AweSums

Marvels and Mysteries of Mathematics

•

LECTURE 6

Peter Lynch
School of Mathematics & Statistics
University College Dublin

Evening Course, UCD, Autumn 2020

Outline

Outline

Meaning and Content of Mathematics

The word Mathematics comes from Greek $\mu\alpha\theta\eta\mu\alpha$ (máthéma), meaning “knowledge” or “study” or “learning”.

It is the study of topics such as

- Quantity (numbers)
- Structure (patterns)
- Space (geometry)
- Change (analysis).

Outline

Prime & Composite Numbers

A prime number is a number that cannot be broken into a product of smaller numbers.

The first few primes are 2, 3, 5, 7, 11, 13, 17 and 19.

There are 25 primes less than 100.

Numbers that are not prime are called composite. They can be expressed as *products of primes.*

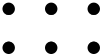
Thus, $6 = 2 \times 3$ **is a composite number.**

The number 1 **is neither prime nor composite.**

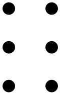
The Atoms of the Number System

A line of six spots

can be arranged in a rectangular array:



or



Note that

$$2 \times 3 = 3 \times 2$$

This is the *commutative law of multiplication.*

The Atoms of the Number System

The primes play a role in mathematics analogous to the elements of Mendeleev's Periodic Table.

Just as a chemical molecule can be constructed from the 100 or so fundamental elements, any whole number be constructed by combining prime numbers.

The primes 2, 3, 5 are the hydrogen, helium and lithium of the number system.

Some History

In 1792 Carl Friedrich Gauss, then only 15 years old, found that the proportion of primes less that n decreased approximately as $1/\log n$.

Around 1795 Adrien-Marie Legendre noticed a similar logarithmic pattern of the primes, but it was to take another century before a proof emerged.

In a letter written in 1823 the Norwegian mathematician Niels Henrik Abel described the distribution of primes as *the most remarkable result in all of mathematics.*

Percentage of Primes Less than N

Table : Percentage of Primes less than N

100	25	25.0%
1,000	168	16.8%
1,000,000	78,498	7.8%
1,000,000,000	50,847,534	5.1%
1,000,000,000,000	37,607,912,018	3.8%

We can see that the percentage of primes is falling off with increasing size.

But the rate of decrease is very slow.

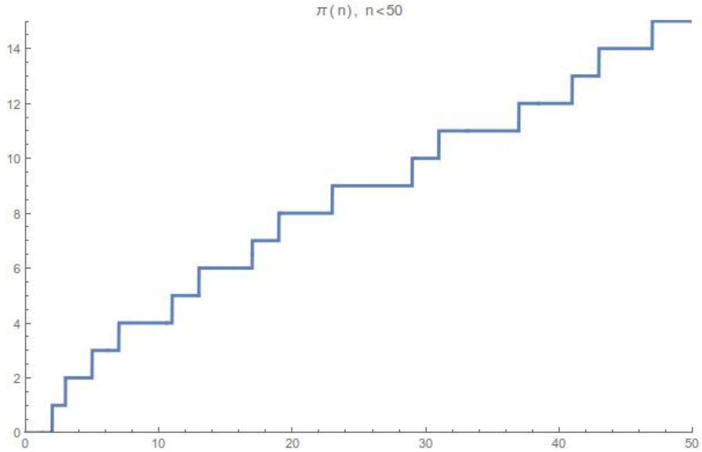


Figure : The prime counting function $\pi(n)$ for $0 \leq n \leq 50$.

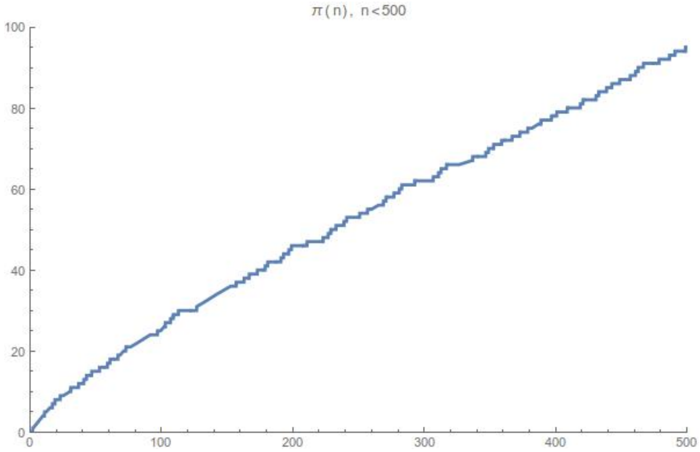


Figure : The prime counting function $\pi(n)$ for $0 \leq n \leq 500$.

Is There a Pattern in the Primes?

It is a simple matter to make a list of
all the primes less that 100 or 1000.

It becomes clear very soon that
no clear pattern is emerging.

The primes appear to be scattered at random.

Figure : Prime numbers up to 100

Is There a Pattern in the Primes?

Do the primes settle down as *n* becomes larger?

Between 9,999,900 and 10,000,000 (100 numbers) there are 9 primes.

Between 10,000,000 and 10,000,100 (100 numbers) there are just 2 primes.

What kind of function could generate this behaviour?

We just do not know.

Is There a Pattern in the Primes?

The gaps between primes are very irregular.

- Can we find a pattern in the primes?
- Can we find a formula that generates primes?
- How can we determine the hundreth prime?
- What is the thousanth? The millionth?

WolframAlpha©

WolframAlpha is a Computational Knowledge Engine.

Wolfram Alpha is based on Wolfram's flagship product Mathematica, a computational platform or toolkit that encompasses computer algebra, symbolic and numerical computation, visualization, and statistics.

It is freely available through a web browser.

Euler's Formula for Primes

No mathematician has ever found a *useful* formula that generates all the prime numbers.

Euler found a beautiful little formula:

$$n^2 - n + 41$$

This gives prime numbers for n between 1 and 40.

But for $n = 41$ we get

$$41^2 - 41 + 41 = 41 \times 41$$

a composite number.

The Infinitude of Primes

Euclid proved that there is no finite limit to the number of primes.

His proof is a masterpiece of symplicity.

(See Dunham book or Wikipedia: *Euclid's Theorem.*)

Some Unsolved Problems

There appear to be an infinite number of prime pairs

$$(2n - 1, 2n + 1)$$

There are also gaps of arbitrary length:

for example, there are 13 consecutive composite numbers between 114 and 126.

We can find gaps as large as we like:

Show that *N! + 1* is followed by a sequence of *N - 1* composite numbers.

Primes have been used as markers of civilization.

In the novel Cosmos, by Carl Sagan,
the heroine detects a signal:

- First 2 pulses
- Then 3 pulses
- Then 5 pulses
- ...
- Then 907 pulses.

In each case, a prime number of pulses.

Could this be due to any natural phenomenon?
Is it evidence of extra-terrestrial intelligence?

Which Primes are Sums of Squares?

```
(* PRINT THE FIRST 100 PRIME NUMBERS *)

primes = {};
For[i = 1, i < 100, i ++, AppendTo[primes, Prime[i]]]
Print["PRIMES"]
primes

PRIMES

Out[51]= {2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43,
 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, 101,
 103, 107, 109, 113, 127, 131, 137, 139, 149, 151,
 157, 163, 167, 173, 179, 181, 191, 193, 197, 199,
 211, 223, 227, 229, 233, 239, 241, 251, 257, 263,
 269, 271, 277, 281, 283, 293, 307, 311, 313, 317,
 331, 337, 347, 349, 353, 359, 367, 373, 379, 383,
 389, 397, 401, 409, 419, 421, 431, 433, 439, 443,
 449, 457, 461, 463, 467, 479, 487, 491, 499, 503,
 509, 521, 523}

(* PRINT THE FIRST 100 SQUARE NUMBERS *)

squares = {};
```

Which Primes are Sums of Squares?

```
509, 521, 523}
(*  PRINT THE FIRST 100 SQUARE NUMBERS   *)
squares = {};
For[i = 1, i < 25, i++, AppendTo[squares, i^2]]
Print["SQUARES"]
squares
SQUARES
Out[63]= {1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121,
 144, 169, 196, 225, 256, 289, 324, 361, 400,
 441, 484, 529, 576}
Prime[1 000 000 000]
Out[69]= 22 801 763 489
```

Which Primes are Sums of Squares?

A Theorem of Fermat states that:

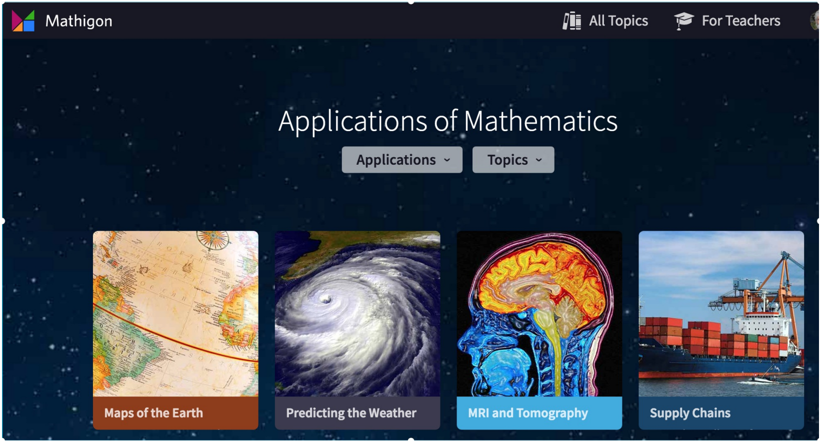
A prime number n may be expressed as
a sum of squares if and only if

$$p \equiv 1 \pmod 4$$

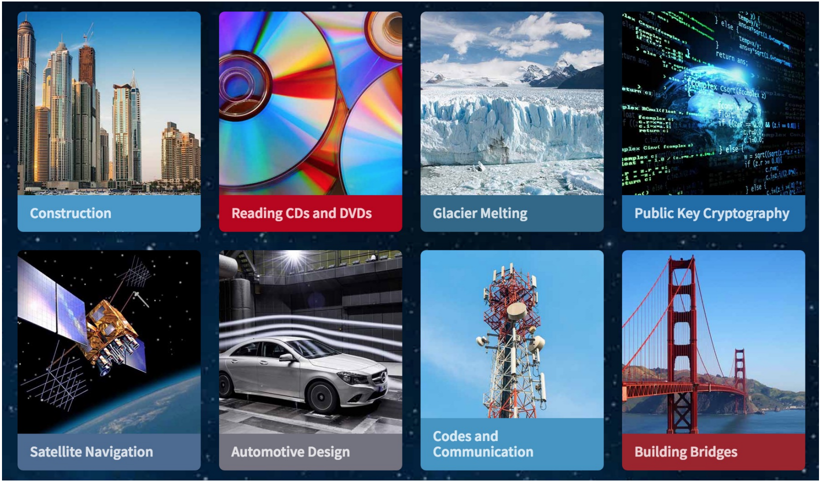
In plain language, if n divided by 4 has remainder 1.

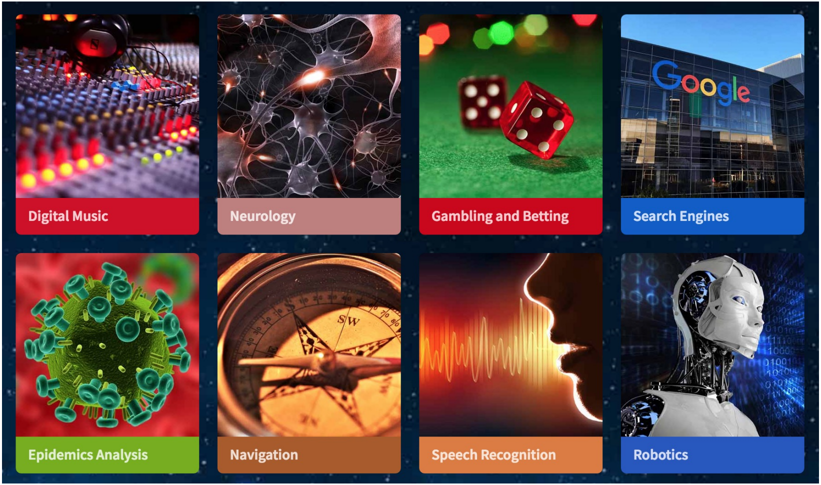
Outline

Applications on mathigon.org

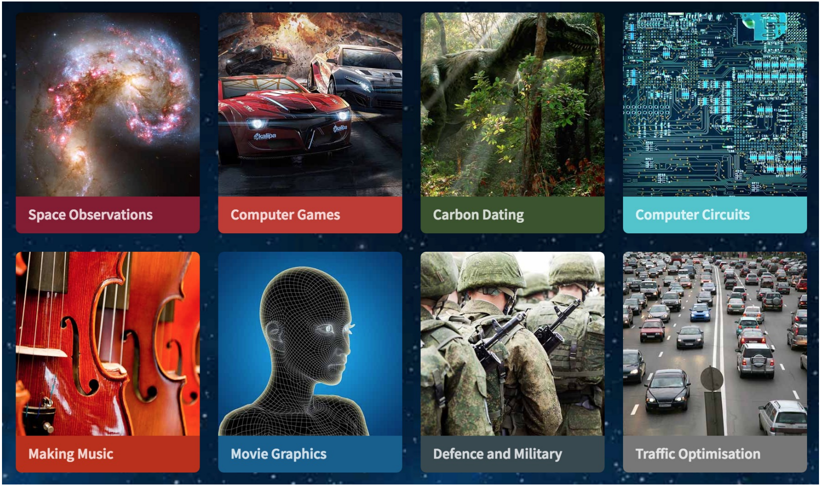


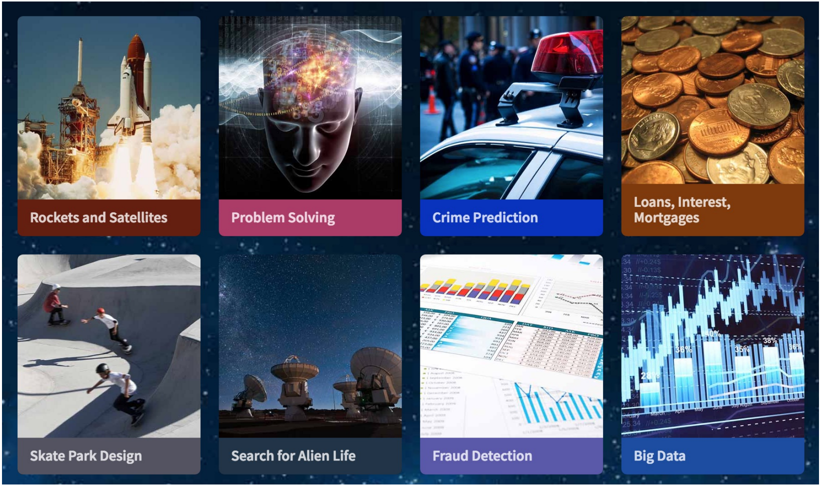

Maps of the Earth
Predicting the Weather
MRI and Tomography
Supply Chains
Finance and Banking
Internet and Phones
Cosmology
Computers

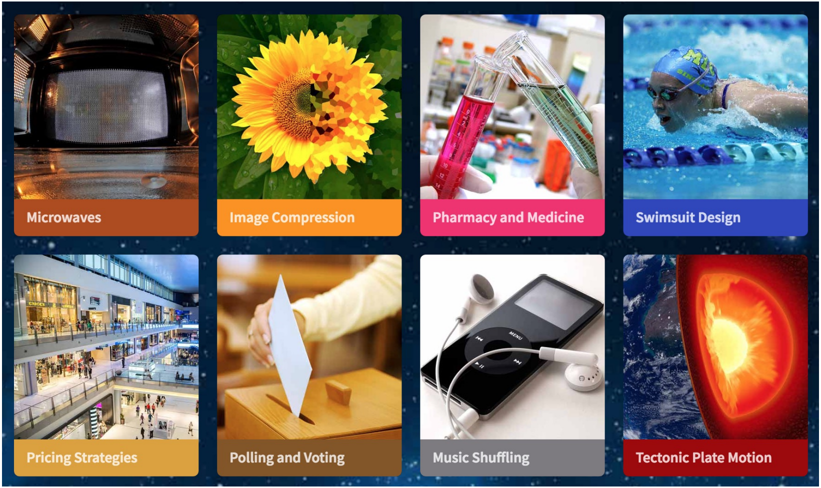

Construction
Reading CDs and DVDs
Glacier Melting
Public Key Cryptography
Satellite Navigation
Automotive Design
Codes and Communication
Building Bridges

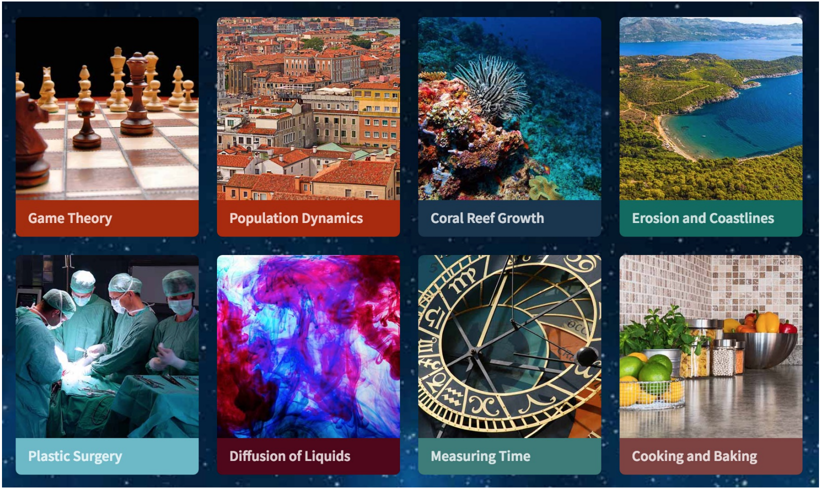

Digital Music
Neurology
Gambling and Betting
Google
Search Engines
Epidemics Analysis
Navigation
Speech Recognition
Robotics

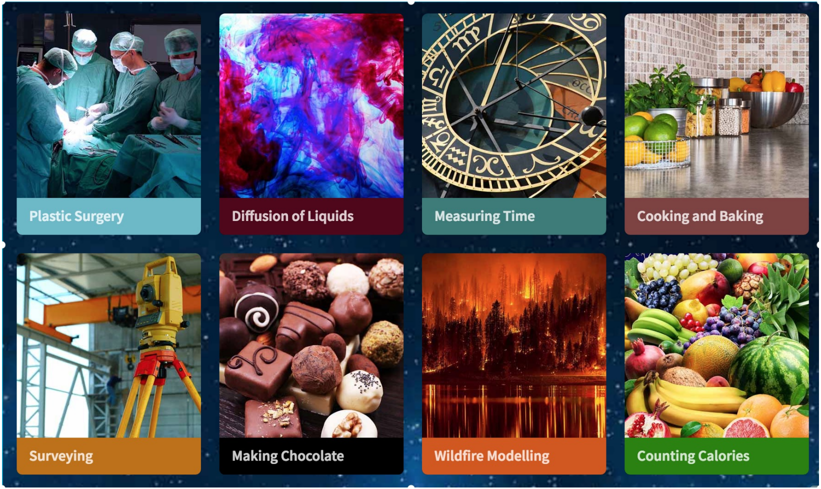

Football Scoring
Volcano Monitoring
Lottery
Roller Coaster Design
Breaking the Enigma
Public Transportation
Crowd Control
Insurance


Space Observations
Computer Games
Carbon Dating
Computer Circuits
Making Music
Movie Graphics
Defence and Military
Traffic Optimisation


Rockets and Satellites
Problem Solving
Crime Prediction
Loans, Interest, Mortgages
Skate Park Design
Search for Alien Life
Fraud Detection
Big Data

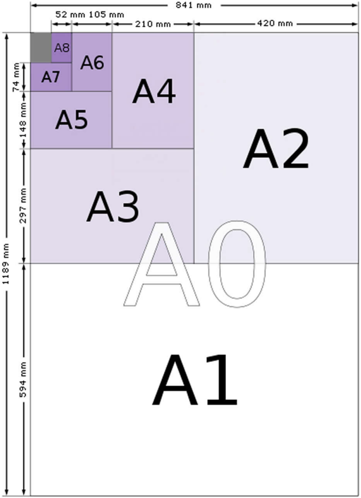

Microwaves
Image Compression
Pharmacy and Medicine
Swimsuit Design
Pricing Strategies
Polling and Voting
Music Shuffling
Tectonic Plate Motion


Game Theory
Population Dynamics
Coral Reef Growth
Erosion and Coastlines
Plastic Surgery
Diffusion of Liquids
Measuring Time
Cooking and Baking


Plastic Surgery
Diffusion of Liquids
Measuring Time
Cooking and Baking
Surveying
Making Chocolate
Wildfire Modelling
Counting Calories

Outline

Standard Paper Sizes



Standard sizes of A-series paper.

The ratio of heights to widths is always $\sqrt{2}$.

Making a Square

The standard sizes of paper are designed so that each has the same shape (or aspect ratio), and the largest, A0, has an area of one square metre.

PUZZLE:
Is it possible to form a square out of sheets of A4 sized paper (without them overlapping)?

Outline

Topology: a Major Branch of Mathematics

Topology is all about continuity and connectivity.

Here are some of the topics in Topology:

- **The Bridges of Königsberg**
- **Doughnuts and Coffee-cups**
- **Knots and Links**
- **Nodes and Edges: Graphs**
- **The Möbius Band**

In this lecture, we look at *Knots and Links.*

Pretzel Puzzle

Look at the two "pretzels" here:

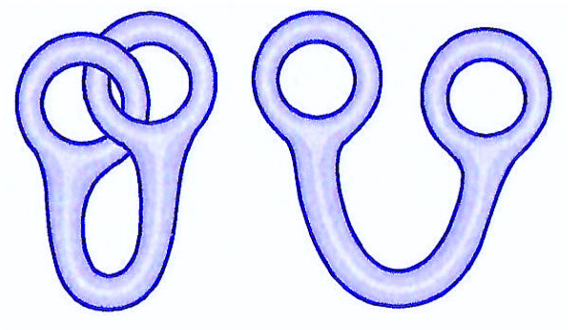


Figure : Two "Pretzels". Are they equivalent?

Knot Theory

A knot is an embedding of the unit circle S^1 into three-dimensional space R^3.

Two knots are equivalent if one can be distorted into the other without breaking it.

A knot is a mapping of the unit circle into three-space.

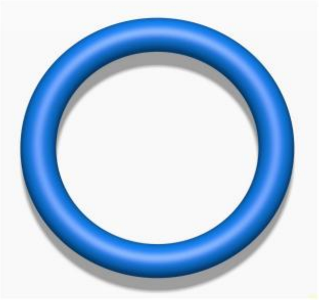
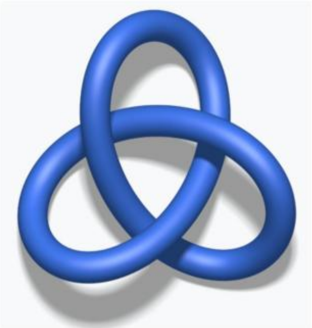
 

Figure : Left: Unknot. Right: Trefoil.

These two knots aren't equivalent: we can't distort the circle into the trefoil without breaking it.

Knots that are Mirror Images

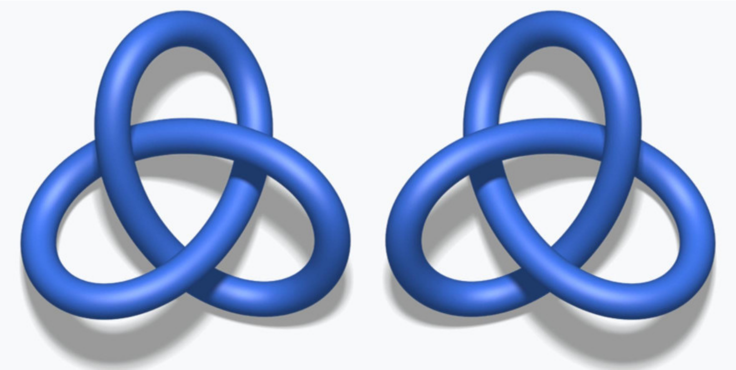


Figure : Levo and Dextro Trefoils.

These knots are not equivalent. We cannot change one into the other without breaking it.

The Simplest Knots and Links

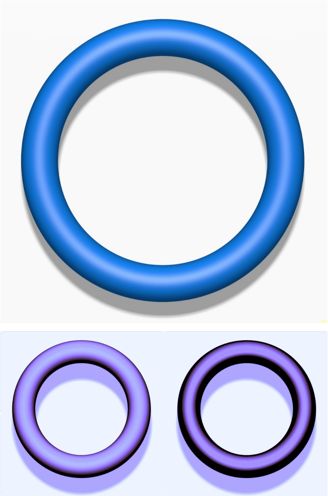


Figure : Top: The Unknot. Bottom: The Unlink.

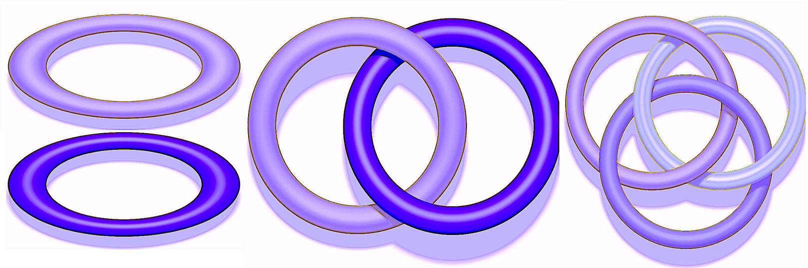


Figure : Unlink, Hopf Link and Borromean Rings.

The Hopf Link

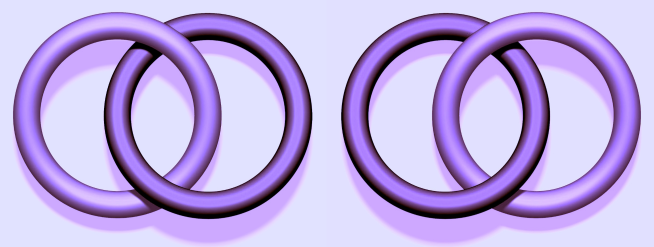


Figure : The Hopf Link and its mirror image. Equivalent?

Rings of Borromeo

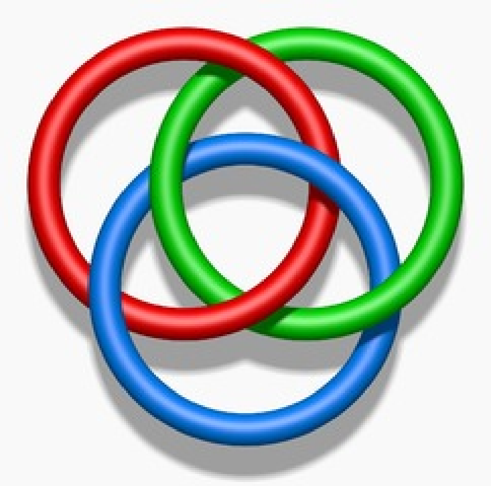


Figure : No two rings are linked! Are the three?

Genus of a Surface

The genus of a topological surface is,
in simple terms, the number of holes in it.

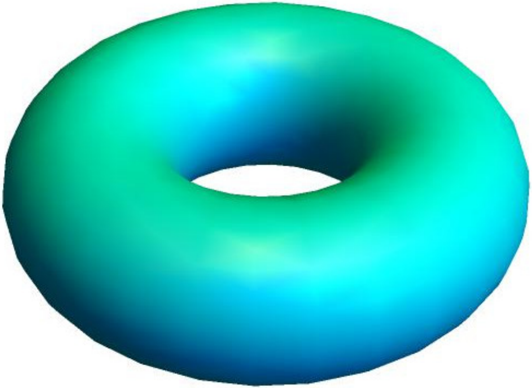
A sphere has no holes, so has genus 0.

A donut has one hole, so has genus 1.

Surfaces can have any number of holes; any genus.

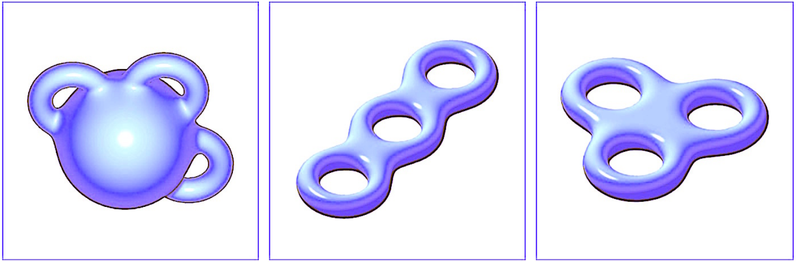
The Sphere, of Genus 0

The Torus, of Genus 1



The Double Torus, of Genus 2

Some Surfaces of Genus 3



Topologists have classified all surfaces in 3-space.

Link between Number Theory and Physics

Forty years ago, physics and and topology had little or nothing to do with one another.

In the 1980s, mathematicians and physicists found ways to use physics to study the properties of shapes.

The field has never looked back.

`http://www.quantamagazine.org/secret-link-uncovered-between-pure-math-and-physics-20171201/`

Triple Torus

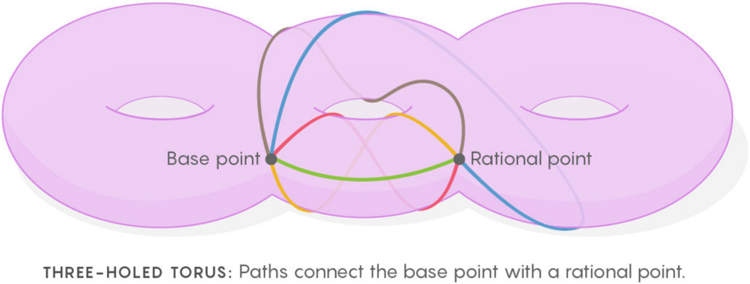


Figure : Rational solutions of $x^4 + y^4 = 1$ are on this surface

Pretzel Puzzle

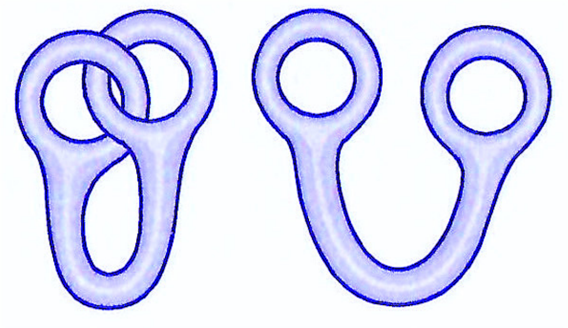


Figure : Two “Pretzels”. Are they equivalent?

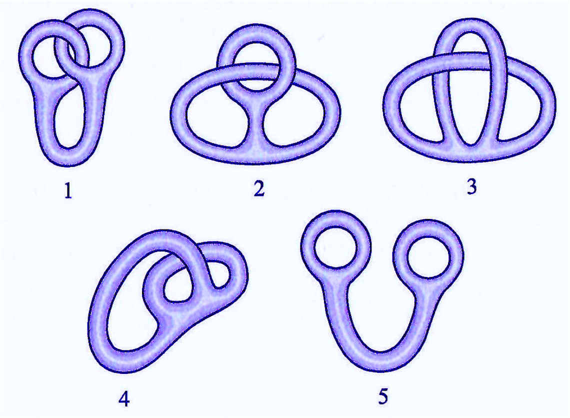


Figure : Equivalence!

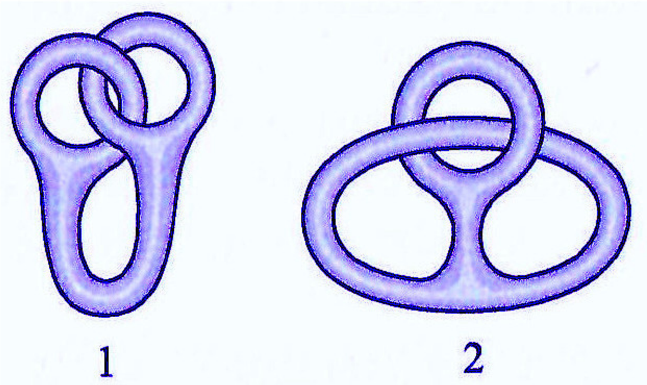


Figure : Make the left-hand loop bigger.

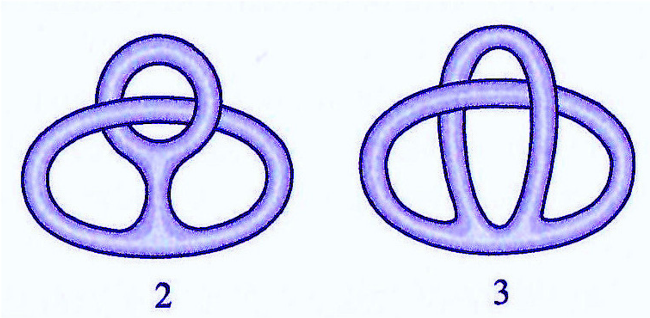


Figure : Make the other loop bigger.

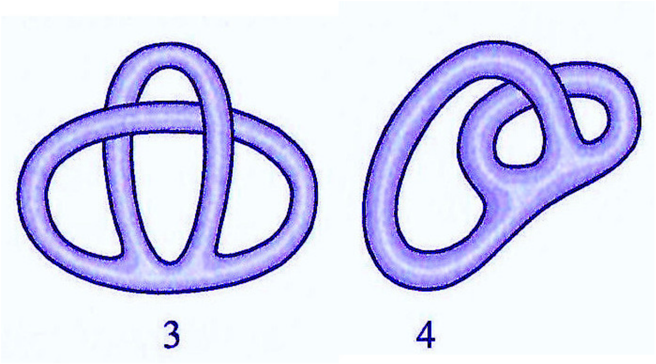


Figure : Pull the top loop away to the side.

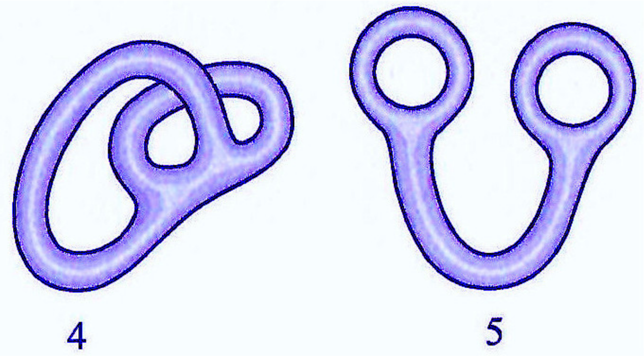


Figure : Smoothly distort to the final form.

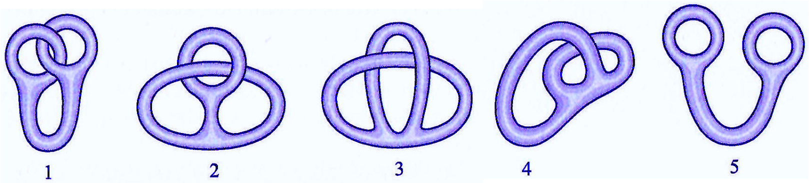


Figure : Combining all the distortions. Equivalence!

Another Surprising Result

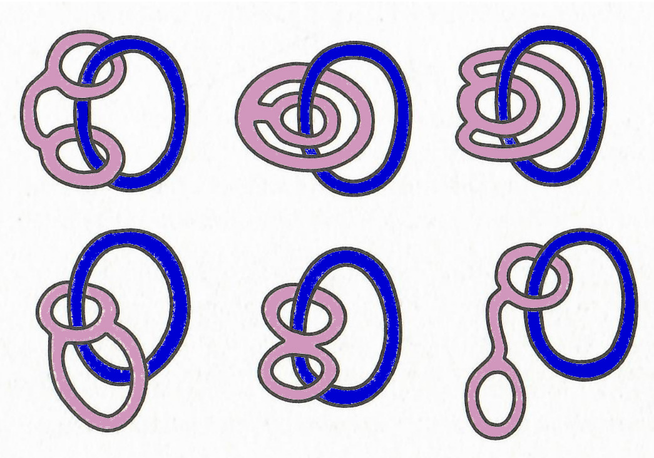


Figure : We can unlink one of the hand-cuffs.

Outline

David Hilbert (1862–1943)

David Hilbert, from a contemporary postcard.

Hilbert's Problems

In August 1900, David Hilbert addresed the *International Congress of Mathematicians* in the Sorbonne in Paris:

> ***"Who of us would not be glad to lift the veil behind which the future lies hidden; to cast a glance at the next advances of our science and at the secrets of its development during future centuries?"***

Hilbert presented 23 problems that challenged mathematicians through the twentieth century.

Hilbert's Problems

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 37, Number 4, Pages 407–436
S 0273-0979(00)00881-8
Article electronically published on June 26, 2000

MATHEMATICAL PROBLEMS

DAVID HILBERT

Lecture delivered before the International Congress of Mathematicians at Paris in 1900.

Hilbert's eighth problem concerned itself with what is called the Riemann Hypothesis (RH).

RH is generally regarded as the deepest and most important unproven mathematical problem.

Anyone who can prove it is assured of lasting fame.

Why is RH Important?

A large number of mathematical theorems (1000's) depend for their validity on the RH.

Were RH to turn out to be false, many of these mathematical arguments would simply collapse.

In 2000, industrialist Landon Clay donated $7M, with $1M for each of 7 problems in mathematics.

The Riemann hypothesis is one of these problems.

`http://www.claymath.org/millennium-problems`

Why is RH Important?

Whoever proves Riemann's hypothesis will have completed thousands of theorems that start like this:

"Assuming that the Riemann hypothesis is true . . . ".

He or she will be assured of lasting fame.

Those who establish fundamental mathematical results probably come closer to immortality than almost anyone else.

Outline

What is Randomness?

Randomness is a *slippery concept,*
defying precise definition.

Toss a coin and get a sequence like 1001110100.

Some uses of Random Numbers:

- Computer simulations of fluid flow.
- Interactions of subatomic particles.
- Evolution of galaxies.

Tossing coins is impractical.
We need more effective methods.

Defining Randomness?

Richard von Mises (1919):
A binary sequence is random if the proportion of zeros and ones approaches 50% and if this is also true for any sub-sequence.
Consider (0101010101).

Andrey Kolmogorov **defined the complexity of a binary sequence as the length of a computer program or algorithm that generates it.**

The phrase a sequence of one million 1s completely defines a sequence.

Non-random sequences are compressible.
Randomness and incompressibility are equivalent.

Pseudo-random versus Truly Random

Pseudo-random number generators are algorithms that use mathematical formulae to produce sequences of numbers.

The sequences appear completely random and satisfy various statistical conditions for randomness.

Pseudo-random numbers **are valuable for many applications but they have serious difficiencies.**

Truly Random Number Generators

True random number generators extract randomness from physical phenomena that are completely unpredictable.

Atmospheric noise is the static generated by lightning [globally there are 40 flashes/sec]. It can be detected by an ordinary radio.

Truly Random Number Generators

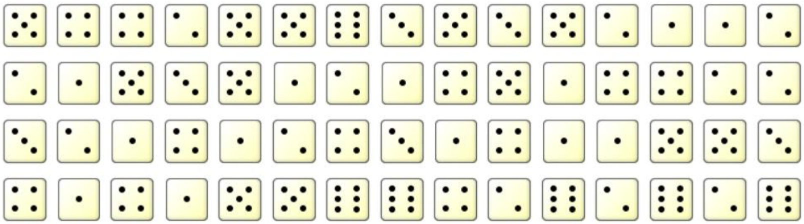
Atmospheric noise passes all the statistical checks for randomness.

Dr Mads Haahr of Trinity College, Dublin uses atmospheric noise to produce random numbers.

Results available on on the website: `random.org`.

20 Random Coin Tosses

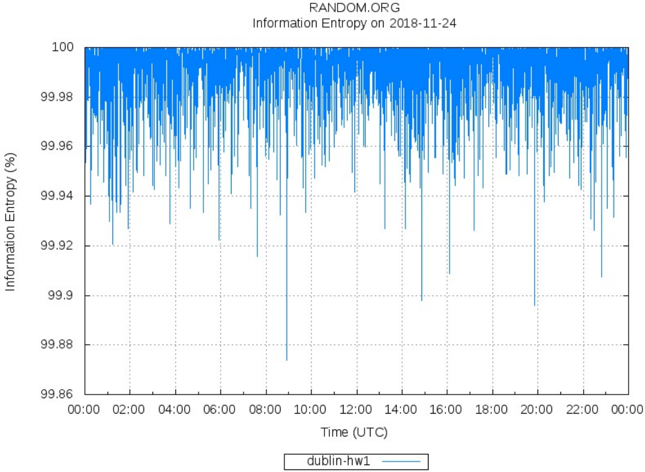
60 Dice Rolls



100 Random Numbers in [0,99]

17	60	57	66	4	71	59	36	8	49
87	64	94	82	6	38	14	87	76	72
97	38	44	59	56	24	20	6	24	97
0	40	14	77	18	98	41	39	6	79
21	59	49	86	91	81	65	64	3	11
92	17	65	6	37	98	84	17	70	93
60	52	1	98	20	2	65	9	57	3
48	86	27	3	71	51	57	56	2	2
13	14	73	65	11	32	17	7	91	37
3	8	10	67	0	72	0	42	15	24

Quality of Random Numbers



PRNG versus TRNG

Characteristic	Pseudo-Random Number Generators	True Random Number Generators
Efficiency	Excellent	Poor
Determinism	Determinstic	Nondeterministic
Periodicity	Periodic	Aperiodic

Outline

The Möbius Band

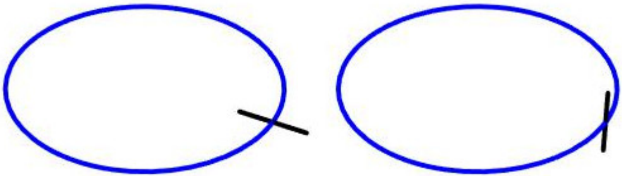
You may be familiar with the Möbius strip or Möbius band. It has one side and one edge.

It was discovered independently by August Möbius and Johann Listing in the same year, 1858.

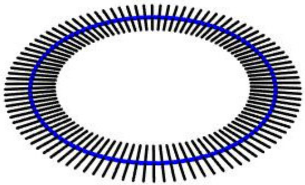
Building the Band

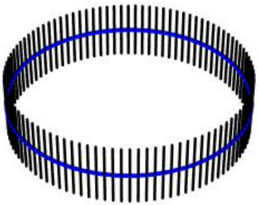
It is easy to make a Möbius band from a paper strip.

For a geometrical construction, we start with a circle and a small line segment with centre on this circle.



Now move the line segment around the circle:





To show the boundary of the surface, we color one end of the line segment red and the other magenta.

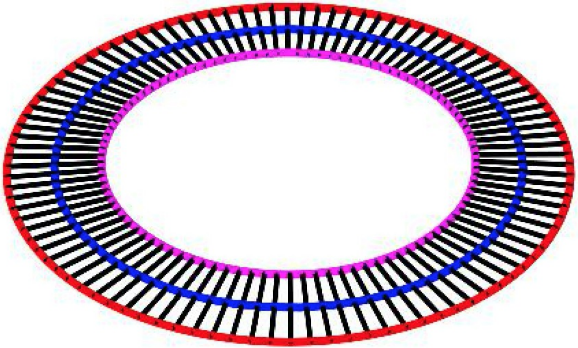


Figure : The boundary comprises two unlinked circles

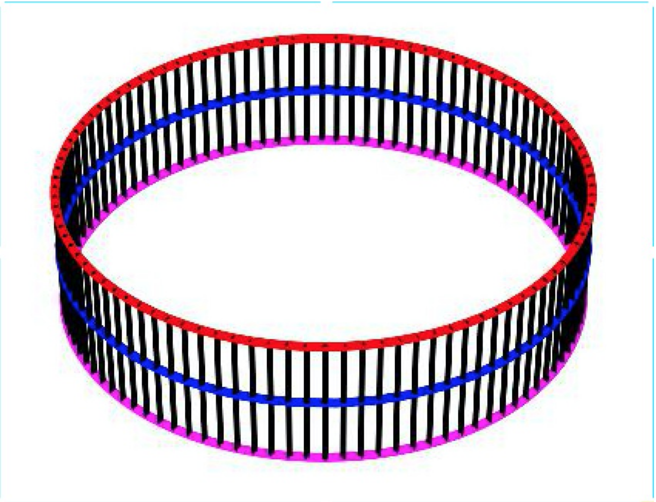
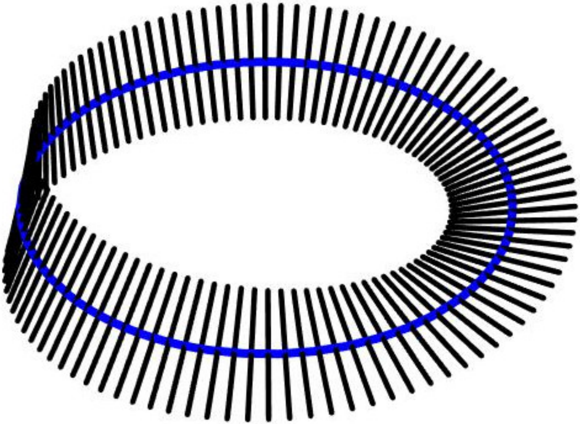


Figure : The boundary comprises two unlinked circles

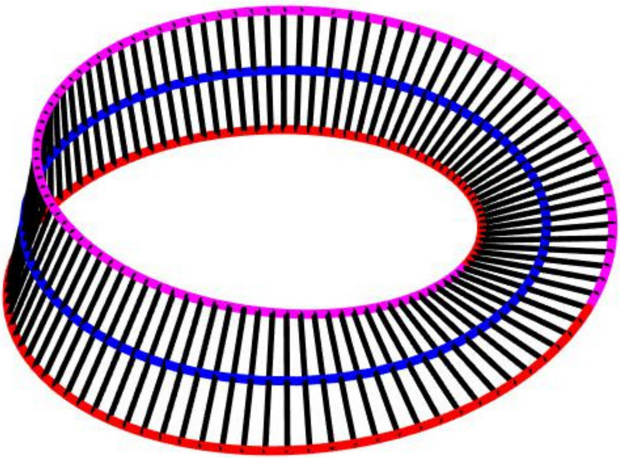
The Möbius Band

Now, as the line moves, we give it a half-twist:



The Möbius Band

The two boundary curves now join up to become one:



The Möbius Band

The Möbius Band has only one side.

It is possible to get from any point on the surface to any other point *without crossing the edge.*

The surface also has just one edge.

Band with a Full Twist

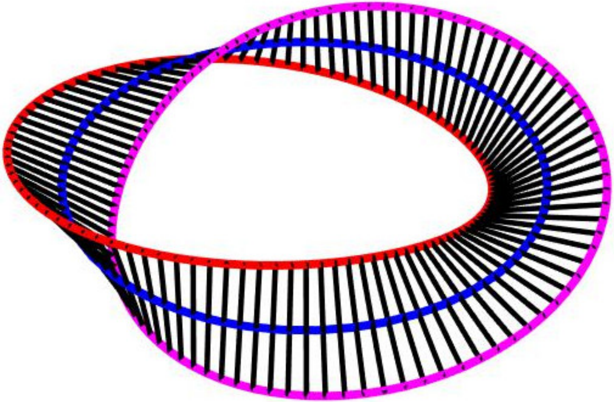


Figure : The boundary comprises two linked circles

Band with Three Half-twists

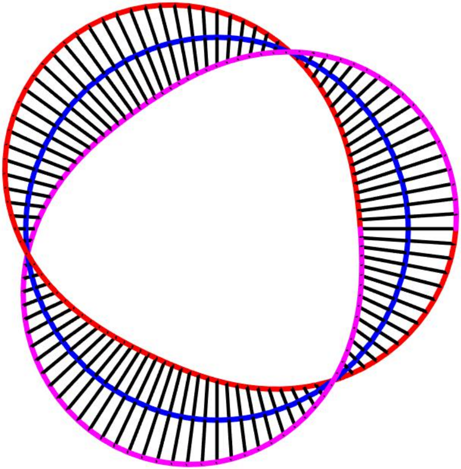
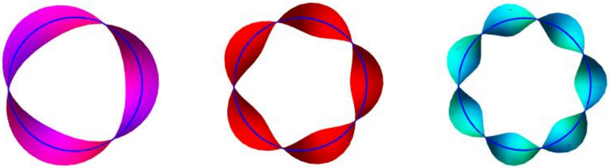
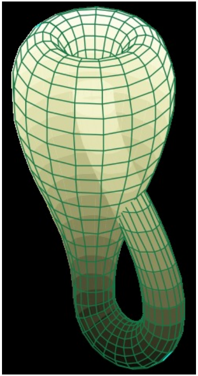


Figure : The boundary is a knot, a trefoil curve



Two Möbius Bands make a Klein Bottle



A mathematician named Klein
Thought the Möbius band was divine.
Said he: "If you glue
The edges of two,
You'll get a weird bottle like mine."

Equations for the Möbius Band

The process of moving the line segment around the circle leads us to the equations for the Möbius band.

In cylindrical polar coordinates the circle is $(r, \theta, z) = (a, \theta, 0)$.

The tip of the segment, relative to its centre, is

$$(r, \theta, z) = (b\cos\phi, 0, b\sin\phi)$$

where $b = \frac{1}{2}\ell$ **is half the segment length and** $\phi = \alpha\theta$, **with** α **determining the amount of twist.**

The tip of the line has $(r, z) = (a + b\cos\alpha\theta, b\sin\alpha\theta)$.

Equations for the Möbius Band

In cartesian coordinates, the equations become

$$\begin{aligned} x &= (a + b\cos\alpha\theta)\cos\theta \\ y &= (a + b\cos\alpha\theta)\sin\theta \\ z &= (b\sin\alpha\theta) \end{aligned}$$

These are the parametric equations for the twisted bands, with $\theta \in [0, 2\pi]$ and $b \in [-\ell, \ell]$.

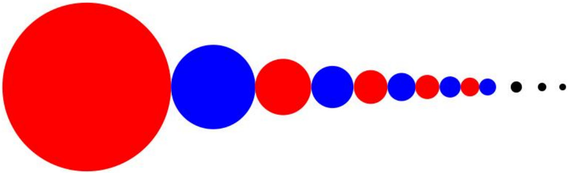
For the Möbius band, $\alpha = \frac{1}{2}$.

Outline

A Surprising Result

Let us consider an infinite row of cookies each smaller than the previous one.

Assume that the radius of the n-th cookie is $1/n$. Then the surface area is π/n^2.



A Surprising Result

The total length of the row of cookies is

$$2\sum_{n=1}^{\infty}\frac{1}{n}$$

This is the divergent harmonic series.

The total surface area of the cookies is

$$\sum_{n=1}^{\infty}\pi\times\left(\frac{1}{n}\right)^2=\pi\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^3}{6}$$

This series is known as the Basel series, and it is convergent, with sum $\pi^2/6$.

Outline

Alfred Moessner's Conjecture

Aus den Sitzungsberichten der Bayerischen Akademie der Wissenschaften
Mathematisch-naturwissenschaftliche Klasse 1951 Nr. 3

Eine Bemerkung über die Potenzen der natürlichen Zahlen

Von Alfred Moessner in Gunzenhausen

Vorgelegt von Herrn O. Perron am 2. März 1951

A Remark on the Powers of the Natural Numbers

Moessner's Construction: n=2

We start with the sequence of natural numbers:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	...

Now we delete *every second number* and calculate the sequence of partial sums:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1		*4*		*9*		*16*		*25*		*36*		*49*		*64*	

The result is the sequence of perfect squares:

$$1^2 \quad 2^2 \quad 3^2 \quad 4^2 \quad 5^2 \quad 6^2 \quad 7^2 \quad 8^2 \quad \ldots$$

Moessner's Construction: n=3

Now we delete *every third number* and calculate the sequence of partial sums.

Then we delete *every second number* and calculate the sequence of partial sums:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	3		7	12		19	27		37	48		61	75		91
1			8			27			64			125			216

The result is the sequence of perfect cubes:

$$1^3 \quad 2^3 \quad 3^3 \quad 4^3 \quad 5^3 \quad 6^3 \quad \ldots$$

Moessner's Construction: n=4

The Moessner Construction also works for larger n:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	3	6		11	17	24		33	43	54		67	81	96	
1	4			15	32			65	108			175	256		
1				16				81				256			

The result is the sequence of fourth powers:

$$1^4 \quad 2^4 \quad 3^4 \quad 4^4 \quad \ldots$$

Moessner's Constructions

Remark:

Using Moessner's construction, we can generate a table of squares, cubes or higher powers.

The only arithmetical operations used are *counting* and *addition!*

Are there any other sequences generated in this way?

Moessner's Construction for n!

We begin by striking out the *triangular numbers*, $\{1, 3, 6, 10, 15, 21, \dots\}$ and form partial sums.

Next, we delete the final entry in each group and form partial sums. This process is repeated indefinitely:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
	2		6	11		18	26	35		46	58	71	85		101
			6			24	50			96	154	225			326
						24				120	274				600
										120					720

This yields the *factorial numbers:*

1! 2! 3! 4! 5! 6! . . .

Beautiful Math

The beauty of maths?
What do mathematicians think?

VIDEO: Beautiful Maths, available at

`http://momath.org/home/beautifulmath/`

Video by James Tanton

Try to disregard the antipodean exuberance!

Wikipedia Mathematics Portal

Topics in mathematics edit

General

- Mathematicians
- History of mathematics
- Philosophy of mathematics
- Mathematical notation
- Mathematical beauty
- Mathematics education
- Areas of mathematics
- Outline of mathematics
- List of mathematical symbols
- Wikipedia Books: Mathematics

Foundations

- Foundations of mathematics
- Mathematical logic
  - Proof theory
    - Gödel's incompleteness theorems
  - Model theory
  - Recursion theory
- Set theory (portal)
  - Naive set theory
  - Axiomatic set theory
- Category theory (portal)
  - Topos theory

Number theory

- Number theory (portal)
- Algebraic number theory
- Analytic number theory
- Arithmetic
  - Fundamental theorem of arithmetic
- Numbers
  - Natural numbers
  - Prime numbers
  - Rational numbers
  - Algebraic numbers

Discrete mathematics

- Discrete mathematics (portal)
- Combinatorics
  - Combinatorial geometry
  - Coding theory
  - Combinatorial design
  - Enumerative combinatorics
  - Combinatorial optimization
- Graph theory
- Order theory
  - Lattice theory
- Digital Signal Processing

Algebra

- Algebra (portal)
- Elementary algebra
- Abstract algebra
  - Group theory
  - Ring theory
  - Field theory
  - Commutative algebra
- Geometric algebra
- Linear algebra
  - Matrix theory
  - Multilinear algebra
- Universal algebra
- Fundamental theorem of algebra

Analysis

- Analysis (portal)
- Calculus
  - Fundamental theorem of calculus
  - Vector calculus
  - Geometric calculus
- Measure theory
- Real analysis
- Complex analysis
- Differential equations
  - Ordinary differential equations
  - Partial differential equations
- Integral equations
- Approximation theory
- Special functions
- Potential theory
- Harmonic analysis
  - Fourier analysis
- Functional analysis
- Operator theory

Geometry and topology

- Geometry (portal)
- Euclidean geometry
  - Trigonometry
- Analytic geometry
- Non-Euclidean geometry
- Affine geometry
- Projective geometry
- Convex geometry
- Discrete geometry
- Algebraic geometry
- Differential geometry
  - Riemannian geometry
  - Lie groups
- Topology (portal)
  - General topology
  - Algebraic topology
  - Geometric topology
  - Differential topology

Applied mathematics

- Applied mathematics
- Mathematical modeling
- Mathematical physics
- Dynamical systems
  - Control theory
    - Calculus of variations
- Optimization
- Mathematical economics
  - Game theory
  - Mathematical finance
- Statistics (portal)
- Probability theory
  - Stochastic processes
- Numerical analysis
- Theoretical computer science
  - Computability theory
  - Complexity theory
- Cryptography (portal)
- Information theory

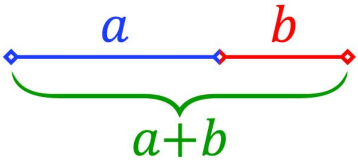
Index of mathematics articles edit

Outline

Golden Rectangle in Your Pocket

Aspect ratio is about $\phi = \frac{1+\sqrt{5}}{2} \approx 1.618$.

Geometric Ratio: $a + b$ is to a as a is to b.



$$\left[\frac{\text{Short Bit}}{\text{Long Bit}}\right] = \left[\frac{\text{Long Bit}}{\text{Full Line}}\right] \qquad \text{or} \qquad \frac{b}{a} = \frac{a}{a+b}$$

Let the blue segment be $a = 1$ and the whole line ϕ.

Then $b = \phi - 1$ and we have

$$\frac{\phi - 1}{1} = \frac{1}{\phi}$$

$$\phi - 1 = \frac{1}{\phi}$$

This means ϕ solves a quadratic equation:

$$\phi^2 - \phi - 1 = 0$$

Recall the two solutions of a quadratic equation

$$ax^2 + bx + c = 0 \qquad \textbf{are} \qquad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

In the present case, this means that the roots are

$$\phi = \frac{1 \pm \sqrt{1+4}}{2}$$

We take the *positive root,* giving

$$\phi = \frac{1+\sqrt{5}}{2} \approx 1.618$$

This is the golden ratio.

Check the Solution

The quadratic equation is

$$\phi^2 - \phi - 1 = 0 \qquad \textbf{or} \qquad \phi^2 = \phi + 1$$

Suppose

$$\phi = \frac{1 + \sqrt{5}}{2}$$

Then

$$\phi + 1 = \frac{3 + \sqrt{5}}{2} \qquad \textbf{and} \qquad \phi^2 = \frac{3 + \sqrt{5}}{2}$$

Golden Rectangle

Ratio of breath to height is $\phi = \frac{1+\sqrt{5}}{2}$.

Golden Rectangle in Your Pocket

Aspect ratio is about $\phi = \frac{1+\sqrt{5}}{2} \approx 1.618$.

Terminology

- Golden Ratio. Golden Number. Golden Mean.
- Golden Proportion. Golden Cut.
- Golden Section. Medial Section.
- *Divine Proportion.* Divine Section.
- Extreme and Mean Ratio.
- Various Other Terms.

Fibonacci Numbers

The Fibonacci sequence is the sequence

$$\{0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots\}$$

where *each number is the sum of the previous two.*

The Fibonacci numbers obey a recurrence relation

$$F_{n+1} = F_n + F_{n-1}$$

with the *starting values* $F_0 = 0$ *and* $F_1 = 1$.

Can we *solve this recurrence relation for all* F_n?

Fibonacci Numbers

The *recurrence relation* is

$$F_{n+1} = F_n + F_{n-1}$$

We assume that the solution is of the form $F_n = k\chi^n$, where we have to find χ (this is called an *Ansatz*).

Substitute this solution into the recurrence relation:

$$k\chi^{n+1} = k\chi^n + k\chi^{n-1}$$

Divide by $k\chi^{n-1}$ to get the quadratic equation

$$\chi^2 = \chi + 1 \qquad \text{or} \qquad \chi^2 - \chi - 1 = 0$$

This is the quadratic we got for the golden number.

Fibonacci Numbers

We found that $F_n = k\phi^n$ where ϕ is a root of

$$\phi^2 - \phi - 1 = 0$$

The two roots are

$$\frac{1+\sqrt{5}}{2} \qquad \text{and} \qquad \frac{1-\sqrt{5}}{2}$$

Then the full solution for the Fibonacci numbers is

$$F_n = \frac{1}{\sqrt{5}}\left[\frac{1+\sqrt{5}}{2}\right]^n - \frac{1}{\sqrt{5}}\left[\frac{1-\sqrt{5}}{2}\right]^n$$

Check that the conditions $F_0 = 0$ and $F_1 = 1$ are true.

Fibonacci Numbers

$$F_n = \frac{1}{\sqrt{5}}\left[\frac{1+\sqrt{5}}{2}\right]^n - \frac{1}{\sqrt{5}}\left[\frac{1-\sqrt{5}}{2}\right]^n$$

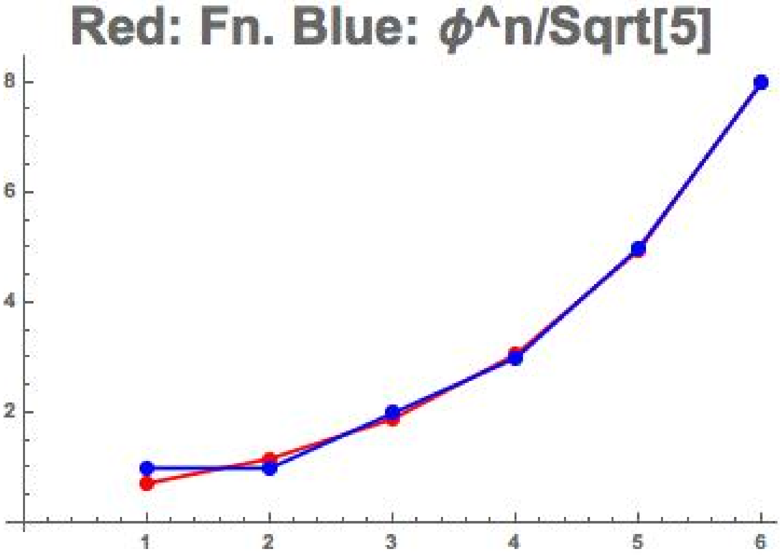
The first term in square brackets is greater than 1, so the powers *grow rapidly with* n.

The second term in square brackets is less than 1, so the powers *become small rapidly with* n.

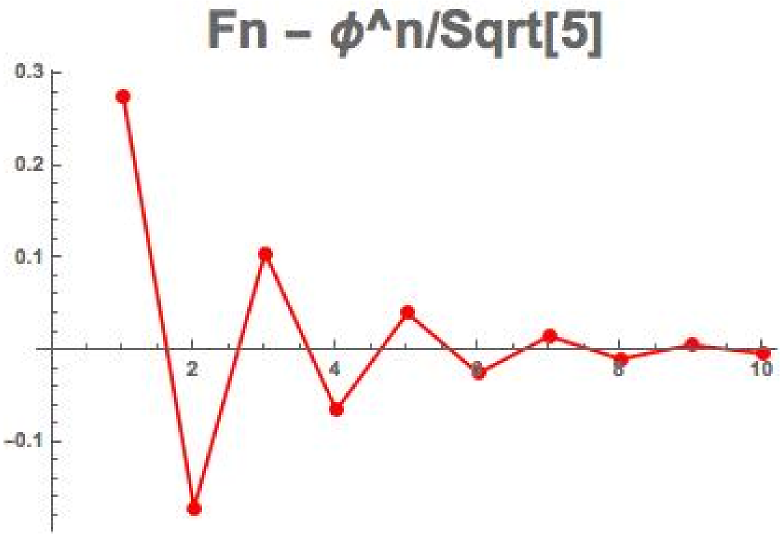
So, we ignore the second term and write

$$F_n \approx \frac{1}{\sqrt{5}}\left[\frac{1+\sqrt{5}}{2}\right]^n \qquad \textbf{or} \qquad F_n \approx \frac{\phi^n}{\sqrt{5}}$$

Approximation to F_n



Oscillating Error of Approximation



Ratio F_n/F_{n-1}

$$F_n \approx \frac{\phi^n}{\sqrt{5}} \implies \frac{F_n}{F_{n-1}} \approx \phi$$

Let's consider the sequence of ratios of terms

$$\frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \frac{21}{13}, \frac{34}{21}, \dots$$

The ratios get closer and closer to ϕ:

$$\frac{F_{n+1}}{F_n} \to \phi \quad \textbf{as} \quad n \to \infty$$

Continued Fraction for ϕ

$$\phi^2 - \phi - 1 = 0 \implies \phi = 1 + \frac{1}{\phi}$$

Now use the equation to replace ϕ on the right:

$$\phi = 1 + \frac{1}{\phi} = 1 + \frac{1}{1 + \frac{1}{\phi}} = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{\phi}}}$$

Eventually

$$\phi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}}$$

Continued Root for ϕ

$$\phi^2 - \phi - 1 = 0 \implies \phi = \sqrt{1+\phi}$$

Now use the equation to replace ϕ on the right:

$$\phi = \sqrt{1+\phi} = \sqrt{1+\sqrt{1+\phi}} = \sqrt{1+\sqrt{1+\sqrt{1+\phi}}}$$

Eventually

$$\phi = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\cdots}}}}}$$

Fibonacci Numbers in Nature

Look at post

Sunflowers and Fibonacci: Models of Efficiency

on the *ThatsMaths* blog.

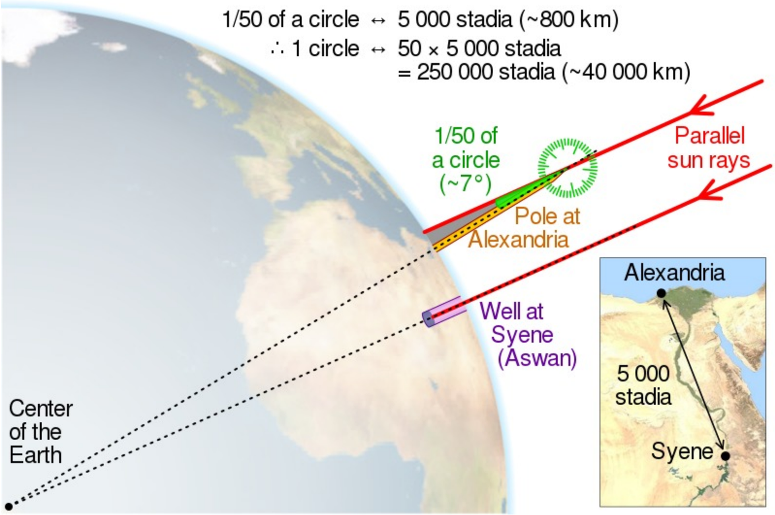
Vi Hart's Videos

Vi Hart has many mathematical videos on YouTube.

- On Fibonacci Numbers: `https://www.youtube.com/watch?v=ahXIMUkSXX0`
- On the Three Utilities Problem: `https://www.youtube.com/watch?v=CruQylWSfoU&feature=youtu.be`
- On Continued Fractions: `https://www.youtube.com/watch?v=a5z-OEIfw3s`

Outline

Eratosthenes Measured the Earth



The Sieve of Eratosthenes

Eratosthenes was the Librarian in Alexandria when Archimedes flourished in Syracuse.

They were "pen-pals".

Eratosthenes estimated size of the Earth.

He devised a systematic procedure for generating the prime numbers: the Sieve of Eratosthenes.

The Sieve of Eratosthenes

The idea:

- List all natural numbers up to n.
- Circle 2 and strike out all multiples of two.
- Move to the next number, 3.
- Circle it and strike out all multiples of 3.
- Continue till no more numbers can be struck out.

The numbers that have been circled are the prime numbers. Nothing else survives.

It is sufficient to go as far as $\sqrt{n}$.

The Sieve of Eratosthenes

	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

The Sieve of Eratosthenes

	2	3		5		7		9	
11		13		15		17		19	
21		23		25		27		29	
31		33		35		37		39	
41		43		45		47		49	
51		53		55		57		59	
61		63		65		67		69	
71		73		75		77		79	
81		83		85		87		89	
91		93		95		97		99	

The Sieve of Eratosthenes

	2	3		5		7			
11		13				17		19	
		23		25				29	
31				35		37			
41		43				47		49	
		53		55				59	
61				65		67			
71		73				77		79	
		83		85				89	
91				95		97			

The Sieve of Eratosthenes

	2	3		5		7			
11		13				17		19	
		23						29	
31						37			
41		43				47		49	
		53						59	
61						67			
71		73				77		79	
		83						89	
91						97			

The Sieve of Eratosthenes

	2	3		5		7			
11		13				17		19	
		23						29	
31						37			
41		43				47			
		53						59	
61						67			
71		73						79	
		83						89	
						97			

The Sieve of Eratosthenes

	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Is There a Pattern in the Primes?

It is a simple matter to make a list of all the primes less that 100 or 1000.

It becomes clear very soon that there is no clear pattern emerging.

The primes appear to be scattered at random.

Figure : Prime numbers up to 100

The grand challenge is to find patterns in the sequence of prime numbers.

This is an enormously difficult problem that has taxed the imagination of the greatest mathematicians for centuries.

Sources to Continue your Interest

- Mathigon.org
- Plus Magazine
- Quanta Magazine
- WOlfram Alpha
- Desmos Graphics Site
- Wikipedia
- ThatsMaths.com

Thank you