 1-1 Correspondence

How do we show that two sets are the same size?

For finite sets, this is straightforward counting.



For infinite sets, we must find a 1-1 correspondence.

Cardinality

The number of elements in a set is called the cardinality of the set.

Cardinality of a set A is written in various ways:

$$|\mathbf{A}| \qquad \|\mathbf{A}\| \qquad \mathrm{card}(\mathbf{A}) \qquad \#(A)$$

For example

$$\#\{\textbf{Irish Counties}\} = 32$$

The Empty Set

We call the set with no elements the empty set.

It is denoted by a special symbol

$$\varnothing = \{ \ \}$$

Clearly

$$\#\{ \ \} = 0 \,.$$

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We could have a philosophical discussion about the empty set. Is it related to a perfect vacuum?

The Greeks regarded the vacuum as an impossibility.

The Natural Numbers $\mathbb{N}$

The counting numbers (positive whole numbers) are

1 2 3 4 5 6 7 8 . . . .

They are also called the Natural Numbers.

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1 2 3 4 5 6 7 8 . . . .

They are also called the **Natural Numbers**.

The set of natural numbers is denoted $\mathbb{N}$.

This is our first **infinite set.**

We use a special symbol to denote its cardinality:

$$\#(\mathbb{N}) = \aleph_0$$

$$\aleph_0$$

The Power Set

For any set, we can form a new one, the **Power Set.**

The Power Set is the **set of all subsets of A.**

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Suppose the set A has just two elements:

$$\mathbf{A} = \{3, 7\}$$

Here are the subsets of A:

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The power set is

$$\mathcal{P}[\mathbf{A}] = \Big\{ \{\ \}, \{3\}, \{7\}, \{3, 7\} \Big\}$$

Cantor's Theorem

Cantor's theorem states that,
for any set A, the power set of A has a
strictly greater cardinality than A itself:

$$\#[\mathcal{P}(\mathbf{A})] > \#[A]$$

This holds for both finite and infinite sets.

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This holds for both finite and infinite sets.

This means that, for every cardinal number,
there is a greater cardinal number.

One-to-one Correspondence

Take all the natural numbers,

$$\mathbb{N} = \{1, 2, 3, ...\}$$

as one set and all the even numbers

$$\mathbb{E} = \{2, 4, 6, ...\}$$

as the other.

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as the other.

By associating each number $n \in \mathbb{N}$ with $2n \in \mathbb{E}$, we have a perfect 1-to-1 correspondence.

By Cantor's argument, the two sets are the same size:

$$\#[\mathbb{N}] = \#[\mathbb{E}]$$

Again,

$$\#[\mathbb{N}] = \#[\mathbb{E}]$$

But this is paradoxical: The set of natural numbers contains all the even numbers:

$$\mathbb{E} \subset \mathbb{N}$$

and also all the odd ones.

In an intuitive sense, $\mathbb{N}$ is larger than $\mathbb{E}$.

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and also all the odd ones.

In an intuitive sense, $\mathbb{N}$ is larger than $\mathbb{E}$.

The same paradoxical result had been deduced by Galileo some 250 years earlier.

Cantor carried these ideas much further:

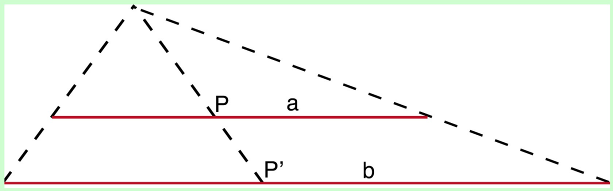
The set of all the real numbers has a degree of infinity, or cardinality, greater than the counting numbers:

$$\#[\mathbb{R}] > \#[\mathbb{N}]$$

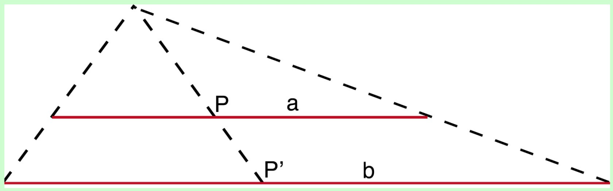
Cantor showed this using an ingenious approach called the diagonal argument.

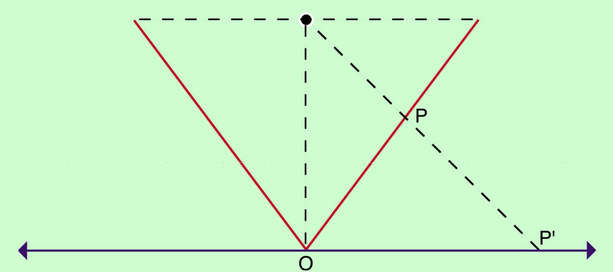
This is a fascinating technique, but we will not give details here.

How Many Points on a Line?



How Many Points on a Line?





There is a 1-1 map between $(-1, +1)$ and $\mathbb{R}$.

Review: Infinities Without Limit

For any set A, the power set $\mathcal{P}$(A) is
the collection of all the subsets of A.

Cantor proved $\mathcal{P}$(A) has cardinality greater than A.

For finite sets, this is obvious;
for infinite ones, it was startling.

The result is now known as Cantor's Theorem,
and Cantor used his diagonal argument in proving it.

He thus developed an entire hierarchy
of transfinite cardinal numbers.

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Books on a Shelf

Six books are arranged on a shelf.
They include an Almanac (A) and a Bible (B).

Suppose A must be to the left of B
(not necssarily beside it).

How many possible arrangements are there?

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Hint: Use the idea of symmetry.

ANSWER NEXT WEEK

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BIG IDEA: SYMMETRY.

Every SOLUTION correponds to a NON-SOLUTION:
Just switch the positions of A and B!

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The total number of arrangements is 6!.
For half of these, A is to the left of B.

So, answer is $\frac{1}{2}(6 \times 5 \times \cdots \times 1) = \frac{1}{2} \times 6! = 360$ Q.E.D.

Thank you