AweSums

Marvels and Mysteries of Mathematics

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LECTURE 5

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Outline

Outline

Meaning and Content of Mathematics

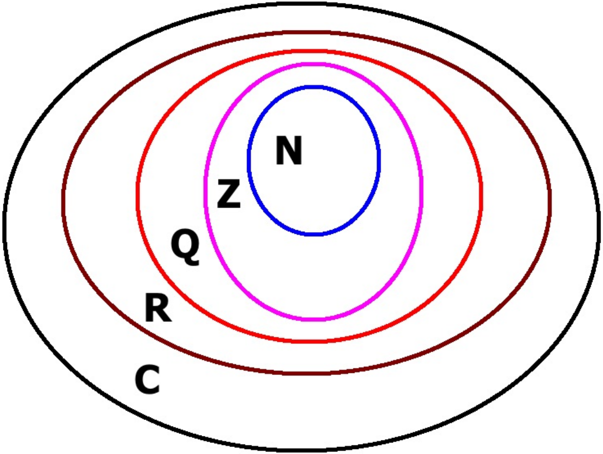
The word Mathematics comes from Greek $\mu\alpha\theta\eta\mu\alpha$ (máthéma), meaning "knowledge" or "study" or "learning".

It is the study of topics such as

- Quantity (numbers)
- Structure (patterns)
- Space (geometry)
- Change (analysis).

Outline

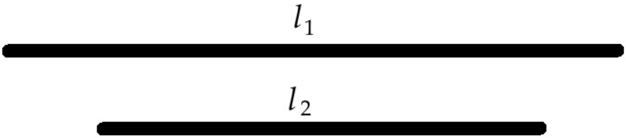
The Hierarchy of Numbers



$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$$

Incommensurability

Suppose we have two line segments



Can we find a unit of measurement such that *both lines are a whole number of units?*

Can they be co-measured? Are they commensurable?

Are ℓ_1 and ℓ_2 commensurable?

If so, let the unit of measurement be λ.

Then

$$\begin{aligned} \ell_1 &= m\lambda\,, \quad m \in \mathbb{N} \\ \ell_2 &= n\lambda\,, \quad n \in \mathbb{N} \end{aligned}$$

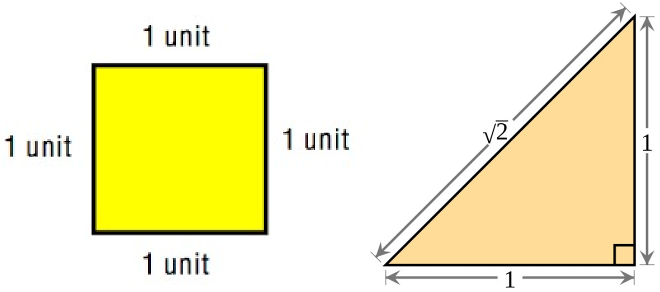
Therefore

$$\frac{\ell_1}{\ell_2} = \frac{m\lambda}{n\lambda} = \frac{m}{n}$$

If not, then ℓ_1 and ℓ_2 are incommensurable.

Irrational Numbers

If the side of a square is of length 1, then the diagonal has length $\sqrt{2}$ (by the Theorem of Pythagoras).



The ratio between the diagonal and the side is:

$$\frac{\textbf{Diagonal}}{\textbf{Side Length}} = \sqrt{2}$$

Irrationality of $\sqrt{2}$

For the Pythagoreans, numbers were of two types:

1. ***Whole numbers***
2. ***Ratios of whole numbers***

There were no other numbers.

Let's suppose that $\sqrt{2}$ is a ratio of whole numbers:

$$\sqrt{2} = \frac{p}{q}$$

We can assume that p and q have no common factors. Otherwise, we just cancel them out.

For example, suppose $p = 42$ and $q = 30$. Then

$$\frac{p}{q} = \frac{42}{30} = \frac{7 \times 6}{5 \times 6} = \frac{7}{5}$$

Remarks on *Reductio ad Absurdum.*

Sherlock Holmes:

“How often have I said to you that when you have eliminated the impossible, whatever remains, however improbable, must be the truth?”

The Sign of the Four (1890)

We say that p and q are relatively prime if they have no common factors.

In particular, p and q cannot both be even numbers.

Now square both sides of the equation $\sqrt{2} = p/q$:

$$2 = \frac{p}{q} \times \frac{p}{q} = \frac{p^2}{q^2} \qquad \textbf{or} \qquad p^2 = 2q^2$$

This means that p^2 is even. Therefore, p is even.

Let $p = 2r$ where r is another whole number.

$$\textbf{Then} \qquad p^2 = (2r)^2 = 4r^2 = 2q^2 \qquad \textbf{or} \qquad 2r^2 = q^2$$

But this means that q^2 is even. So, q is even.

Both p and q are even. This is a contradiction.

The supposition was that $\sqrt{2}$ is a ratio of two integers that have no common factors:

$$\sqrt{2} = \frac{p}{q}$$

This assumption has led to a contradiction.

By *reductio ad absurdum*, $\sqrt{2}$ is *irrational.*

It is *not a ratio of whole numbers.*

To the Pythagoreans, $\sqrt{2}$ was not a number.

$$\kappa\rho\iota\sigma\eta \qquad \kappa\alpha\tau\alpha\sigma\tau\rho\mathbf{O}\phi\eta\mathbf{!}$$

$\sqrt{2}$ and the Development of Mathematics

The discovery of irrational quantities had a dramatic effect on the development of mathematics.

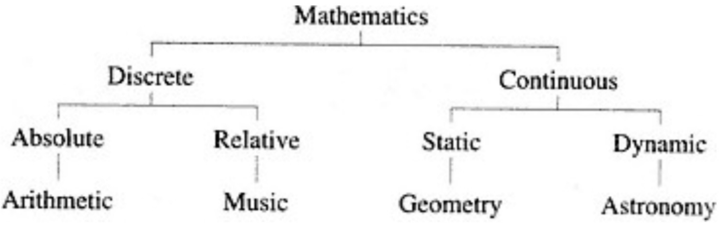
Legend has it that the discoveror of this fact was thrown from a ship and drowned.

The result was that focus now fell on geometry, and arithmetic or number theory was neglected.

The problems were not resolved for many centuries.

Outline

The Quadrivium



The Pythagorean model of mathematics

The Ancient Greeks

Mathematics and Astronomy are intimately linked.

Two of the strands of the Quadrivium were Geometry (static) and Cosmology (dynamic space).

Greek astronomer *Claudius Ptolemy* (c.90–168AD) placed the Earth at the centre of the universe.

The Sun and planets move around the Earth in orbits that are of the most perfect of all shapes: *circles.*

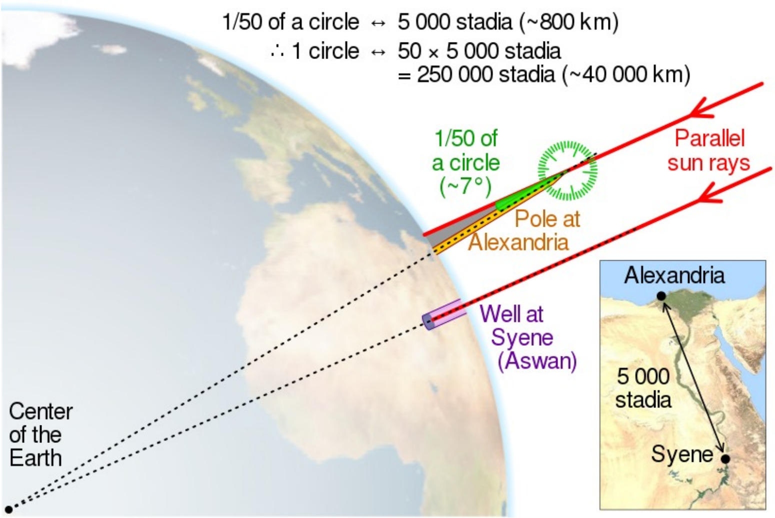
Aristarchus of Samos (c.310–230 BC)

Aristarchus of Samos (’A$\rho\iota\sigma\tau\alpha\rho\chi o\varsigma$), astronomer and mathematician, presented the first model that placed the Sun at the center of the universe.

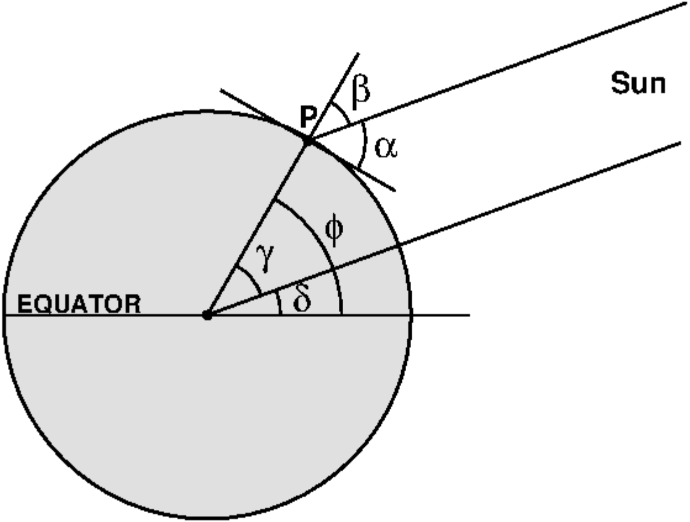
The original writing of Aristarchus is lost, but Archimedes wrote in his *Sand Reckoner:*

“His hypotheses are that the fixed stars and the Sun remain unmoved, that the Earth revolves about the Sun on the circumference of a circle, ... "

Eratosthenes (c.276–194 BC)



Eratosthenes (c.276–194 BC)



Hipparchus (c.190–120 BC)

Hipparchus of Nicaea ($'I\pi\pi\alpha\rho\chi o\varsigma$) was a Greek astronomer, geographer, and mathematician.

Regarded as the greatest astronomer of antiquity.

Often considered to be the *founder of trigonometry.*

Famous for

- **Precession of the equinoxes**
- **First comprehensive *star catalog***
- **Invention of the *astrolabe***
- **Invention (perhaps) of the *armillary sphere.***

Claudius Ptolemy (c.AD 100–170)

Claudius Ptolemy was a Greco-Roman astronomer, mathematician, geographer and astrologer.

He lived in the city of Alexandria.

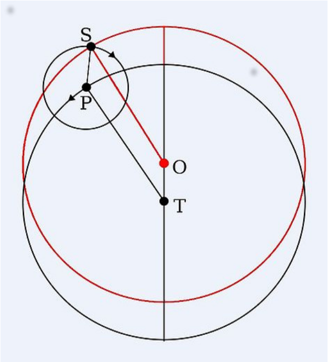
Ptolemy wrote several scientific treatises:

- **An astronomical treatise (*the Almagest*) originally called Mathematical Treatise (Mathematike Syntaxis).**
- **A book on geography.**
- **An astrological treatise.**

Ptolemy's Almagest is the only surviving comprehensive ancient treatise on astronomy.

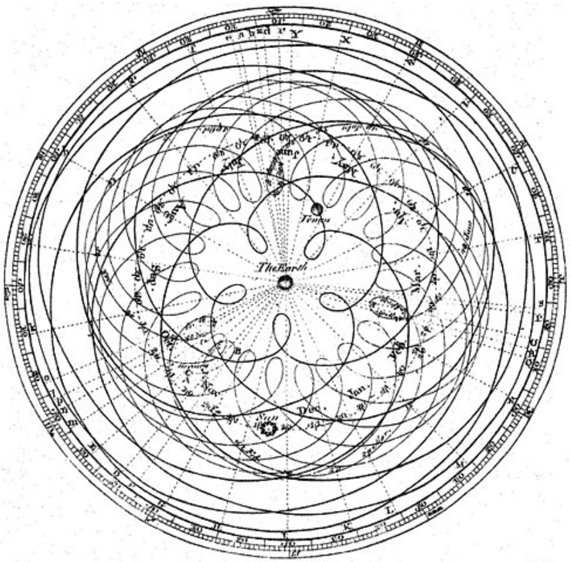
Ptolemy's Model

Ptolemy's model was universally accepted until the appearance of simpler heliocentric models during the scientific revolution.



O is the earth and S the planet.

Ptolemaic Epicycles



“Adding Epicycles”

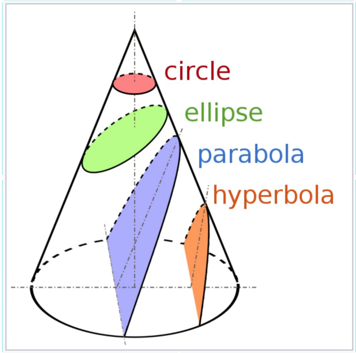
According to *Norwood Russell Hanson* (science historian):

There is no bilaterally symmetrical, nor eccentrically periodic curve used in any branch of astrophysics or observational astronomy which could not be smoothly plotted as the resultant motion of a point turning within a constellation of epicycles, finite in number, revolving around a fixed deferent.

“The Mathematical Power of Epicyclical Astronomy", 1960

Any path — periodic or not, closed or open — can be approximated by a sum of epicycles.

Conic Sections



Circles are special cases of conic sections.

They are formed by a plane cutting a cone at an angle.

Conics were studied by *Apollonius of Perga* (late 3rd – early 2nd centuries BC).

`https://en.wikipedia.org/wiki/Conic_section`

The Scientific Revolution

TRAILER

Next week, we will look at developments in the sixteenth and seventeenth centuries.

Figure from mathigon.org

Outline

The Real Numbers

We need to be able to assign a number to a line of any length.

The Pythagoreans found that no number known to them gave the diagonal of a unit square.

It is as if there are gaps in the number system.

We look at the rational numbers and show how to complete them: how to fill in the gaps.

The set $\mathbb{N}$ is infinite, but each element is isolated.

The set $\mathbb{Q}$ is infinite and also dense:
between any two rationals there is another rational.

PROOF: Let $r_1 = p_1/q_1$ and $r_2 = p_2/q_2$ be rationals.

$$\bar{r} = \tfrac{1}{2}(r_1 + r_2) = \frac{1}{2}\left(\frac{p_1}{q_1} + \frac{p_2}{q_2}\right) = \frac{p_1 q_2 + q_1 p_2}{2 q_1 q_2}$$

is another rational between them: $r_1 < \bar{r} < r_2$.

Although $\mathbb{Q}$ is dense, there are gaps.
The line of rationals is discontinuous.

We complete it—filling in the gaps—by *defining* the limit of any sequence of rationals as a real number.

WARNING:
We are glossing over a number of fundamental ideas of mathematical analysis:

- **What is an infinite sequence?**
- **What is the limit of a sequence?**

To give a particular example, we know that

$$\sqrt{2} = 1.41421356\ldots$$

We construct a sequence of rational numbers

$$\{1,\ 1.4,\ 1.41,\ 1.414,\ 1.4142,\ 1.41421,\ldots\}$$

In terms of fractions, this is the sequence

$$\left\{\frac{1}{1},\ \frac{14}{10},\ \frac{141}{100},\ \frac{1414}{1000},\ \frac{14142}{10000},\ \frac{141421}{100000},\ldots\right\}$$

These rational numbers get *closer and closer to* $\sqrt{2}$.

EXERCISE:
Construct a sequence in $\mathbb{Q}$ that tends to π.

The Real Number Line

The set of Real Numbers, $\mathbb{R}$, contains all the rational numbers in $\mathbb{Q}$ and also all the limits of sequences of rationals [technically, all *'Cauchy sequences'*].

We may assume that

- **Every point on the number line corresponds to a real number.**
- **Every real number corresponds to a point on the number line.**

PHYSICS: There are unknown aspects of the microscopic structure of spacetime! These go beyond our 'Universe of Discourse'.

Now we have the chain of sets:

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$$

We can also consider the prime numbers $\mathbb{P}$. They are subset of the natural numbers, so

$$\mathbb{P} \subset \mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$$

Outline

Pascal’s Triangle

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & 1 & & 1 & & & & & & \\
 & & & & & 1 & & 2 & & 1 & & & & & \\
 & & & & 1 & & 3 & & 3 & & 1 & & & & \\
 & & & 1 & & 4 & & 6 & & 4 & & 1 & & & \\
 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \\
 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\
1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1
\end{array}
$$

Combinatorial Symbol

$$\binom{n}{r} \quad \text{“}\boldsymbol{n}\ \text{choose}\ \boldsymbol{r}\text{”}$$

This symbol represents the number of combinations of r objects selected from a set of n objects.

$\binom{n}{r}$ **are also called Binomial coefficients.**

Pascal's Triangle: Combinations

$$\binom{0}{0}$$

$$\binom{1}{0} \quad \binom{1}{1}$$

$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$

$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$

$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$

Pascal’s Triangle

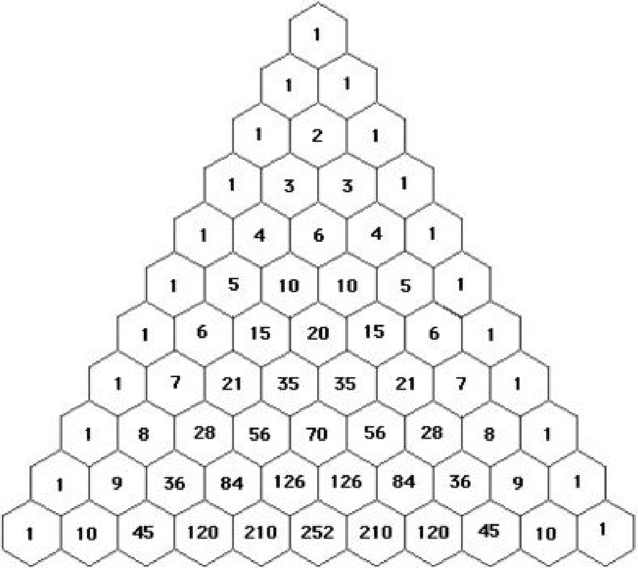
Pascal’s triangle is a triangular array
of the binomial coefficients.

It is named after French mathematician *Blaise Pascal.*

It was studied centuries before him in:

- India (Pingala, C2BC)
- Persia (Omar Khayyam, C11AD)
- China (Yang Hui, C13AD).

Pascal’s *Traité du triangle arithmétique* (Treatise
on Arithmetical Triangle) was published in 1665.


1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1

Pascal's Triangle

The rows of Pascal's triangle are numbered starting with row n = 0 at the top (0-th row).

The entries in each row are numbered from the left beginning with k = 0.

The triangle is easily constructed:

- A single entry 1 in row 0.
- Add numbers above for each new row.

The entry in the nth row and k-th column of Pascal's triangle is denoted $\binom{n}{k}$.

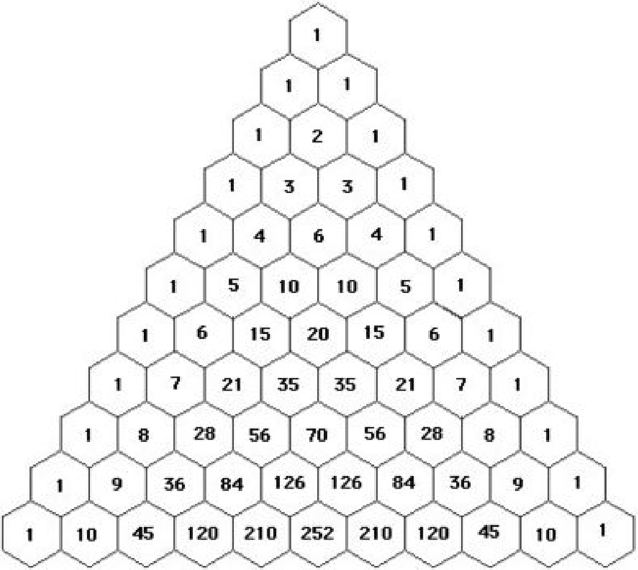
The entry in the topmost row is $\binom{0}{0} = 1$.

Pascal's Identity

The construction of the triangle may be written:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

This relationship is known as Pascal's Identity.


1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1

See Mathigon website

`https://mathigon.org/course/sequences/pascals-triangle`

Pascal's Triangle & Fibonacci Numbers.

1							
1	1						
1	2	1					
1	3	3	1				
1	4	6	4	1			
1	5	10	10	5	1		
1	6	15	20	15	6	1	
1	7	21	35	35	21	7	1

Figure : Pascal's Triangle and Fibonacci Numbers

Where are the Fibonacci Numbers hiding here?

Sierpinski's Gasket

Sierpinski Gasket: usual construction by trisection.

Sierpinski's Gasket is constructed by starting with an equilateral triangle, and successively removing the central triangle at each scale.

Sierpinski's Gasket at Stage 6

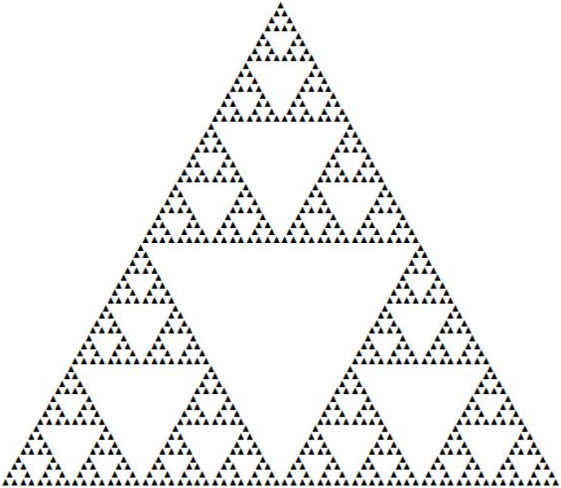


Figure : Result after 6 subdivisions

Sierpinski's Gasket in Pascal's Triangle

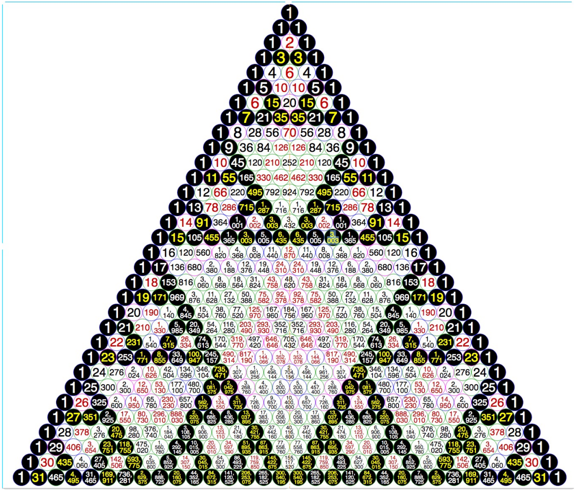


Figure : Odd numbers are in black

Remember Walking in Manhattan?

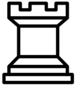
	1	1	1
1	2	3	4
1	3	6	10
1	4	10	20

Figure : Number of routes for a rook in chess.

Geometric Numbers in Pascal's Triangle

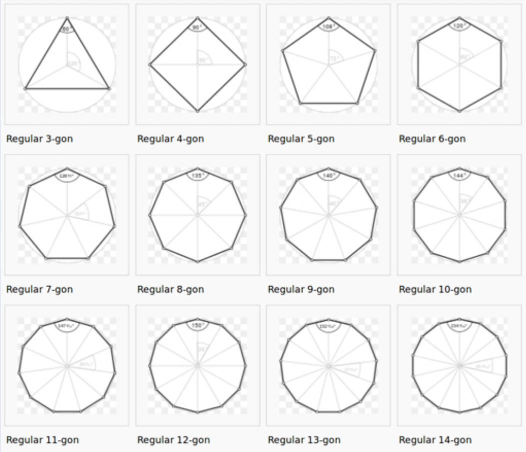
1	Natural numbers, $n = C(n, 1)$								
1	1	Triangular numbers, $T_n = C(n+1, 2)$							
1	2	1	Tetrahedral numbers, $Te_n = C(n+2, 3)$						
1	3	3	1	Pentatope numbers $= C(n+3, 4)$					
1	4	6	4	1	5-simplex ({3,3,3,3}) numbers				
1	5	10	10	5	1	6-simplex ({3,3,3,3,3}) numbers			
1	6	15	20	15	6	1	7-simplex ({3,3,3,3,3,3}) numbers		
1	7	21	35	35	21	7	1		
1	8	28	56	70	56	28	8	1	

Outline

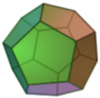
Euler's polyhedron formula.

Carving up the globe.

Regular Polygons



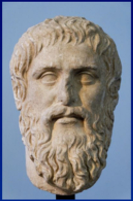
The Platonic Solids
(polyhedra)

Tetrahedron (four faces)	Cube or hexahedron (six faces)	Octahedron (eight faces)	Dodecahedron (twelve faces)	Icosahedron (twenty faces)
				

These five regular polyhedra were discovered in ancient Greece, perhaps by Pythagoras.

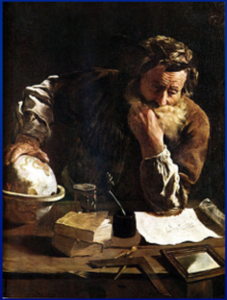
Plato used them as models of the universe.

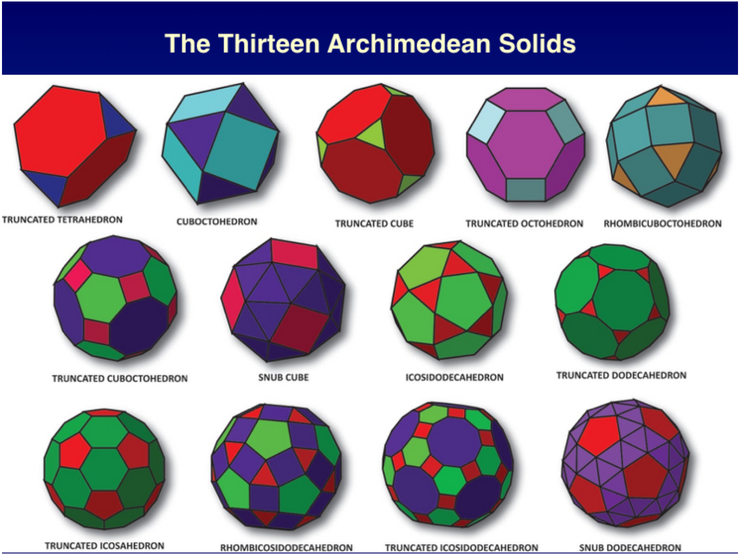
They are analysed in Book XIII of Euclid's Elements.



There are only five **Platonic** solids.

But **Archimedes** found, using different types of polygons, that he could construct 13 new solids.





Check V - E + F for the Truncated Cube

Euler's Polyhedron Formula

The great Swiss mathematician, **Leonard Euler**, noticed that, for all (convex) polyhedra,

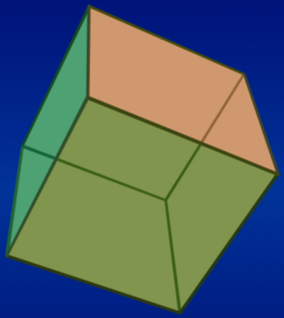
$$V - E + F = 2$$

where

- V = Number of vertices
- E = Number of edges
- F = Number of faces

Mnemonic: *Very Easy Formula*

For example, a Cube

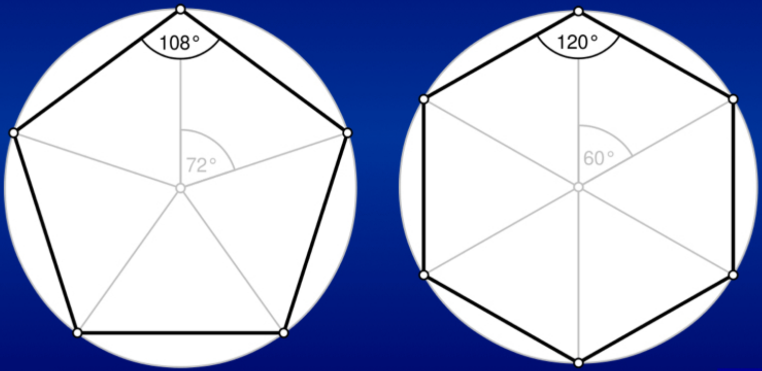


Number of vertices: V = 8
Number of edges: E = 12
Number of faces: F = 6

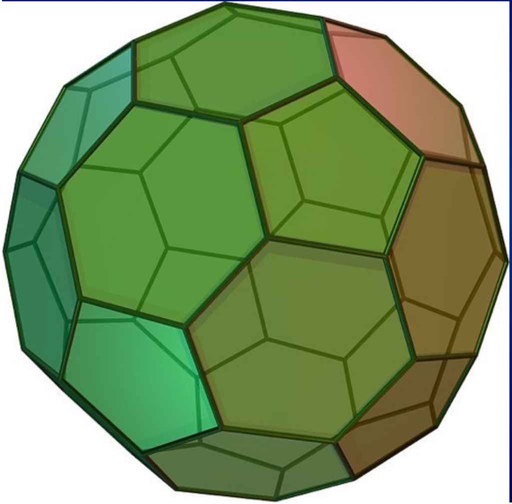
$$(V - E + F) = (8 - 12 + 6) = 2$$

Mnemonic: *Very Easy Formula*

Pentagons and Hexagons



The Truncated Icosahedron

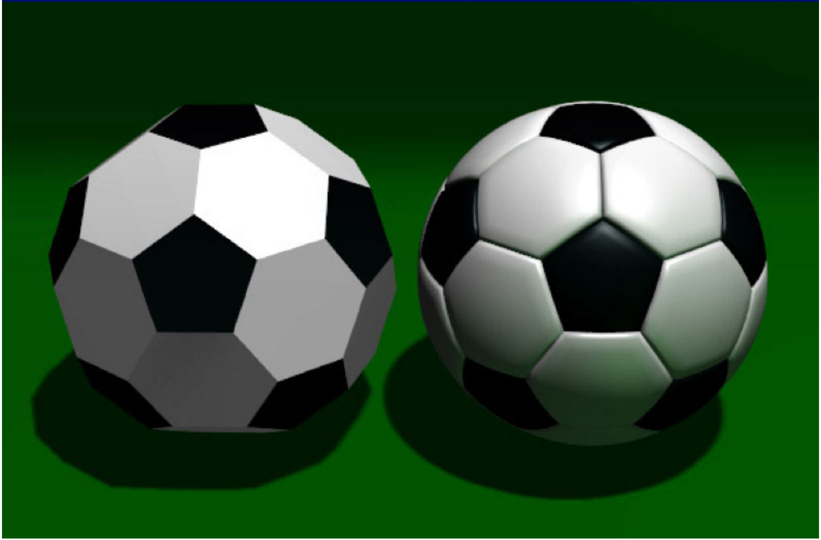


An Archimedean solid
with
pentagonal and
hexagonal faces.

The Truncated Icosahedron

Whare have you seen this before?

The Truncated Icosahedron

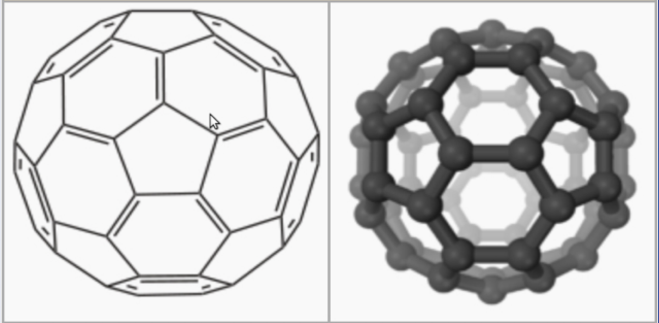


The "Buckyball", introduced at the the 1970 World Cup Finals in Mexico.

It has 32 panels: 20 hexagons and 12 pentagons.


A Geodesic Dome designed by the American architect
Richard Buckminster "Bucky" Fuller.

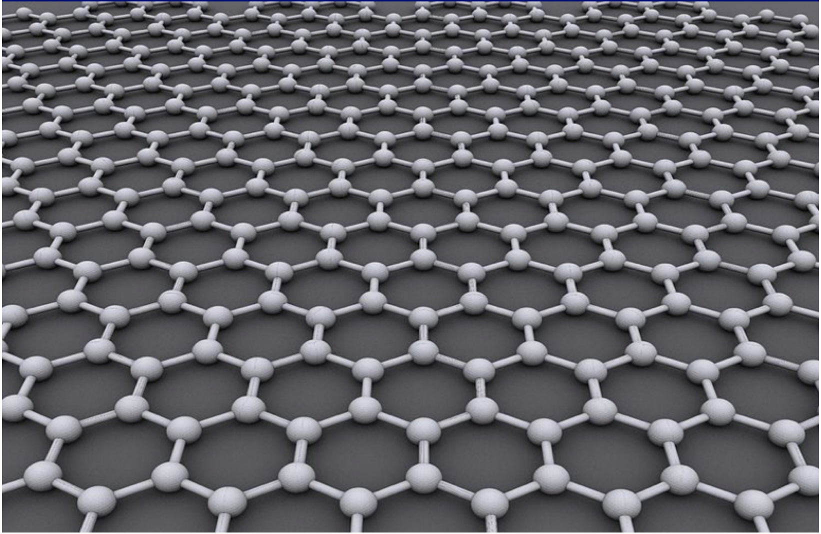
Buckminsterfullerene

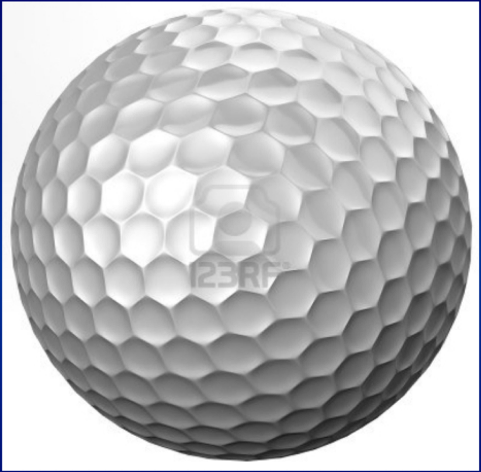


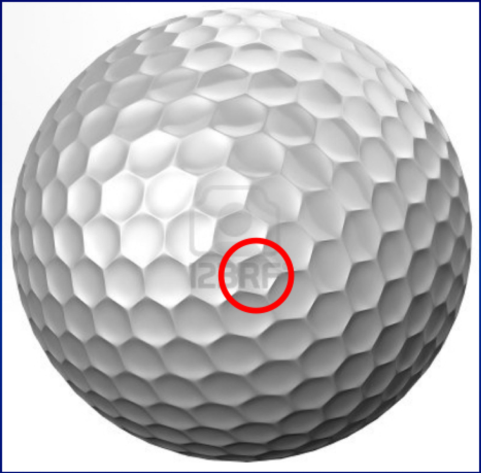
Buckminsterfullerene is a molecule with formula C_{60}

It was first synthesized in 1985.

Graphene
A hexagonal pattern of carbon one atom thick



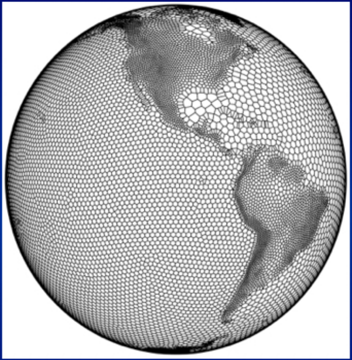

123RF

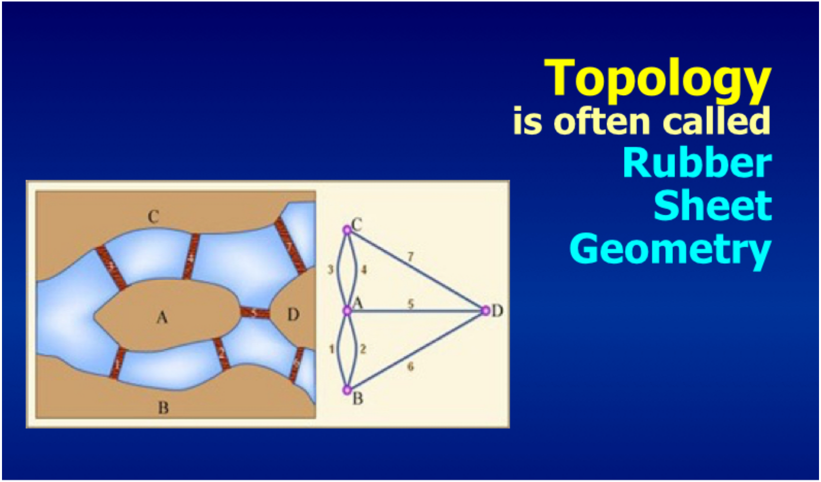


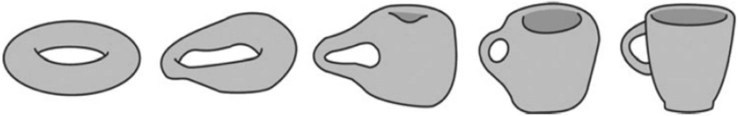
Euler's Polyhedron Formula

$$V - E + F = 2$$

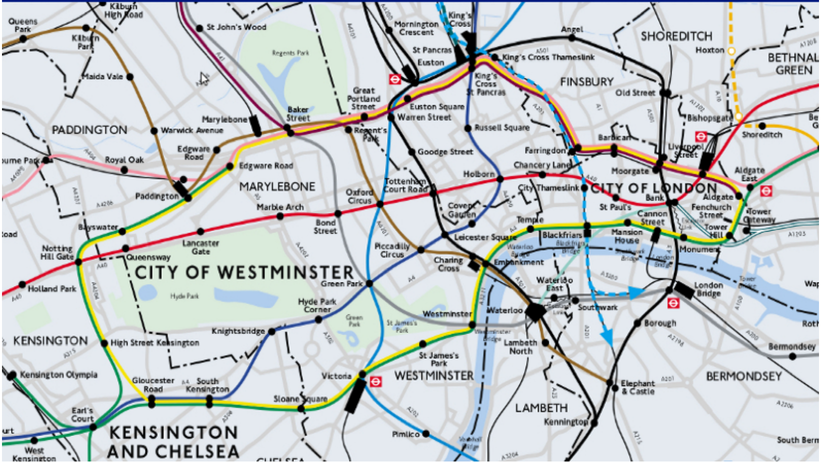
still holds.



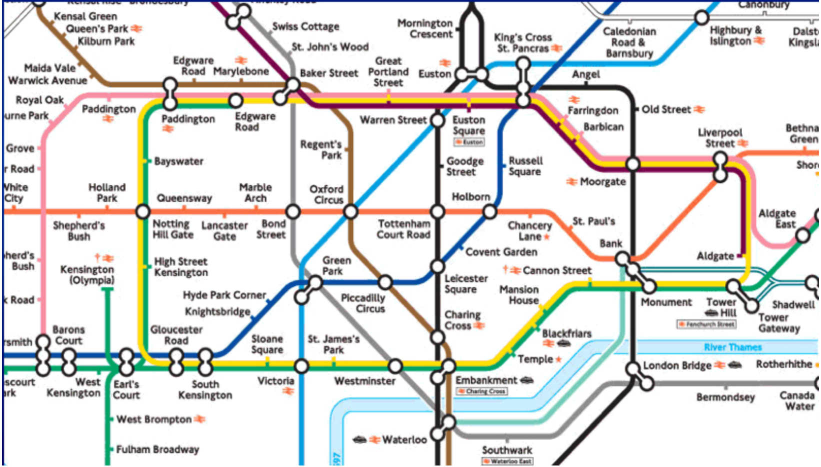

Topology
is often called
Rubber
Sheet
Geometry
C
A
D
B
C
3
4
7
A
5
D
1
2
6
B



Topology and the London Underground
Topographical Map

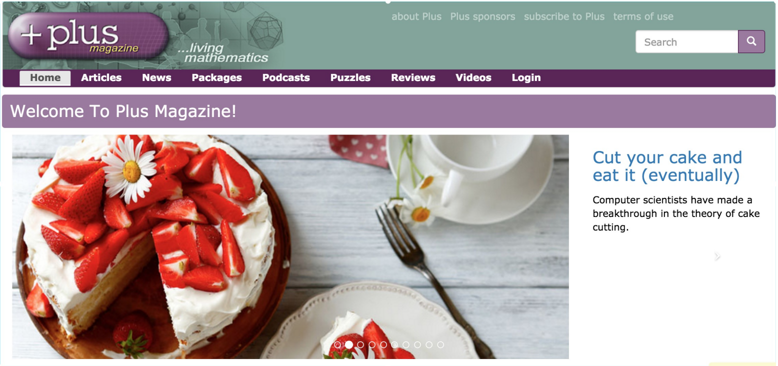


Topology and the London Underground
Topological Map



Outline

Distraction 7: Plus Magazine



PLUS: The Mathematics e-zine

`https://plus.maths.org/`

Outline

The Scientific Revolution

INTRODUCTION

This week, we will look at developments in the sixteenth and seventeenth centuries.

Nicolaus Copernicus	Tycho Brahe	Johannes Kepler	Galileo Galilei
1473 – 1543	1546 – 1601	1571 – 1630	1564 – 1642

Figure from mathigon.org

The Heliocentric Model

In 1543, *Nicolaus Copernicus* (1473–1543) published *“On the Revolutions of the Celestial Spheres”*.

He explained that the Sun is at the centre of the universe and that the Earth and planets move around it in circular orbits.

Danish astronomer *Tycho Brahe* (1546–1601) made very accurate observations of the movements of the planets, and developed his own model of the solar system.

Johannes Kepler (1571–1630)

Johannes Kepler (1571–1630) succeeded Brahe as imperial mathematician.

After many years of struggling, Kepler succeeded in formulating his three Laws of Planetary Motion.

Kepler's Laws describe the solar system much as we know it to be true today.

Kepler's Laws

- **The planets move on elliptical orbits, with the Sun at one of the two foci.**
  This explains why the Sun appears larger at

  some times of the year and smaller at others.
- **A line joining the planet and the Sun sweeps out equal areas in equal times.**
  This means that a planet moves faster when

  close to the Sun, and slower when further away.
- **The square of the orbital period is proportional to the cube of the mean radius of the orbit.**
  This law allows us to find the orbital time

  of a planet if we know the size of the orbit.

Jovian Year from Kepler's Third Law

- **Distance from Sun to Earth: 1.0 AU**
- **Distance from Sun to Jupiter: 5.2 AU**
-
- **Rotation Period of Earth: 1 Year**
- **Rotation Period of Jupiter: To Be Found**

$$\frac{{P_J}^2}{{P_E}^2} = \frac{{R_J}^3}{{R_E}^3}$$

$${P_J}^2 = {R_J}^3$$

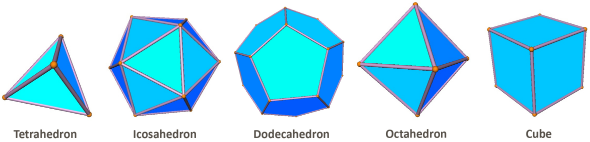
$$P_J = {R_J}^{\frac{3}{2}} \qquad P_J = (5.2)^{\frac{3}{2}} \approx 12 \textbf{ Years}$$

Do this in reverse: get distance from period.

The *Mysterium Cosmographicum*

There were six known planets in Kepler's time: Mercury, Venus, Earth, Mars, Jupiter, Saturn.

There are precisely five platonic solids:



This gave Kepler an extraordinary idea!

`https://thatsmaths.com/2016/10/13/`

`\keplers-magnificent-mysterium-cosmographicum/`

Galileo Galelii (1564–1630)

Galileo introduced the *telescope* to astronomy, and made some dramatic discoveries.

He observed the four large moons of Jupiter *revolving around that planet.*

He established the laws of inertia that underlie Newton's dynamical laws.

Four Remarkable Scientists

Figure from mathigon.org

Isaac Newton (1642–1727)

In 1687, Isaac Newton published the Principia Mathematica. He established the mathematical foundations of dynamics.

Between any two masses there is a force:

$$F = \frac{GMm}{r^2}$$

This is the force of gravity and gravity is what makes the planets move around the Sun.

Newton's Laws imply and explain Kepler's laws.

Outline

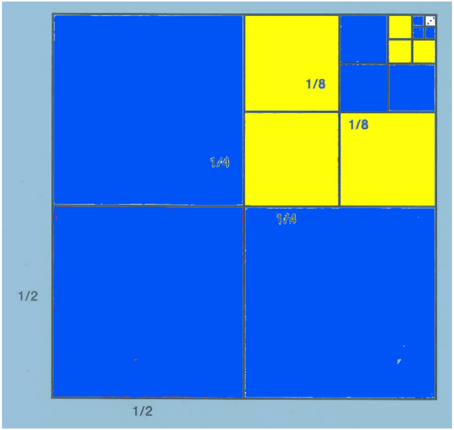
Distraction 8: Sum by Inspection

Can you guess the sum of this series:

$$\left(\frac{1}{2}\right)^2 + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{8}\right)^2 + \left(\frac{1}{16}\right)^2 + \cdots$$

Distraction 8: Sum by Inspection

We will find the shaded area without calculation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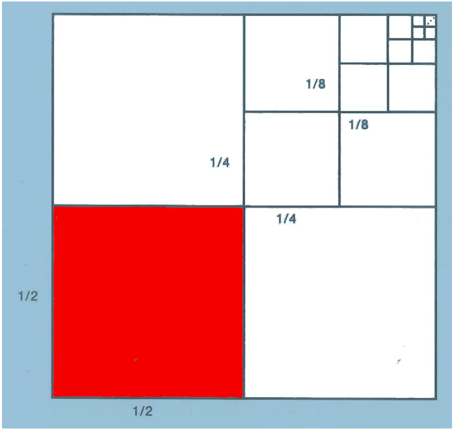
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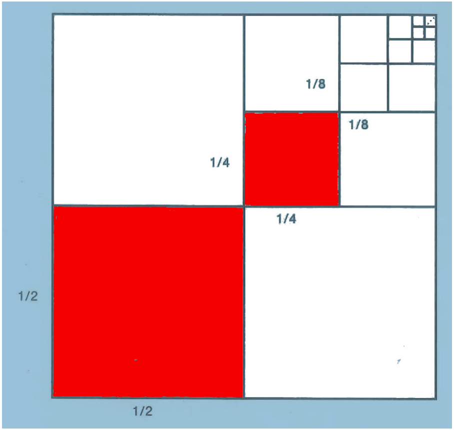
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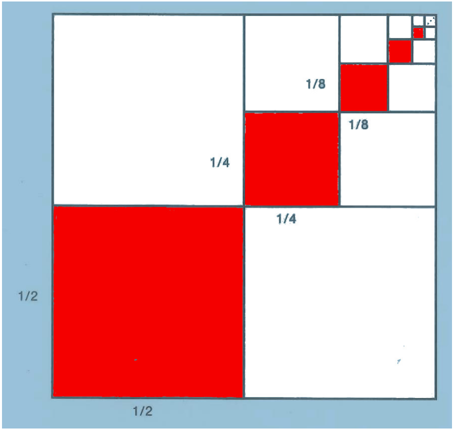
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Proof by Inspection

Look at the figure in two different ways

At each scale, we have three squares the same size, and we keep one of them (red) and omit the others.

So, the area of the shaded squares is $\frac{1}{3}$.

However, it is also given by the series

$$\left(\frac{1}{2}\right)^2 + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{8}\right)^2 + \left(\frac{1}{16}\right)^2 + \cdots$$

Therefore we can sum the series:

$$\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \cdots \quad = \quad \frac{1}{3}$$

Outline

Carl Friedrich Gauss (1777–1855)

Carl Friedrich Gauss (1777–1855)

A German mathematician who made profound contributions to many fields of mathematics:

- **Number theory**
- **Algebra**
- **Statistics**
- **Analysis**
- **Differential geometry**
- **Geodesy & Geophysics**
- **Mechanics & Electrostatics**
- **Astronomy**

One of the greatest mathematicians of all time.

Gauss Outsmarts his Teacher

Gauss was a genius. He was known as

The Prince of Mathematicians.

When very young, Gauss outsmarted his teacher.

I can now reveal a fact unknown to historians:

The teacher got his own back. Ho! ho! ho!

Gauss Outsmarts his Teacher

Gauss's school teacher tasked the class:

- **Add up all the whole numbers from 1 to 100.**

Gauss solved the problem in a flash.

He wrote the correct answer,

$$5,050$$

on his slate and handed it to the teacher.

How did Gauss do it?

First, Gauss wrote the numbers in a row:

$$1 \quad 2 \quad 3 \quad \ldots \quad 98 \quad 99 \quad 100$$

Next he wrote them again, *in reverse order:*

$$\begin{array}{ccccccc} 1 & 2 & 3 & \ldots & 98 & 99 & 100 \\ 100 & 99 & 98 & \ldots & 3 & 2 & 1 \end{array}$$

Then he added the two rows, column by column:

$$\begin{array}{ccccccc} 1 & 2 & 3 & \ldots & 98 & 99 & 100 \\ 100 & 99 & 98 & \ldots & 3 & 2 & 1 \\ --- & --- & --- & --- & --- & --- & --- \\ 101 & 101 & 101 & \ldots & 101 & 101 & 101 \end{array}$$

Clearly, the total for the two rows is 10,100.

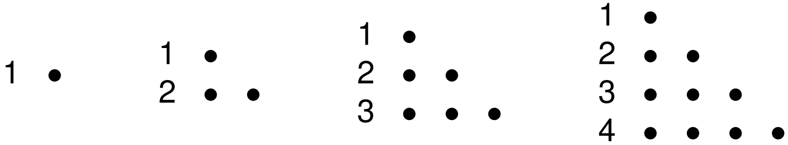
But every number from 1 to 100 is counted twice.

$$\therefore \ \mathbf{1 + 2 + 3 + \cdots + 98 + 99 + 100 = 5{,}050}$$

Triangular Numbers

Gauss had calculated the 100-th triangular number.

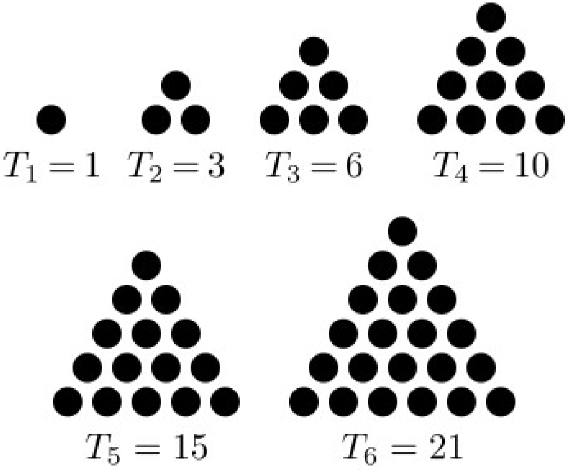
Let us take a geometrical look at the sums of the first few natural numbers:



We see that the sums can be arranged as triangles.

Triangular Numbers

The first few *triangular numbers* are $\{1, 3, 6, 10, 15, 21\}$**.**



Let's look at the 10th triangular number.

For $n = 10$ the pattern is:

$$
\begin{array}{rcccccccccc}
1 & \bullet \\
2 & \bullet & \bullet \\
3 & \bullet & \bullet & \bullet \\
4 & \bullet & \bullet & \bullet & \bullet \\
5 & \bullet & \bullet & \bullet & \bullet & \bullet \\
6 & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
7 & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
8 & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
9 & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
10 & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet
\end{array}
$$

How do we compute its value? Gauss's method!

It is easy to show that the n-th triangular number is

$$T_n = (1 + 2 + 3 + \cdots + n) = \tfrac{1}{2} n(n+1)$$

We do just as Gauss did, and list the numbers twice:

$$\begin{array}{cccccc} 1 & 2 & 3 & \ldots & n-1 & n \\ n & n-1 & n-2 & \ldots & 2 & 1 \\ --- & --- & --- & \ldots & --- & --- \\ n+1 & n+1 & n+1 & \ldots & n+1 & n+1 \end{array}$$

There are n columns, each with total $n+1$.

So the grand total is $n \times (n+1)$.

Each number has been counted twice, so

$$T_n = \tfrac{1}{2} n(n+1)$$

Let's check this for Gauss's problem of $n = 100$:

$$T_{100} = 1 + 2 + 3 + \cdots + 100 = \frac{100 \times 101}{2} = 5,050$$

Gauss's approach was to look at the problem from a new angle.

Such *lateral thinking* is very common in mathematics:

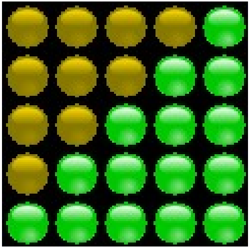
Problems that look difficult can sometimes be solved easily when tackled from a different angle.

Two Triangles Make a Square

A nice property of *consecutive* triangular numbers:

$$\begin{array}{ccc} \star & & \\ \star & \star & \\ \star & \star & \star \end{array} \quad + \quad \begin{array}{cccc} \bullet & & & \\ \bullet & \bullet & & \\ \bullet & \bullet & \bullet & \\ \bullet & \bullet & \bullet & \bullet \end{array} \quad = \quad \begin{array}{cccc} \bullet & \star & \star & \star \\ \bullet & \bullet & \star & \star \\ \bullet & \bullet & \bullet & \star \\ \bullet & \bullet & \bullet & \bullet \end{array}$$

$$T_3 + T_4 = 6 + 10 = 16 = 4^2$$



Triangular Numbers

We have seen, by means of geometry that the sum of two consecutive triangular numbers is a square.

Now let us prove this algebraically:

$$
\begin{aligned}
T_n + T_{n+1} &= \tfrac{1}{2}n(n+1) + \tfrac{1}{2}(n+1)(n+2) \\
&= \tfrac{1}{2}(n+1)[n+(n+2)] \\
&= \tfrac{1}{2}(n+1)[2(n+1)] \\
&= (n+1)^2
\end{aligned}
$$

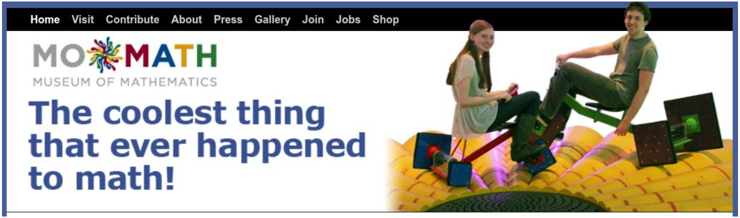
The result is *a perfect square.*

Puzzle

What is the sum of all the numbers from 1 up to 100 and back down again?

The answer is in the video coming up now.

A Video from the Museum of Mathematics



VIDEO: Beautiful Maths, available at

`http://momath.org/home/beautifulmath/`

Video by James Tanton

Gauss Outsmarted by his Teacher

The teacher thought that he would have a half-hour of peace and quiet while the pupils grappled with the problem of adding up the first 100 numbers.

He was annoyed when Gauss came up almost immediately with the correct answer 5,050.

So, he said:

"Oh, you zink you are zo zmart!
Zo, multiply ze first 100 numbers."

EXERCISE: Zink about that!

Thank you