AweSums

Marvels and Mysteries of Mathematics

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LECTURE 4

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University College Dublin

Evening Course, UCD, Autumn 2020

Outline

Outline

Meaning and Content of Mathematics

The word Mathematics comes from Greek $\mu\alpha\theta\eta\mu\alpha$ (máthéma), meaning "knowledge" or "study" or "learning".

It is the study of topics such as

- Quantity (numbers)
- Structure (patterns)
- Space (geometry)
- Change (analysis).

Outline

Distraction 13: Conway's Puzzle

Find a 10-digit number ABCDEFGHIJ such that:

1. A is divisible by 1
2. AB is divisible by 2
3. ABC is divisible by 3
4. ABCD is divisible by 4
5. ABCDE is divisible by 5
6. ABCDEF is divisible by 6
7. ABCDEFG is divisible by 7
8. ABCDEFGH is divisible by 8
9. ABCDEFGHI is divisible by 9
10. ABCDEFGHIJ is divisible by 10

Each letter is a digit (1,2,3,4,5,6,7,8,9,0).

Distraction 13. Solution

(1): Try every possible permutation:

$$10! = 3{,}628{,}800$$

(2) Use division rules to reduce this number.

Distraction 13. Solution: 3 816 547 290

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$$10! = 3{,}628{,}800$$

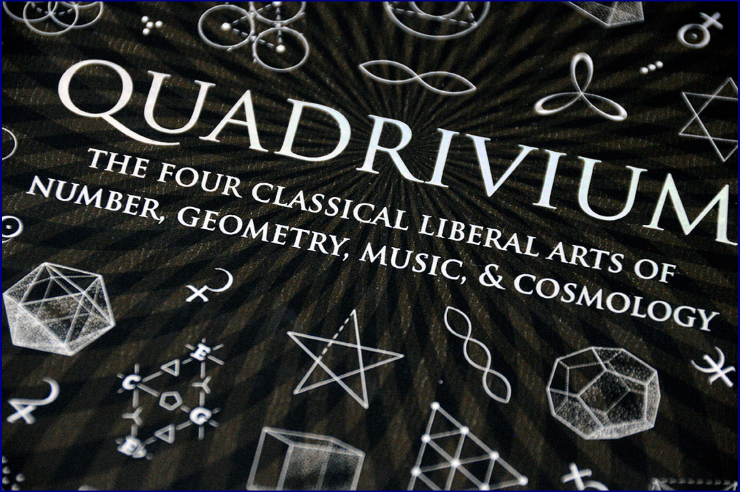
(2) Use division rules to reduce this number.

(3) Go to this page in *The Guardian:* `https://www.theguardian.com/science/2020/apr/20/can-you-solve-it-john-horton-conway-playful-maths-genius`
(or Google for Conway's number puzzle Bellos)

(4) Go to this page in *Quanta Magazine:* `https://www.quantamagazine.org/three-math-puzzles-inspired-by-john-horton-conway-20201015/`

Outline

The Quadrivium



The Quadrivium

The Quadrivium originated with the Pythagoreans around 500 BC.

The Pythagoreans' quest was to find the eternal laws of the Universe, and they organized their studies into the scheme later known as the Quadrivium.

It comprised four disciplines:

- Arithmetic
- Geometry
- Music
- Astronomy

The Quadrivium

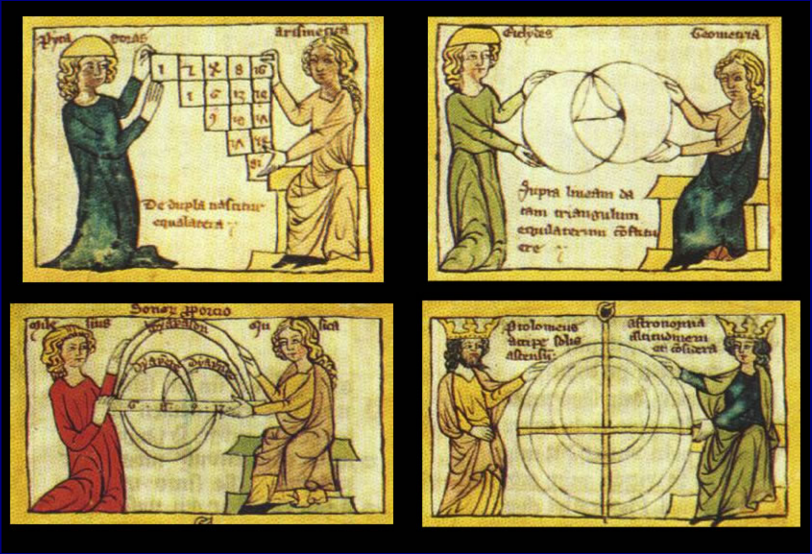
First comes Arithmetic, concerned with the infinite linear array of numbers.

Moving beyond the line to the plane and 3D space, we have Geometry.

The third discipline is Music, which is an application of the science of numbers.

Fourth comes Astronomy, the application of Geometry to the world of space.

The Quadrivium



Static/Dynamic. Pure/Applied

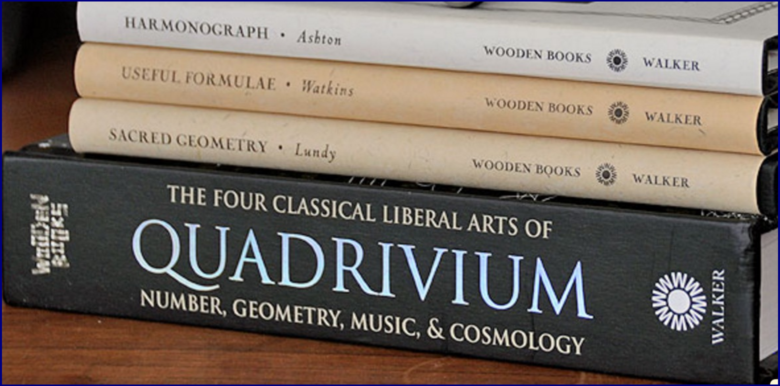
- Arithmetic (static number)
- Music (moving number)
- Geometry (measurement of static Earth)
- Astronomy (measurement of moving Heavens)

Arithmetic represents numbers at rest,
Geometry is magnitudes at rest,

Music is numbers in motion and
Astronomy is geometry in motion.

The first two are pure in nature,
while the last two are applied.

The Quadrivium



For the Greeks, **Mathematics** embraced all four areas.

The Pythagoreans

Pythagoras distinguished between
quantity and magnitude.

Objects that can be counted
yield quantities or numbers.

Substances that are measured provide magnitudes.

Thus, cattle are counted whereas milk is measured.

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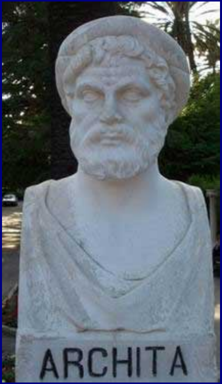
Arithmetic studies quantities or numbers and
Music involves the relationship between numbers
and their evolution in time.

Geometry deals with magnitudes, and
Astronomy with their distribution in space.

Archytas (428–350 BC): $AP X \Upsilon TA\Sigma$



$A\rho\chi\upsilon\tau\alpha\varsigma$.

Born in Tarentum, son of Hestiaeus.

Mathematician and philosopher.

Pythagorean, student of Philolaus.

Provided a solution for the Delian problem of doubling the cube.

Said to have tutored Plato in mathematics(?)

Archytas (428–350 BC)

Archytas lived in Tarentum (now in Southern Italy).

One of the last scholars of the Pythagorean School and was a good friend of Plato.

The designation of the four disciplines of the Quadrivium was ascribed to Archytas.

His views were to dominate pedagogical thought for over two millennia.

Partly due to Archytas, mathematics has played a prominent role in education ever since.

Plato’s Academy

According to Plato, mathematical knowledge was essential for an understanding of the Universe. The curriculum was outlined in Plato’s *Republic.*

Inscription over the entrance to Plato’s Academy:

"Let None But Geometers Enter Here".

This indicated that the Quadrivium was a prerequisite for the study of philosophy in ancient Greece.

Boethius (AD 480–524)

The Western Roman Empire was in many ways static for centuries.

No new mathematics between the conquest of Greece and the fall of the Roman Empire in AD 476.

Boethius, the 6th century Roman philosopher, was one of the last great scholars of antiquity.

The organization of the Quadrivium was formalized by Boethius.

It was the mainstay of the medieval monastic system of education.

The Quadrivium

Typus Arithmeticae

A woodcut from the book *Margarita Philosophica*, by Gregor Reisch, Freiburg, 1503.

The central figure is Dame Arithmetic, watching a competition between Boethius, using pen and Hindu-Arabic numerals, and Pythagoras, using a counting board or *tabula*.

She looks favourably toward Boethius.

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But how did Boethius know about Hindu-Arabic numerals?

The Liberal Arts

The seven liberal arts comprised
the Trivium and the Quadrivium.

The Trivium was centred on three arts of language:

- Grammar: proper structure of language.
- Logic: for arriving at the truth.
- Rhetoric: the beautiful use of language.

Aim of the Trivium: Goodness, Truth and Beauty.

Aristotle traced the origin of the Trivium back to Zeno.

The Ultimate Goal

The goal of studying the Quadrivium was
an insight into the nature of reality,
an understanding of the Universe.

The Quadrivium offered the seeker of wisdom
an understanding of the integral nature of
the Universe and the role of humankind within it.

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That is our aim in AweSums!

Outline

Theorem of Pythagoras

The Theorem of Pythagoras is of fundamental importance in Euclidean geometry

It encapsulates the structure of space.

In the BBC series, The Ascent of Man, Jacob Bronowski said

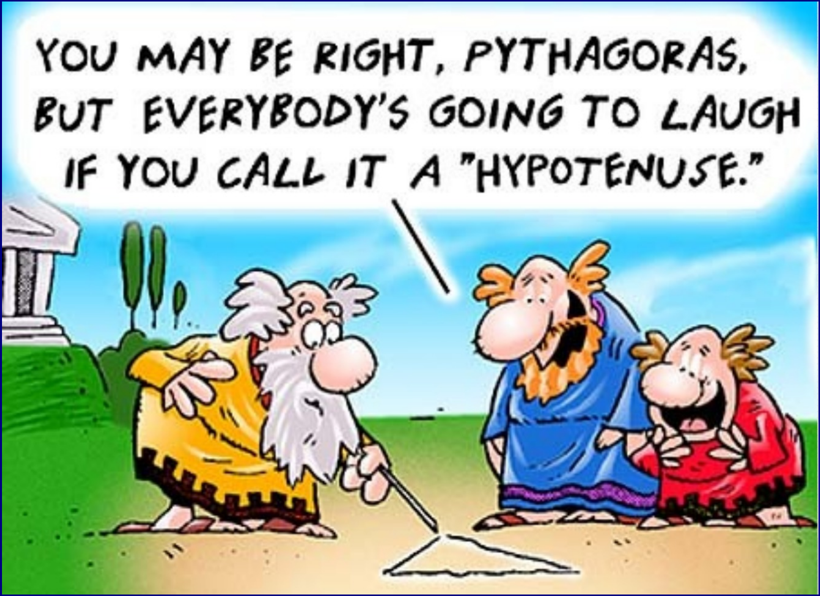
"The theorem of Pythagoras remains the most important single theorem in mathematics."

Theorem of Pythagoras

YouTube video with Danny Kaye

Google search for "Danny Kaye Hypotenuse"

https://www.youtube.com/watch?v=oeRCsAGQVy8


YOU MAY BE RIGHT, PYTHAGORAS, BUT EVERYBODY'S GOING TO LAUGH IF YOU CALL IT A "HYPOTENUSE."

Hypotenuse

The side of a right triangle opposite to the right angle.

1570s, from Late Latin hypotenusa, from Greek hypoteinousa “stretching under" (the right angle).

Fem. present participle of hypoteinein, from hypo- “under" + teinein “to stretch"

From Online Etymology Dictionary: `http://www.etymonline.com/`

Mathigon.org

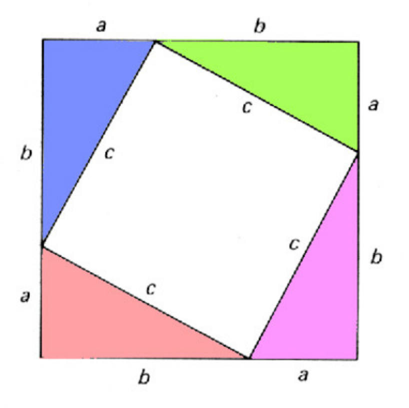
Mathigon.org video on Proofs without Formulas:

- What is the sum of the angles in a triangle?
- What is the sum of the angles in a polygon?
- What is the area of a triangle?
- How does Pythagoras' Theorem work?

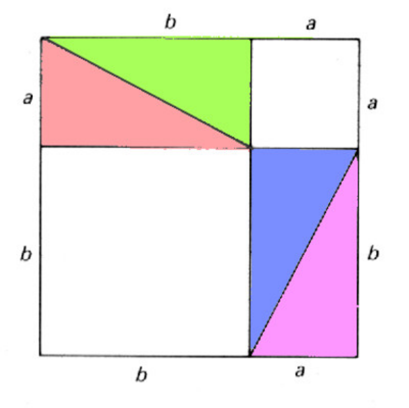
In the video below, these and other important concepts are explained in only two minutes using nothing but graphics.

https://youtu.be/IUCK8bk0xPo

Proof without Formulae



Proof without Formulae



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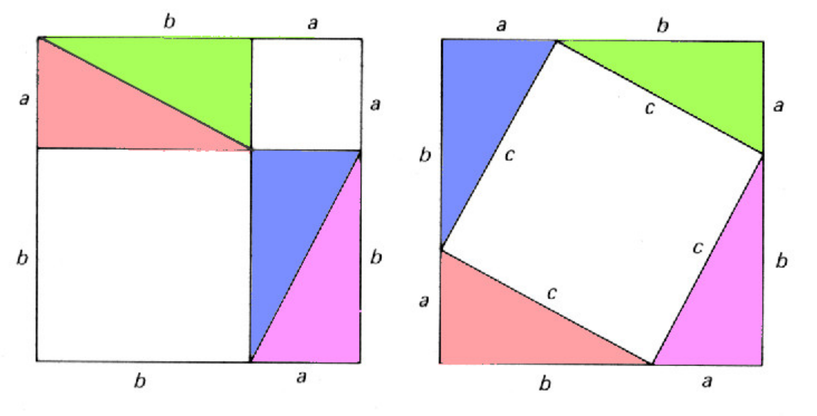

$$a^2 + b^2 = c^2$$

Why is this Important / Interesting?

Squares on the sides of triangles don't seem much.

But the theorem gives us distances.

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If one point is at $(0,0)$ and another at (x,y), the theorem gives the distance:

$$r^2 = x^2 + y^2 \qquad \text{or} \qquad r = \sqrt{x^2 + y^2}$$

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This tells us about the structure of space.

I should expand on this topic (e.g., SAR)

Outline

The Unary System

The simplest numeral system is the unary system.

Each natural number is represented by
a corresponding number of symbols.

If the symbol is “ | ” , the number seven
would be represented by | | | | | | | .

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Tally marks represent one such system,
which is still in common use.

The unary system is only useful for small numbers.

The unary notation can be abbreviated,
with new symbols for certain values.

Sign-Value Notation

The **five-bar gate** system groups 5 strokes together.

Normally, distinct symbols are used for powers of 10.

If “ | ” stands for one, “ Λ ” for ten and “ ϒ ” for 100, then the number **123** becomes ϒ ΛΛ | | |

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If “ | ” stands for one, “ Λ ” for ten and “ ϒ ” for 100, then the number 123 becomes ϒ ΛΛ | | |

There is no need for a symbol for zero.

This is called sign-value notation.

Ancient Egyptian numerals were of this type.

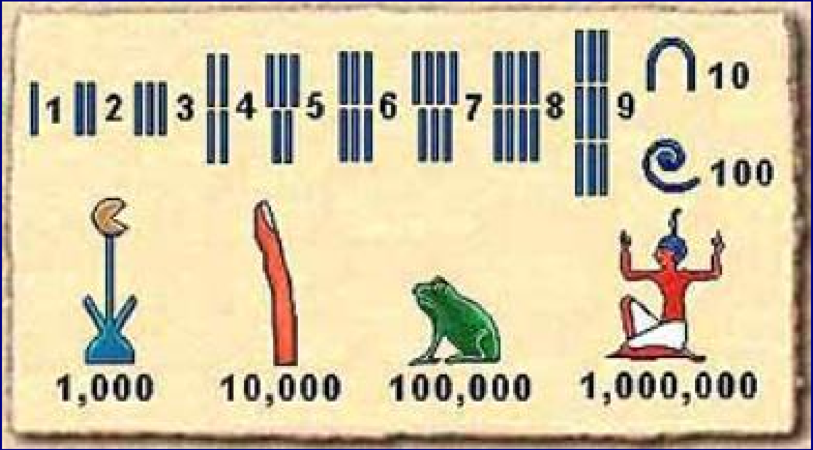
Roman numerals were a modification of this idea.

Egypyian Numerals

Value	1	10	100	1,000	10,000	100,000	1 million, or many
Hieroglyph						 or 	
Description	Single stroke	Heel bone	Coil of rope	Water lily (also called Lotus)	Bent Finger	Tadpole or Frog	Man with both hands raised, perhaps Heh.[3]

Figure: From Wikipedia page `https://en.wikipedia.org/wiki/Egyptian_numerals`

Egypyian Numerals



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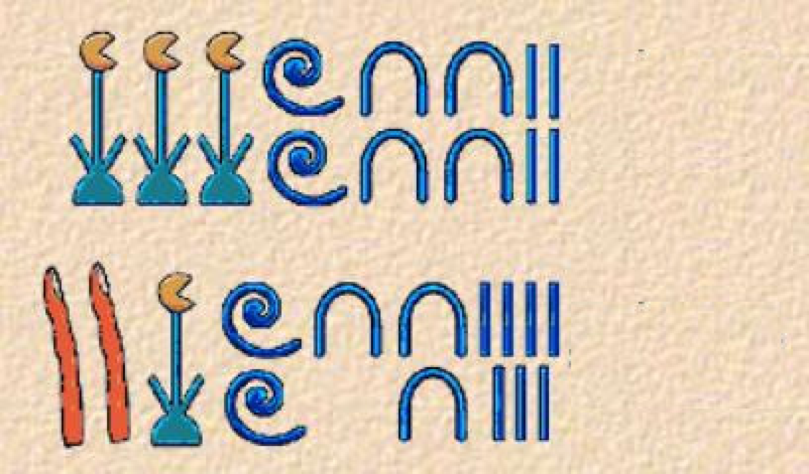
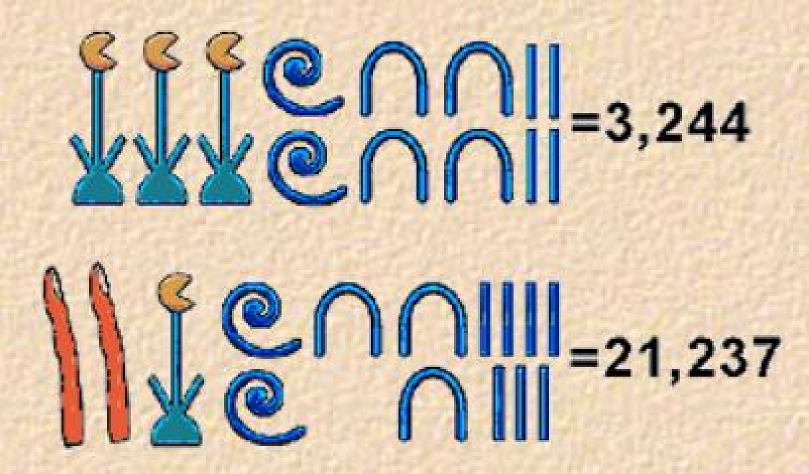


Egypyian Numerals



Hebrew Numerals

Hebrew Number Values

א	Aleph -	1	ל	Lamed -	30
ב	Beth -	2	מ	Mem -	40
ג	Gimel -	3	נ	Nun -	50
ד	Daleth -	4	ס	Samekh -	60
ה	Heh -	5	ע	Ayin -	70
ו	Vav -	6	פ	Peh -	80
ז	Zain -	7	צ	Tzaddi -	90
ח	Cheth -	8	ק	Qoph -	100
ט	Teth -	9	ר	Resh -	200
י	Yod -	10	ש	Shin -	300
כ	Kaph -	20	ת	Tau -	400

The 22 letters of the Hebrew alphabet were used also as numerals.

Each letter corresponded to a numerical value.

Greek Numerals

	Units	Tens	Hundreds
1	α alpha	ι iota	ρ rho
2	β beta	κ kappa	σ sigma
3	γ gamma	λ lambda	τ tau
4	δ delta	μ mu	υ upsilon
5	ε epsilon	ν nu	φ phi
6	ϝ digamma	ξ xi	χ chi
7	ζ zeta	ο omicron	ψ psi
8	η eta	π pi	ω omega
9	θ theta	ϙ koppa	ϡ sampi

The 24 letters of the Greek alphabet had corresponding numerical values.

They were supplemented by three additional letters, which are now archaic.

$$\sigma\mu\gamma = ?$$

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$$\sigma\mu\gamma = ?$$

$$243 = \sigma\mu\gamma$$

Greek Numerals

Arabic number	1	2	3	4	5	6	7	8	9
Greek number	α	β	γ	δ	ε	F	ζ	η	θ
Greek name	alpha	beta	gamma	delta	epsilon	digamma	zeta	eta	theta
Sound	a	b	g	d	short e		z	long e	th
Arabic number	10	20	30	40	50	60	70	80	90
Greek number	ι	κ	λ	μ	ν	ξ	ο	π	ϙ
Greek name	iota	kappa	lambda	mu	nu	xi	omicron	pi	koppa
Sound	i	k/c	l	m	n	x	short o	p	
Arabic number	100	200	300	400	500	600	700	800	900
Greek number	ρ	σ	τ	υ	φ	χ	ψ	ω	ϡ
Greek name	rho	sigma	tau	upsilon	phi	chi	psi	omega	sampi
Sound	r	s	t	u	f/ph	ch	ps	long o	

Outline

Topology: a Major Branch of Mathematics

Topology is all about continuity and connectivity,
but the meaning of that will appear later.

We will look at a few aspects of Topology.

- The Bridges of Königsberg
- Doughnuts and Coffee-cups
- Knots and Links
- Nodes and Edges: Graphs
- The Möbius Band

In this lecture, we study The Bridges of Königsberg.

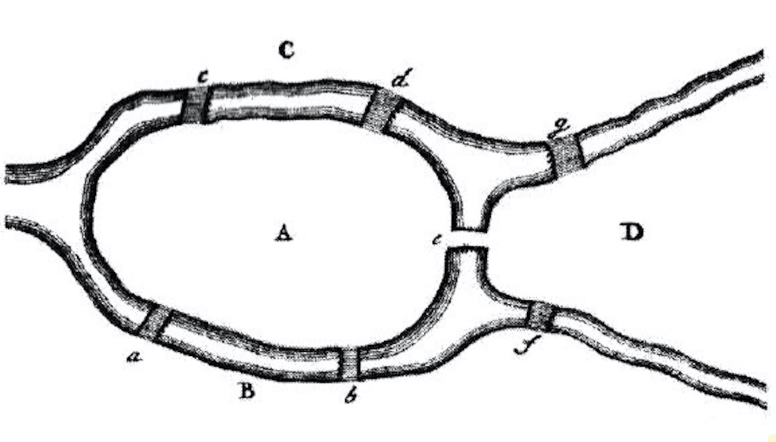
The Bridges of Königsberg

One of the earliest topological puzzles was studied by the renowned Swiss mathematician **Leonard Euler.**

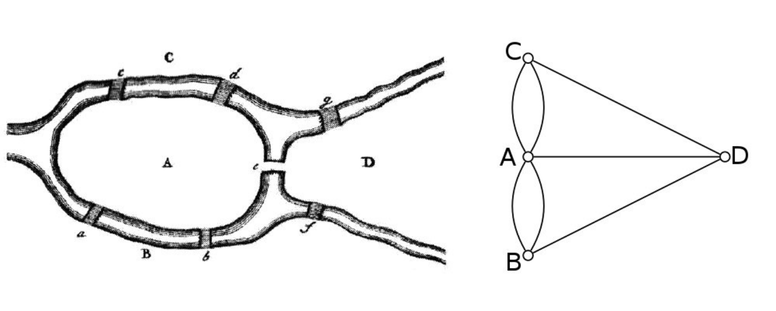
It is called 'The Seven Bridges of Königsberg'.

The goal is to find a route through that city, crossing each of seven bridges exactly once.

The Bridges of Königsberg



The Bridges of Königsberg



Euler reduced the problem to its essentials, removing all extraneous details.

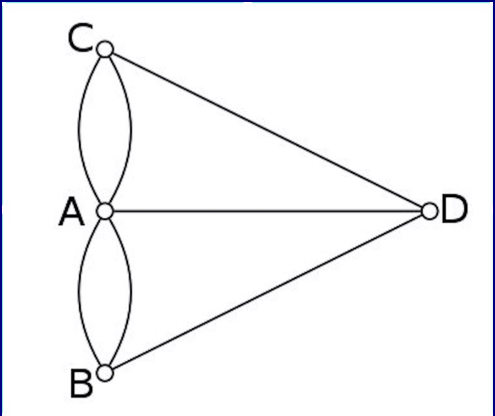
He replaced the map above by the graph on the right.

A simple argument showed that no journey that crosses each bridge exactly once is possible.

Except at the termini of the route, each path arriving at a vertex must have a corresponding path leaving it.

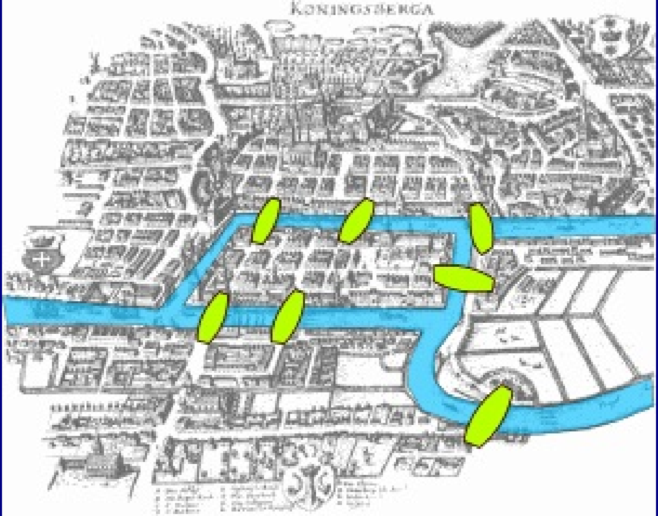
Only two vertices with an odd number of edges are possible for a solution to exist.

The Bridges of Königsberg

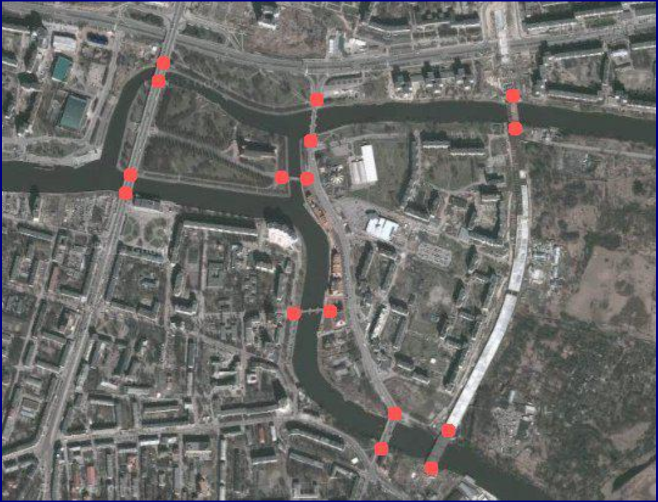


Exercise: Draw the diagram with *A*, *B*, *C* and *D* arranged clockwise at the corners of a square.

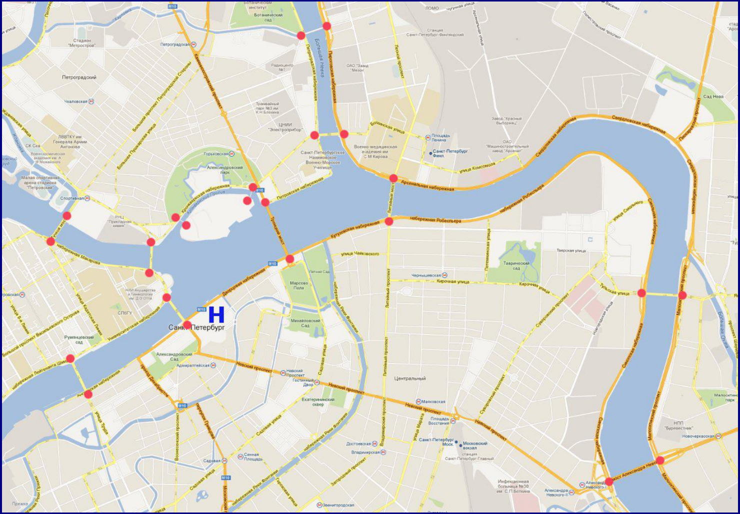
The Bridges of Königsberg



Königsberg Today



The Bridges of St Petersburg



The Bridges of St Petersburg

Euler spend much of his life in St Petersburg, a city with many rivers, canals and bridges.

Did he think about another problem like the Königsberg Bridges problem while there?

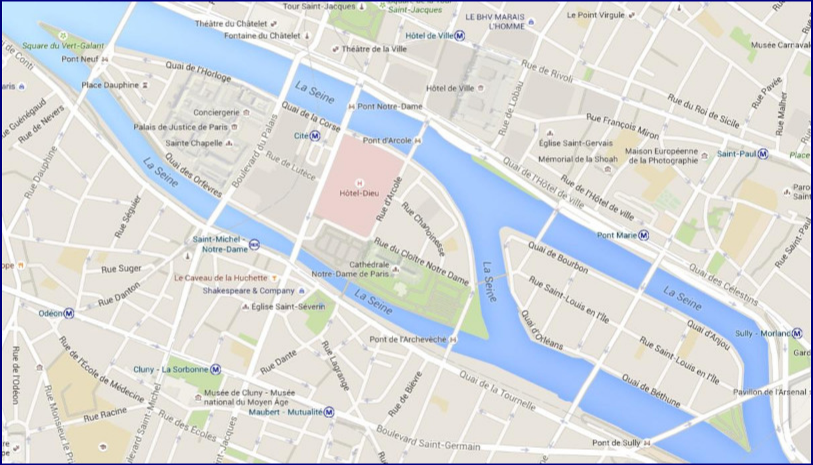
The map of central St Petersburg has twelve bridges.

An Euler cycle is a route that crosses all bridges exactly once and returns to the starting point?

Is there an Euler cycle starting at the Hermitage (marked "H" on the map)?

The Bridges of Paris

Cue romantic music



The Bridges of Paris

In central Paris, two small islands, Île de la Cité and Île Saint-Louis, are linked to the Left and Right Banks of the Seine and to each other.

The number of bridges for each land-mass are:

- Left Bank: 7 bridges
- Right Bank: 7 bridges
- Île de la Cité: 10 bridges
- Île Saint-Louis: 6 bridges

The total is 30. How many bridges are there?

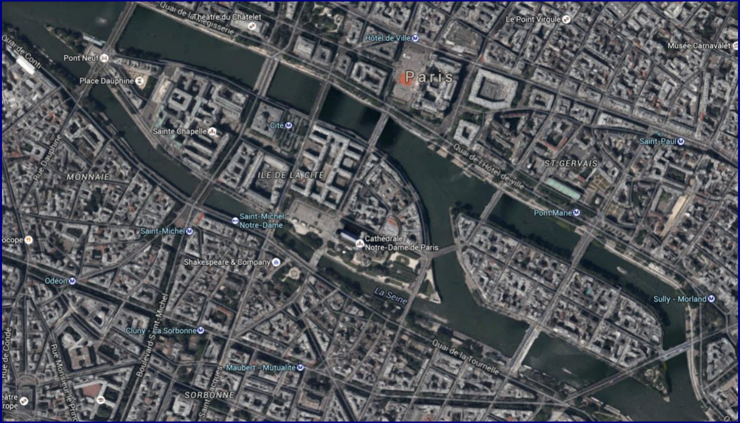
The Bridges of Paris

The Bridges of Paris

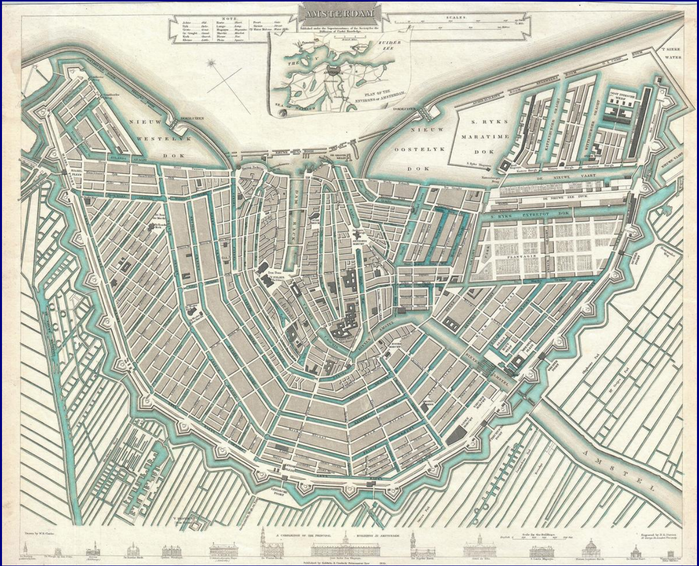
1. Starting from Saint-Michel on the Left Bank, walk continuously so as to cross each bridge once.
2. Start at Saint-Michel but find a closed route that ends back at the starting point.
3. Start at Notre-Dame Cathedral, on Île de la Cité, and cross each bridge exactly once.
4. Find a closed route that crosses each bridge once and arrives back at Notre-Dame.

Try these puzzles yourself. Use logic, not brute force!

The Bridges of Paris



The Bridges of Amsterdam



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Seven Bridges of Königsberg

From Wikipedia, the free encyclopedia

Coordinates: 54°42′12″N 20°30′56″E

This article is about an abstract problem. For the historical group of bridges in the city once known as Königsberg, and those of them that still exist, see § Present state of the bridges.

This article **needs additional citations for verification**. Please help improve this article by adding citations to reliable sources. Unsourced material may be challenged and removed. *(July 2015) (Learn how and when to remove this template message)*

The **Seven Bridges of Königsberg** is a historically notable problem in mathematics. Its negative resolution by Leonhard Euler in 1736 laid the foundations of graph theory and prefigured the idea of topology.[1]

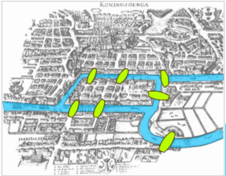
The city of Königsberg in Prussia (now Kaliningrad, Russia) was set on both sides of the Pregel River, and included two large islands which were connected to each other, or to the two mainland portions of the city, by seven bridges. The problem was to devise a walk through the city that would cross each of those bridges once and only once.

By way of specifying the logical task unambiguously, solutions involving either

1. reaching an island or mainland bank other than via one of the bridges, or
2. accessing any bridge without crossing to its other end

are explicitly unacceptable.

Euler proved that the problem has no solution. The difficulty he faced was the development of a suitable technique of analysis, and of subsequent tests that established this assertion with mathematical rigor.



Map of Königsberg in Euler's time showing the actual layout of the seven bridges, highlighting the river Pregel and the bridges

Outline

Volume of a Sphere

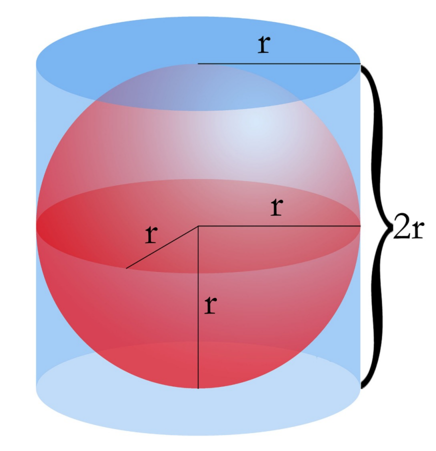


Figure: Archimedes found a formula for V_{SPHERE}

Who First Proved that C/D is Constant?

For every circle, the distance around it is just over three times the distance across it.

This has been "common knowledge" since the earliest times.

But mathematicians don't trust common knowledge.

They demand proof.

Who was first to prove that the ratio of circumference C to diameter D has the same value for all circles?

What about Euclid?

You might expect to find a proof in Euclid's *Elements of Geometry*. But Euclid couldn't prove it.

Euclid's Prop. XII.2 says the areas of circles are to one another as the squares of their diameters:

$$\frac{A_1}{D_1^2} = \frac{A_2}{D_2^2} \,.$$

We would expect to find an analogous theorem: The circumferences of circles vary as their diameters:

$$\frac{C_1}{D_1} = \frac{C_2}{D_2}$$

but we do not find this anywhere in Euclid.

Archimedes Rules OK!

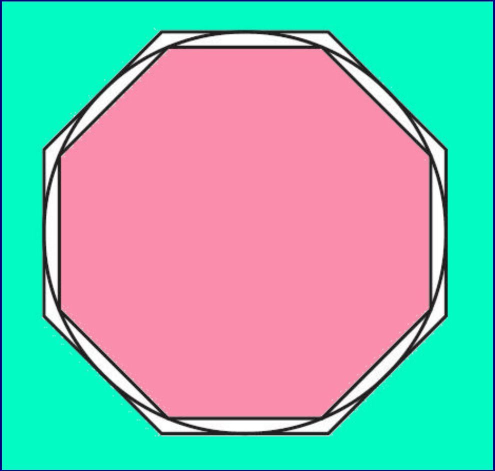
It required the genius of Archimedes
to prove that C/D is the same for all circles.

He needed axioms beyond those of Euclid.

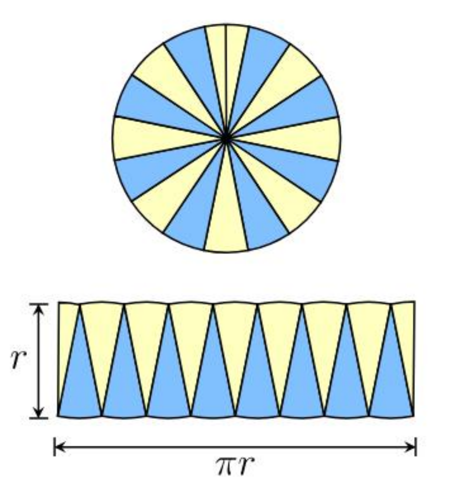
In his work *Measurement of a Circle,*
Archimedes found the area of a circle.

It is equal to the area of a right-angled triangle
with one leg equal to R and the other equal to C:

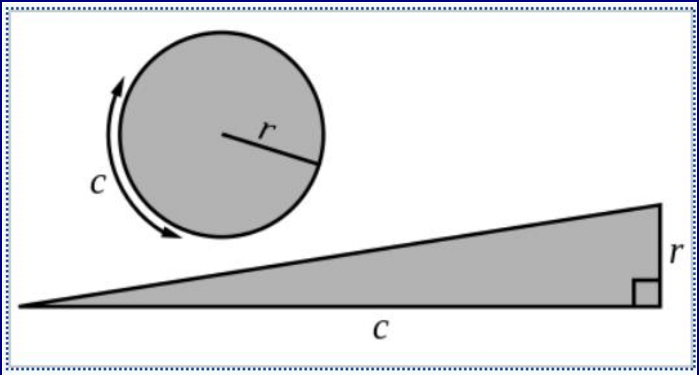
$$A = \tfrac{1}{2} R C \,.$$



Archimedes determined π accurately by considering polygons within and around a circle.



He determined the area of a circle by slicing it up into small triangles.



“Unzipping” the circle,
Archimedes obtained a triangle.

Lengths and Areas both involve π

Archimedes' theorem, together with Euclid's Proposition XII.2, implies that

$$\frac{C}{D} = \pi$$

is the same for every circle.

It also follows that the area constant is also π:

$$\frac{A}{R^2} = \frac{C}{2R} = \frac{C}{D} = \pi\,.$$

Sphere+Cone=Cylinder

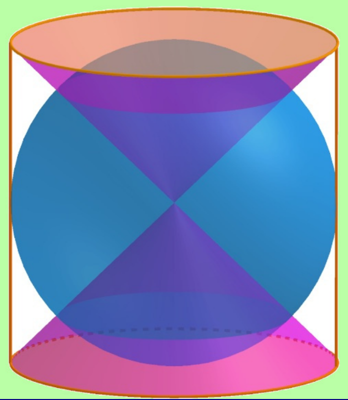


Figure: Volume: Sphere plus Cone equals Cylinder

On the Sphere and Cylinder

One of the most remarkable and important mathematical results obtained by Archimedes was the formula for the volume of a sphere.

Archimedes used a technique of sub-dividing the volume into slices and adding up, or integrating, the volumes of the slices.

This was essentially an application of the integral calculus formulated by Newton and Leibniz.

On the Sphere and Cylinder

Archimedes considered three volumes, a cylinder, cone and sphere, all on bases with the same area.

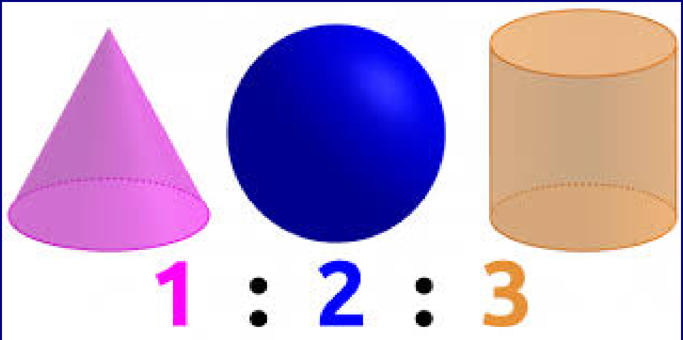


Figure: Cone, sphere and cylinder on the same base.

On the Sphere and Cylinder

Archimedes showed that the three volumes are in the ratio $1 : 2 : 3$.

Thus, in particular, the volume of the sphere is two thirds of the volume of the cylinder.

If we 'rearrange' the volume of the cone, things become much clearer:

We replace the cone by two cones, each of height r.

On the Sphere and Cylinder

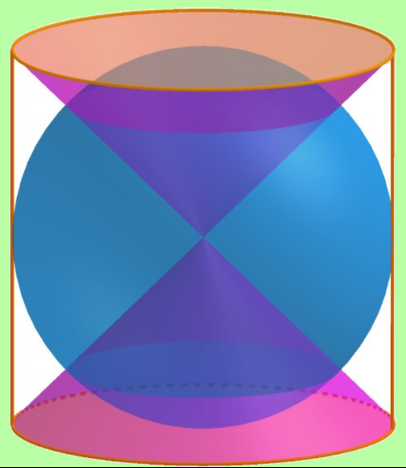
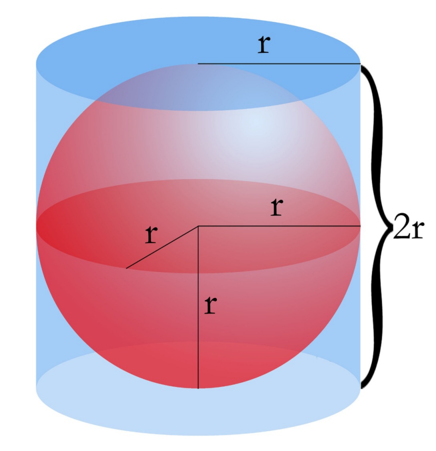


Figure: Cone, sphere and cylinder on the same base.

On the Sphere and Cylinder



This result was carved on Archimedes' tomb.

Archimedes' Tomb as it appears today

Addendum: *On the Sphere and Cylinder*

We let z denote the vertical coordinate,
and Δz be a small increment of height.

The cross-sections of the cone and sphere are

$$\begin{aligned} \Delta V_{\mathrm{CON}} &= \pi z^2 \Delta z \\ \Delta V_{\mathrm{SPH}} &= \pi(\sqrt{r^2 - z^2})^2 \Delta z = \pi(r^2 - z^2)\Delta z\,. \end{aligned}$$

Add to get the cross-sectional area of the cylinder:

$$\Delta V_{\mathrm{CON}} + \Delta V_{\mathrm{SPH}} = \Delta V_{\mathrm{CYL}} = \pi r^2 \Delta z\,,$$

This does not vary with height z.
It is the same as for the cylinder.

Addendum: *On the Sphere and Cylinder*

Adding up the volumes of all slices:

$$\Delta V_{\mathrm{CON}} + \Delta V_{\mathrm{SPH}} = \Delta V_{\mathrm{CYL}} = \pi r^2 H = 2\pi r^3 \,.$$

It is not quite so simple to show that

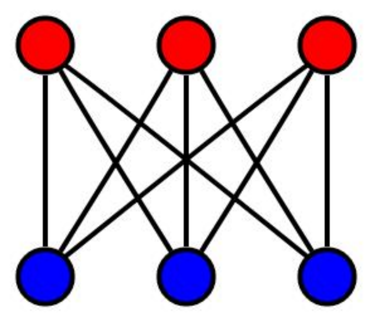
$$\begin{aligned} \Delta V_{\mathrm{CON}} &= \tfrac{1}{3}\Delta V_{\mathrm{CYL}} = \tfrac{1}{3}\pi r^2 H = \tfrac{2}{3}\pi r^3 \\ \Delta V_{\mathrm{SPH}} &= \tfrac{2}{3}\Delta V_{\mathrm{CYL}} = \tfrac{2}{3}\pi r^2 H = \tfrac{4}{3}\pi r^3 \,. \end{aligned}$$

However, this was well within the capability of the brilliant mathematician Archimedes.

Outline

Three Utilities Problem: Abstract

Is the complete 3×3 bipartite graph $K_{3,3}$ planar?



Three Utilities Problem: Abstract

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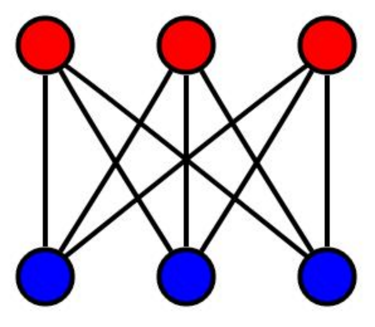


This is an abstract, jargon-filled question in topological graph theory.

We look at a simple, concrete version.

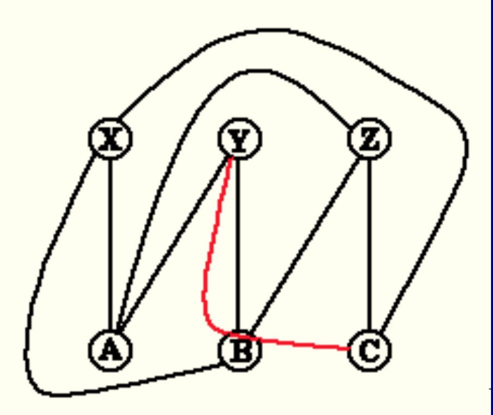
Three Utilities Problem: Concrete

We have to connect 3 utilities to 3 houses.

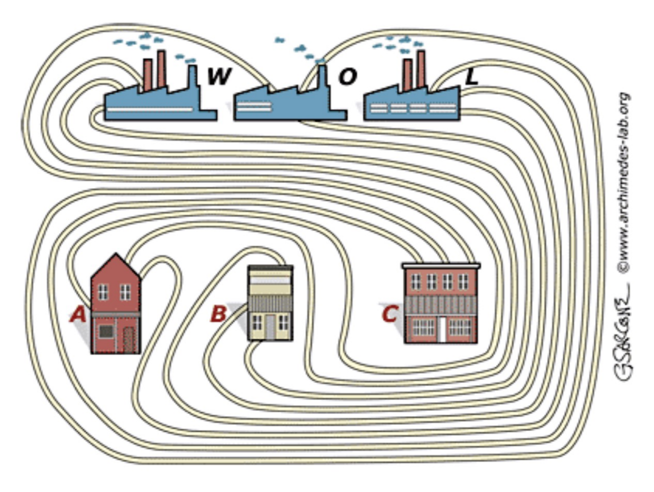
- Electricity
- Water
- Gas

The lines must not cross.

Three Utilities Problem: Have a Go

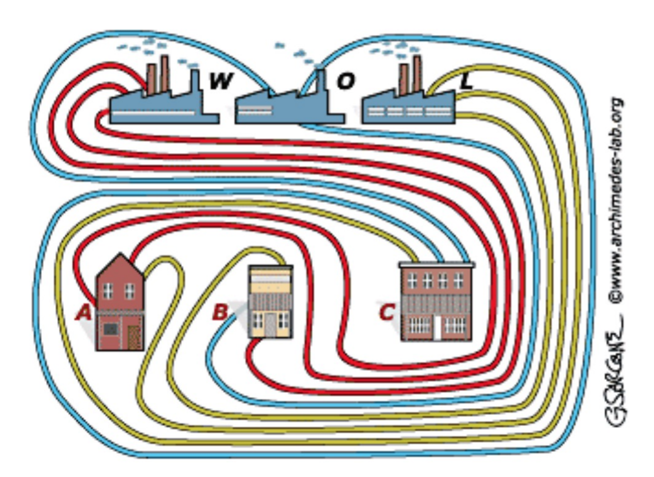


Three Utilities Problem: Solution!



http://www.archimedes-lab.org/How_to_Solve/Water_gas.html

Three Utilities Problem: No Solution!



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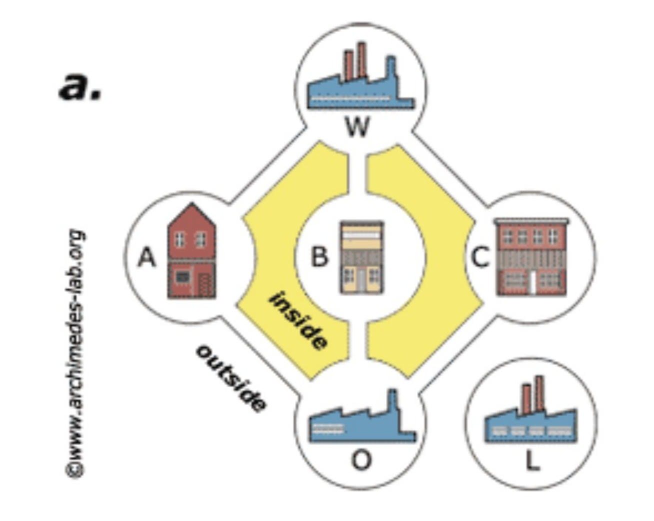
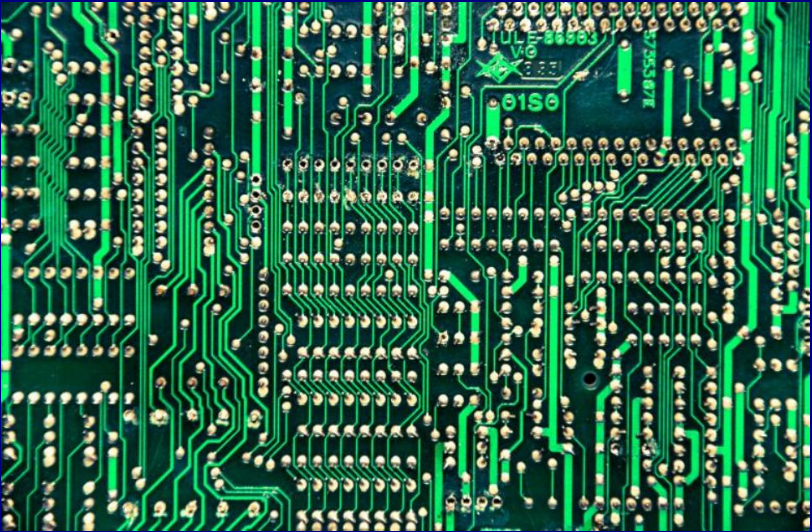
Three Utilities Problem

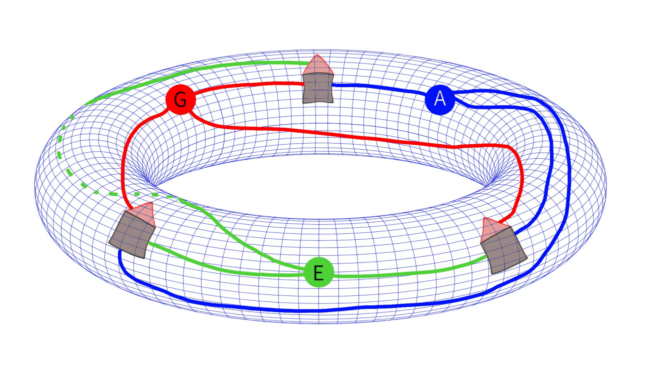


Three Utilities Problem: Application



Three Utilities Problem for Mugs

Three Utilities Problem on a Torus

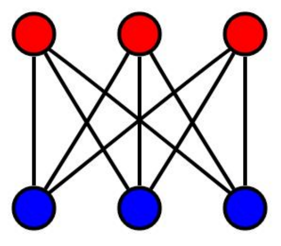


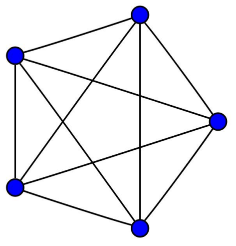
$K_{3,3}$ **is a toroidal graph.**

Vi Hart: `https://www.youtube.com/watch?v=CruQylWSfoU&feature=youtu.be&t=9`

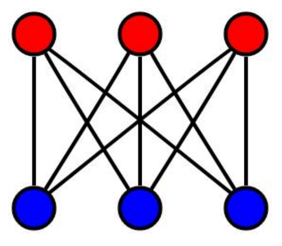
Three Utilities: Kuratowski's Theorem

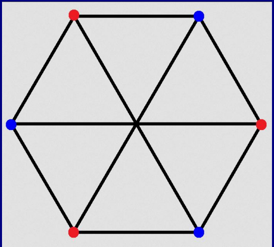
If a graph contains $K_{3,3}$ or K_5 as a sub-graph, it is non-planar. If it does not contain either, it is planar.





Three Utilities: Equivalent Graphs



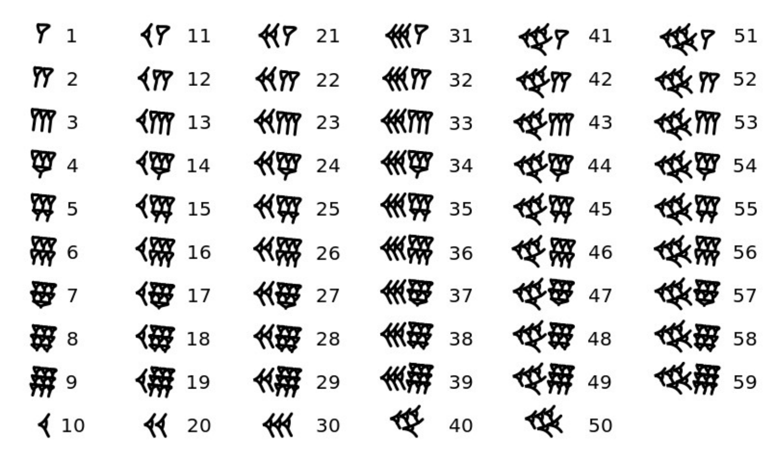


The two forms shown are equivalent.

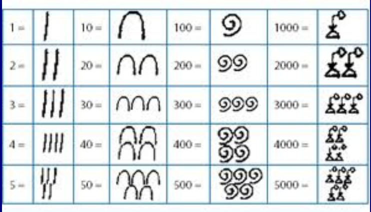
There are crossings in both.

Outline

Babylonian Numerals



Ancient Egyptian Numerals



Ancient Hebrew and Greek Numerals

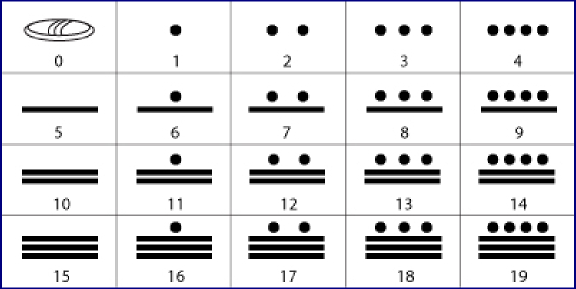
8	7	6	5	4	3	2	1
ח	ז	ו	ה	ד	ג	ב	א
Chet	Zayin	Vav	Hey	Dalet	Gimmel	Bet	Aleph

70	60	50	40	30	20	10	9
ע	ס	נ	מ	ל	כ	י	ט
Ayin	Samekh	Nun	Mem	Lamed	Kaf	Yod	Tet

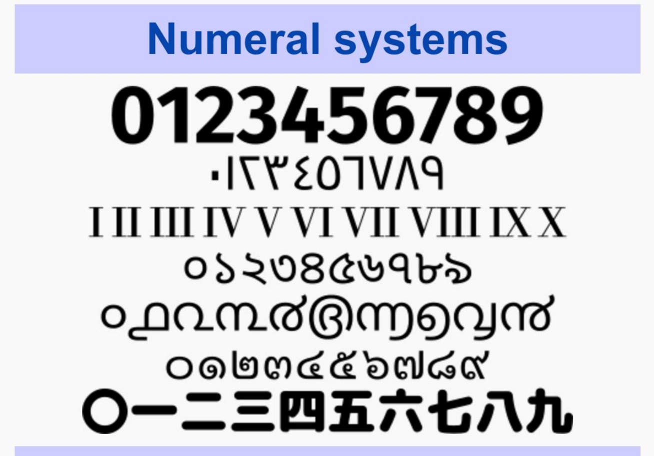
1	α	alpha	10	ι	iota	100	ρ	rho
2	β	beta	20	κ	kappa	200	σ	sigma
3	γ	gamma	30	λ	lambda	300	τ	tau
4	δ	delta	40	μ	mu	400	υ	upsilon
5	ϵ	epsilon	50	ν	nu	500	ϕ	phi
6	ϛ	vau*	60	ξ	xi	600	χ	chi
7	ζ	zeta	70	o	omicron	700	ψ	psi
8	η	eta	80	π	pi	800	ω	omega
9	θ	theta	90	ϟ	koppa*	900	ϡ	sampi

*vau, koppa, and sampi are obsolete characters

Mayan Numerals



Various Numeral Systems



Wikipedia: Hindu-Arabic Numeral System

Roman Numerals

I	1	XXI	21	XLI	41
II	2	XXII	22	XLII	42
III	3	XXIII	23	XLIII	43
IV	4	XXIV	24	XLIV	44
V	5	XXV	25	XLV	45
VI	6	XXVI	26	XLVI	46
VII	7	XXVII	27	XLVII	47
VIII	8	XXVIII	28	XLVIII	48
IX	9	XXIX	29	XLIX	49
X	10	XXX	30	L	50
XI	11	XXXI	31	LI	51
XII	12	XXXII	32	LII	52
XIII	13	XXXIII	33	LIII	53
XIV	14	XXXIV	34	LIV	54
XV	15	XXXV	35	LV	55
XVI	16	XXXVI	36	LVI	56
XVII	17	XXXVII	37	LVII	57
XVIII	18	XXXVIII	38	LVIII	58
XIX	19	XXXIX	39	LIX	59
XX	20	XL	40	LX	60

In order: $M\ D\ C\ L\ X\ V\ I\ = 1666$

How to Multiply Roman Numbers

Table: Multiplication Table for Roman Numbers.

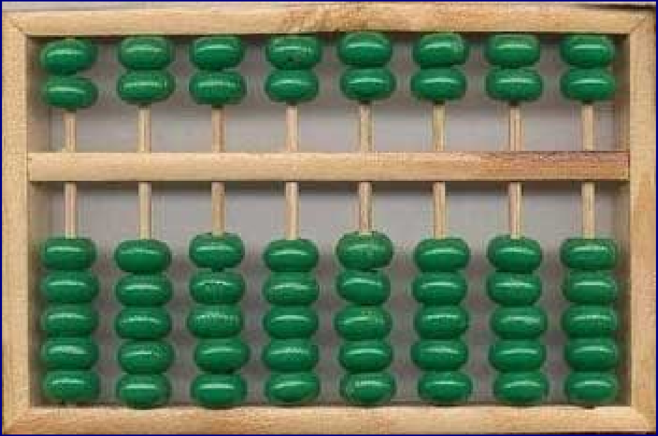
	I	**V**	**X**	**L**	**C**	**D**	**M**
I	*I*	*V*	*X*	*L*	*C*	*D*	*M*
V	*V*	*XXV*	*L*	*CCL*	*D*	*MMD*	$\overline{V}$
X	*X*	*L*	*C*	*D*	*M*	$\overline{V}$	$\overline{X}$
L	*L*	*CCL*	*D*	*MMD*	$\overline{V}$	$\overline{XXV}$	$\overline{L}$
C	*C*	*D*	*M*	$\overline{V}$	$\overline{X}$	$\overline{L}$	$\overline{C}$
D	*D*	*MMD*	$\overline{V}$	$\overline{XXV}$	$\overline{L}$	$\overline{CCL}$	$\overline{D}$
M	*M*	$\overline{V}$	$\overline{X}$	$\overline{L}$	$\overline{C}$	$\overline{D}$	$\overline{M}$

A Roman Abacus

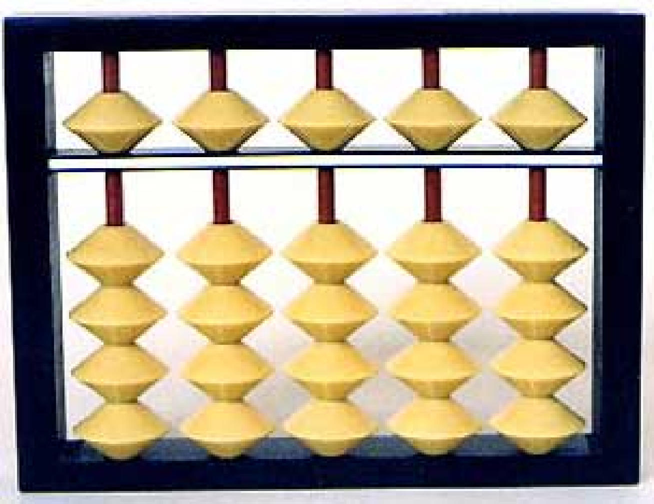
Replica of a Roman abacus from 1st century AD.

Abacus is a Latin word, which comes from the Greek $\alpha\beta\alpha\kappa\alpha\varsigma$ (board or table).

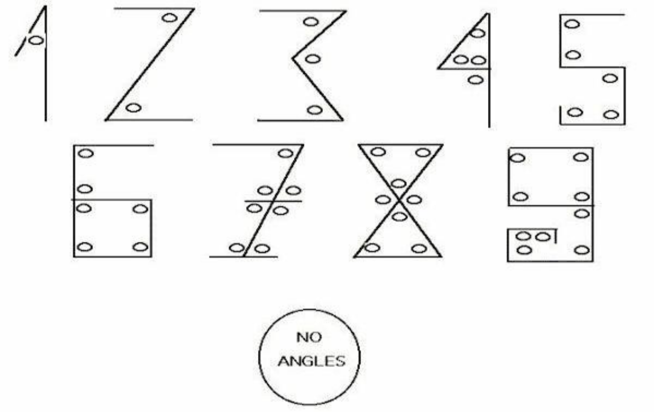
A Chinese Abacus: *Suan Pan*



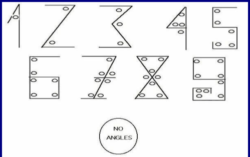
A Japanese Abacus: *Soroban*



A Different Angle on Numerals

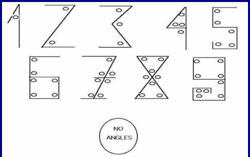


Delightful theory. Almost certainly wrong.



Arguments "for"

1. It is a very simple idea
2. It links symbols to numerical values



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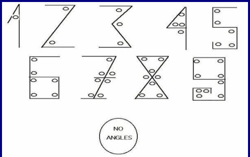
Arguments “against”

1. Number forms modified to fit model
2. Complete lack of historical evidence.



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2. Complete lack of historical evidence.

The great tragedy of science —

the slaying of a beautiful hypothesis by an ugly fact (T H Huxley)

Outline

Estimating π with Series

There are many ways of estimating π.

For example, we can sum up the Basel Series:

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots .$$

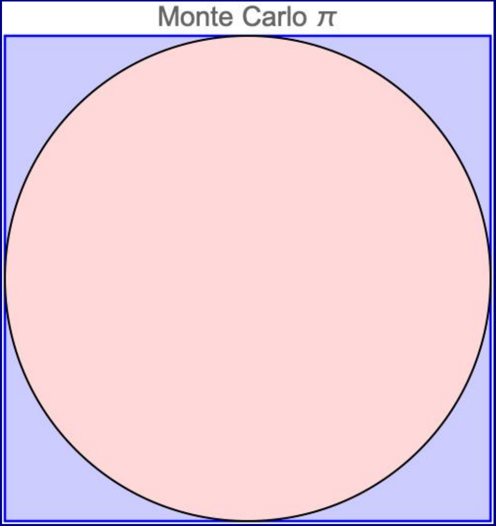
Another way is with the Gregory-Leibniz series, discovered much earlier by Madhava (c. 1340–1425).

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots .$$

We have already seen Archimedes' method.

We now give a completely different approach.

Estimating π with Probability



Estimating π with Probability

Area of Square: 4

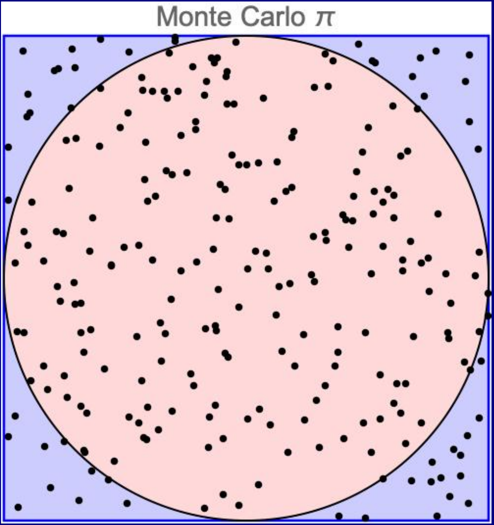
Area of Circle: π

Probability point is within circle: $\frac{\pi}{4}$

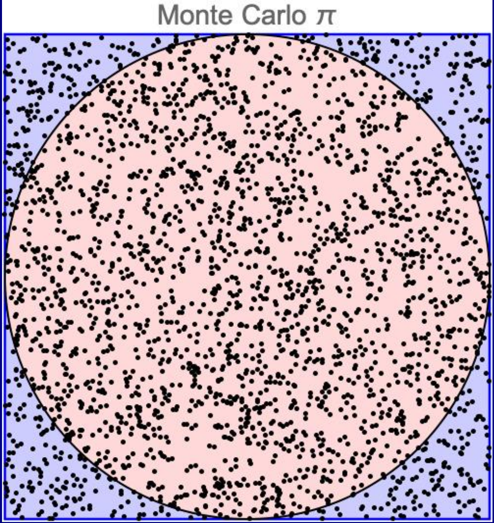
Thus, the following ratio should approach π**:**

$$4 \times \frac{\text{Number of points within Circle}}{\text{Number of points within Square}} \to \pi \, .$$

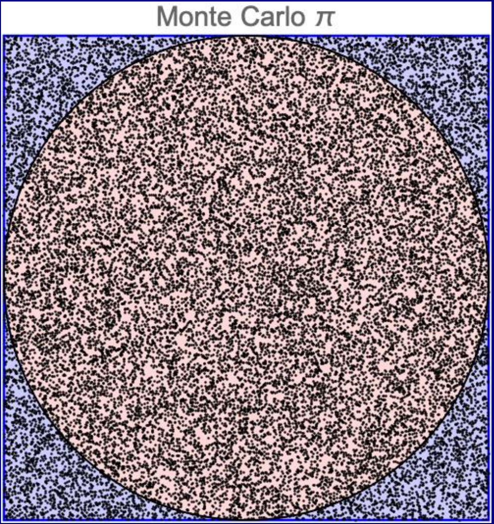
Estimating π with $n = 250$



Estimating π with $n = 2500$



Estimating π with $n = 25000$



Numerical Results

Table: Estimates of π

250	3.23506...
2500	3.15407...
25000	3.13177...
$\vdots$	$\vdots$
∞	3.14159...

Comment on uses of Monte Carlo method.

Outline

A Hierarchy of Numbers

We will introduce a **hierarchy of numbers.**

Each set is contained in the next one.

They are like a set of nested Russian Dolls:

Matryoshka

The Natural Numbers $\mathbb{N}$

The counting numbers were the first to emerge:

$$1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \ \ldots$$

They are also called the Natural Numbers.

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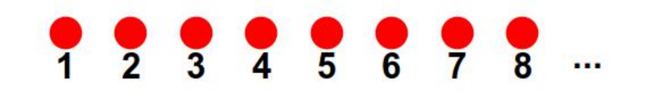
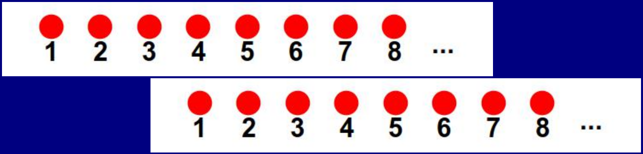


We can arange the natural numbers in a list.

This list is like a toy computer.

A Primitive Sliderule



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The set of natural numbers is denoted $\mathbb{N}$.

If n is a natural number, we write $n \in \mathbb{N}$.

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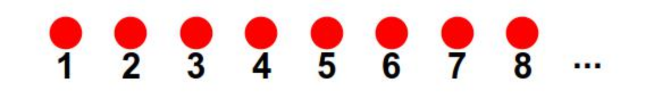
Natural numbers can be added: $4 + 2 = 6 \in \mathbb{N}$



But not always subtracted: $4 - 6 = -2 \notin \mathbb{N}$.

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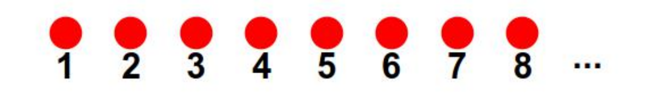

But not always subtracted: $4 - 6 = -2 \notin \mathbb{N}$.

To allow for subtraction we have to extend $\mathbb{N}$.

The Integers $\mathbb{Z}$

We extend the set of counting numbers
by including the negative whole numbers:

$$\ldots \quad -3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad \ldots$$

The whole numbers are also called the **Integers**.

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. . . -3 -2 -1 0 1 2 3 4 . . .

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The set of integers is denoted $\mathbb{Z}$.

If k is an integer, we write $k \in \mathbb{Z}$.

Clearly,

$$\mathbb{N} \subset \mathbb{Z}$$

Integers can be added and subtracted.

They can also multiplied:

$$6 \times 4 = 24 \in \mathbb{Z}\,.$$

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To allow for division we have to extend $\mathbb{Z}$.

The Rational Numbers $\mathbb{Q}$

We extend the integers by including fractions:

$$r = \frac{p}{q} \qquad \text{where } p \text{ and } q \text{ are integers.}$$

These rational numbers are ratios of integers.

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The set of rational numbers is denoted $\mathbb{Q}$.

If r is a rational number, we write $r \in \mathbb{Q}$.

Clearly,

$$\mathbb{Z} \subset \mathbb{Q}$$

With the Rational Numbers, we can:

Add, Subtract, Multiply and Divide

That is, for any $p \in \mathbb{Q}$ and $q \in \mathbb{Q}$, all of

$$\{\ p+q \qquad p-q \qquad p\times q \qquad p\div q\ \}$$

are rational numbers.

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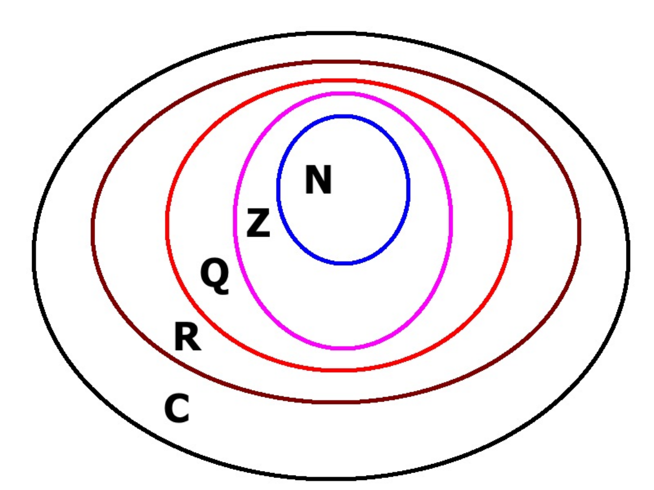
We say that $\mathbb{Q}$ is **closed under addition, subtraction, multiplication and division.**

But we are not yet finished. $\mathbb{R}$ is yet to come.

The Hierarchy of Numbers



$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$$

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Thank you