AweSums

Marvels and Mysteries of Mathematics

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LECTURE 3

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Evening Course, UCD, Autumn 2020

Outline

Outline

Meaning and Content of Mathematics

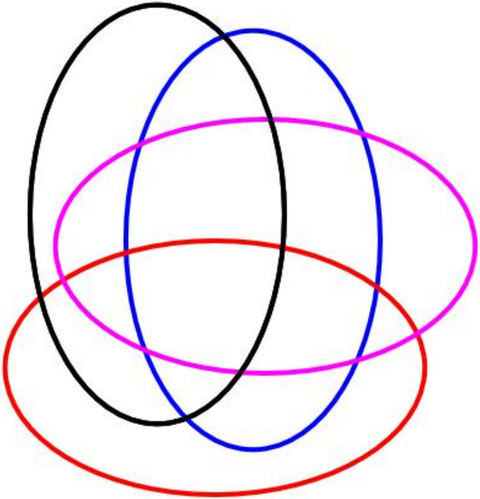
The word Mathematics comes from Greek $\mu\alpha\theta\eta\mu\alpha$ (máthéma), meaning "knowledge" or "study" or "learning".

It is the study of topics such as

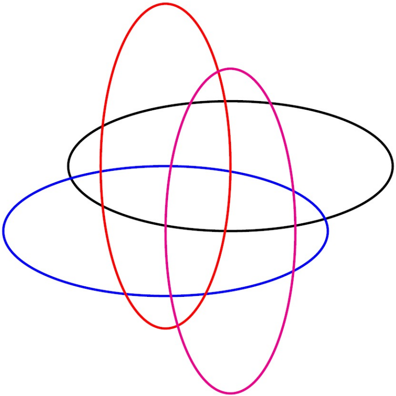
- Quantity (numbers)
- Structure (patterns)
- Space (geometry)
- Change (analysis).

Outline

Venn Diagram for 4 Sets

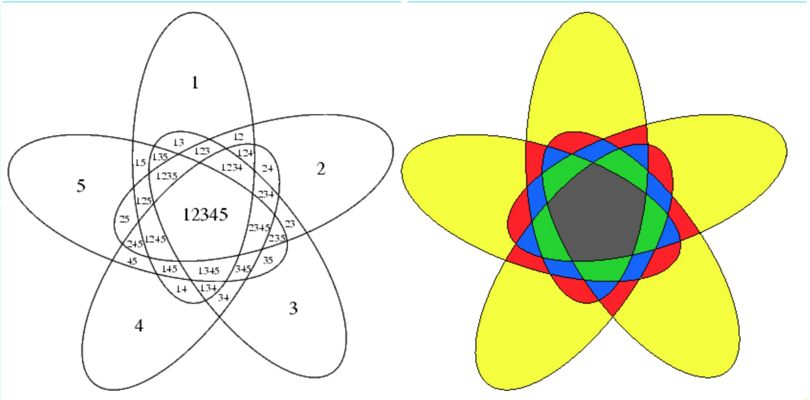


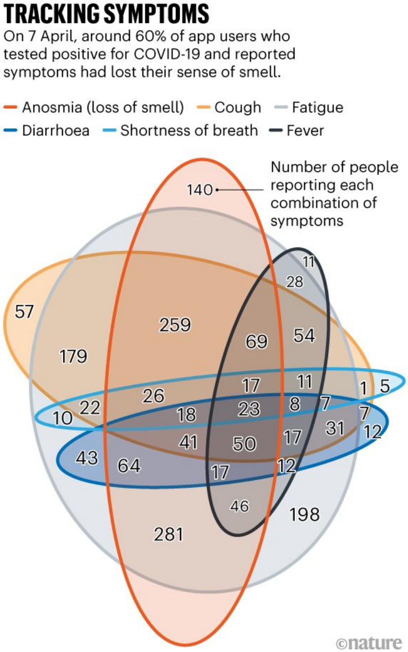
Venn-4 Diagram: Symmetric



Challenge: Construct a symmetric Venn-4 diagram.

Venn Diagram for 5 Sets

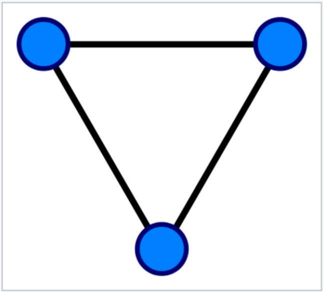


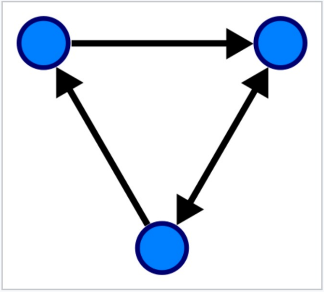


From Science journal *Nature.*

A diagram that is very poorly designed and difficult to understand.

Mathematial Graphs

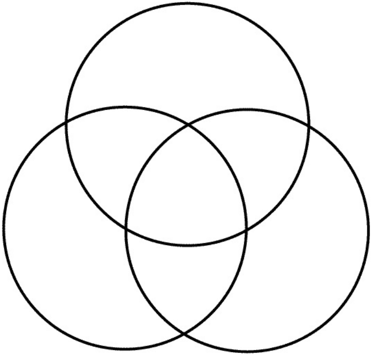


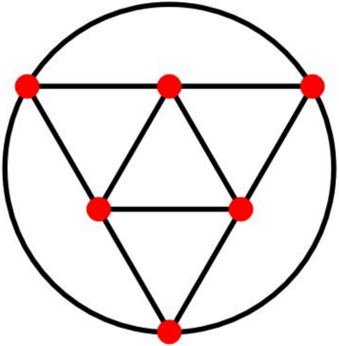


Undirected Graph **Directed Graph**

A *graph* is a set of vertices joined by edges.

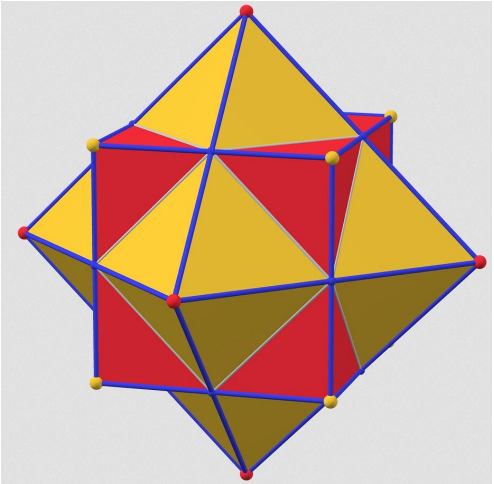
Venn Diagram as a Graph



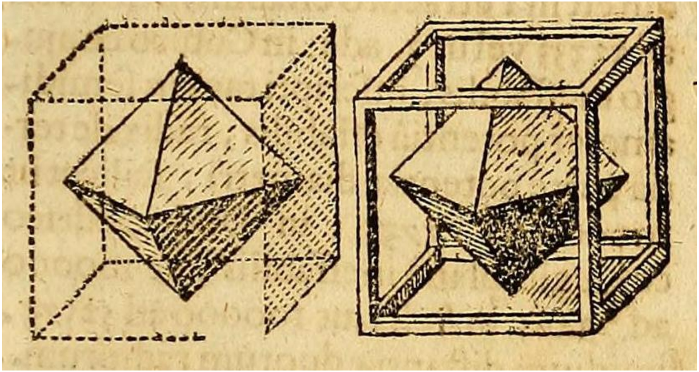


Graph is equivalent to an octahedron

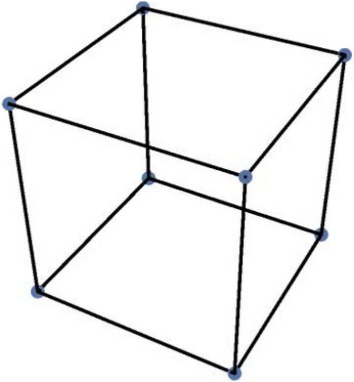
Cube and Octahedron are Duals

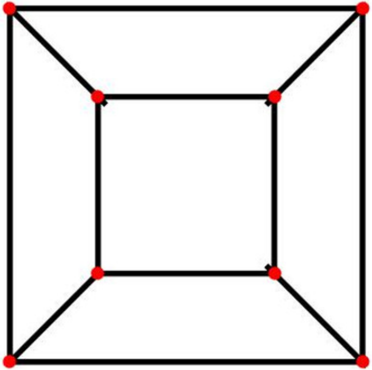


From Kepler’s *Harmonices Mundi*

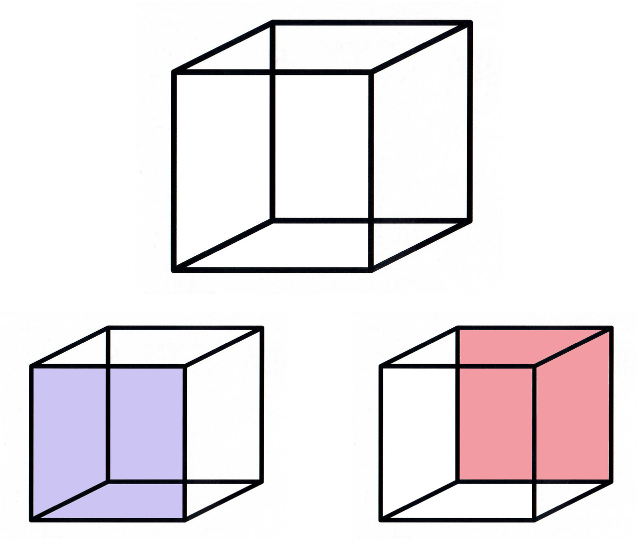


Venn3 Dual as a Cube





The Necker Cube



See blog post

Venn Again's Awake

on my mathematical blog `thatsmaths.com`

Counting Infinite Sets

There is no Largest Number

Children often express bemusement at the idea that there is no largest number.

Given any number, 1 can be added to it to give a larger number.

But the implication that there is no limit to this process is perplexing.

The concept of infinity has exercised the greatest minds throughout the history of human thought.

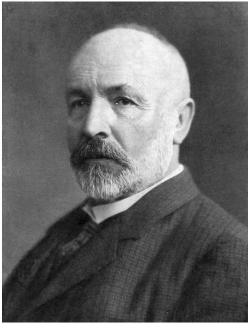
Degrees of Infinity

In the late 19th century, Georg Cantor showed that there are different degrees of infinity.

In fact, there is an infinite hierarchy of infinities.

Cantor brought into prominence several paradoxical results that had a profound impact on the development of logic and of mathematics.

Georg Cantor (1845–1918)

Cantor discovered many remarkable properties of infinite sets.

Cardinality

Finite Sets have a finite number of elements.

Example: The Counties of Ireland form a finite set.

$$\text{Counties} = \{\text{Antrim, Armagh, } \ldots, \text{ Wexford, Wicklow}\}$$

For a finite set A, the *cardinality* of A is:

The number of elements in A

One-to-one Correspondence

A particular number, say 5, is associated with all the sets having five elements.

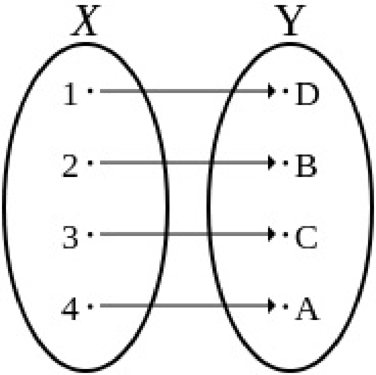
For any two of these sets, we can find a 1-to-1 correspondence between the elements of the two sets.

The number 5 is called the cardinality of these sets.

Generalizing this:

Any two sets are the same size (or cardinality) if there is a 1-to-1 correspondence between them.

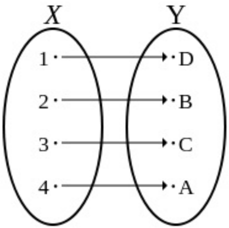
One-to-one Correspondence



Equality of Set Size: 1-1 Correspondence

How do we show that two sets are the same size?

For finite sets, this is straightforward counting.



For infinite sets, we must find a 1-1 correspondence.

Cardinality

The number of elements in a set is called the cardinality of the set.

Cardinality of a set A is written in various ways:

$$|\mathbf{A}| \qquad \|\mathbf{A}\| \qquad \mathrm{card}(\mathbf{A}) \qquad \#(A)$$

For example

$$\#\{\textbf{Irish Counties}\} = 32$$

The Empty Set

We call the set with *no elements* the empty set.

It is denoted by a special symbol

$$\varnothing = \{ \ \}$$

Clearly

$$\#\{ \ \} = 0 \,.$$

We could have a philosophical discussion about the empty set. Is it related to a perfect vacuum?

The Greeks regarded the vacuum as an impossibility.

The Natural Numbers $\mathbb{N}$

The *counting numbers* (positive whole numbers) are

$$1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \ \ldots .$$

They are also called the *Natural Numbers*.

The set of natural numbers is denoted $\mathbb{N}$.

This is our first infinite set.

We use a special symbol to denote its cardinality:

$$\#(\mathbb{N}) = \aleph_0$$

$$\aleph_0$$

The Power Set

For any set, we can form a new one, the Power Set.

The Power Set is the *set of all subsets of* A.

Suppose the set A has just two elements:

$$\mathbf{A} = \{3, 7\}$$

Here are the subsets of A:

$$\{\ \} \qquad \{3\} \qquad \{7\} \qquad \{3, 7\}$$

The *power set* is

$$\wp[\mathbf{A}] = \Big\{\{\ \}, \{3\}, \{7\}, \{3, 7\}\Big\}$$

Cantor's Theorem

Cantor's theorem states that,
for any set A, the power set of A has a strictly greater cardinality than A itself:

$$\#[\wp(\mathbf{A})] > \#[A]$$

This holds for both finite and infinite sets.

This means that, *for every cardinal number, there is a greater cardinal number.*

One-to-one Correspondence

Take all the natural numbers,

$$\mathbb{N} = \{1, 2, 3, \ldots\}$$

as one set and all the even numbers

$$\mathbb{E} = \{2, 4, 6, \ldots\}$$

as the other.

By associating each number $n \in \mathbb{N}$ with $2n \in \mathbb{E}$, we have a perfect 1-to-1 correspondence.

By Cantor's argument, the two sets are the same size:

$$\#[\mathbb{N}] = \#[\mathbb{E}]$$

Again,

$$\#[\mathbb{N}] = \#[\mathbb{E}]$$

But this is *paradoxical:* The set of natural numbers contains all the even numbers:

$$\mathbb{E} \subset \mathbb{N}$$

and also all the odd ones.

In an intuitive sense, $\mathbb{N}$ is larger than $\mathbb{E}$.

The same paradoxical result had been deduced by *Galileo* some 250 years earlier.

Cantor carried these ideas much further:

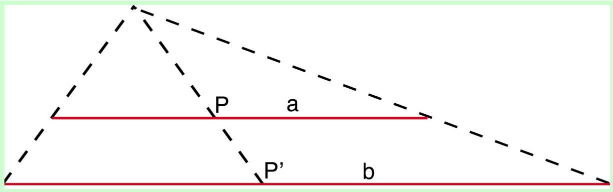
The set of all the real numbers has a degree of infinity, or cardinality, greater than the counting numbers:

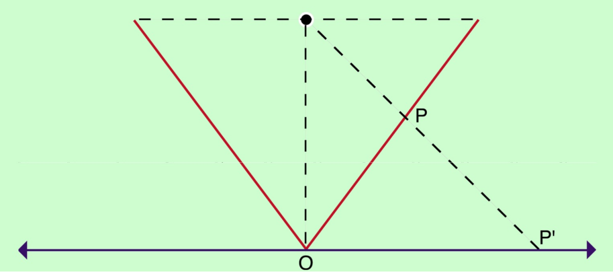
$$\#[\mathbb{R}] > \#[\mathbb{N}]$$

Cantor showed this using an ingenious approach called the diagonal argument.

This is a fascinating technique, but we will not give details here.

How Many Points on a Line?





There is a 1-1 map between $(-1, +1)$ and $\mathbb{R}$.

Review: Infinities Without Limit

For any set A, the power set $\wp$(A) is
the collection of all the subsets of A.

Cantor proved $\wp$(A) has cardinality greater than A.

For finite sets, this is obvious;
for infinite ones, it was startling.

The result is now known as Cantor's Theorem,
and Cantor used his diagonal argument in proving it.

He thus developed an entire hierarchy
of transfinite cardinal numbers.

Outline

The Greek Alphabet

Ελληνικό αλφάβητο

Some Motivation

- Greek letters are used extensively in maths.
- Greek alphabet is the basis of the Roman one.
- Also the basis of the Cyrillic and others.
- A great advantage for touring in Greece.
- You already know several of the letters.
- It is simple to learn in small sections.

Ursa Major

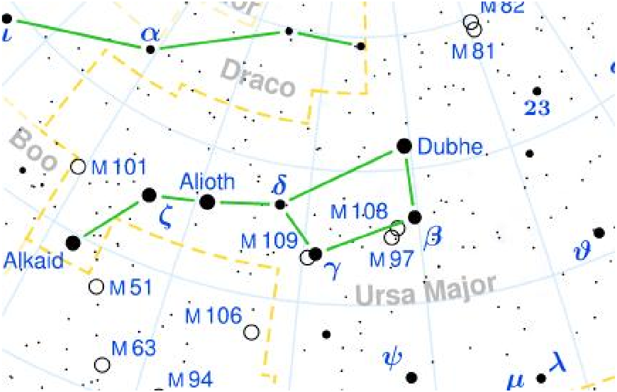


Figure : The Great Bear: Dubhe is α-Ursae Majoris.

Letter	Name	Sound	
		Ancient[5]	Modern[6]
Α α	alpha, άλφα	[a] [aː]	[a]
Β β	beta, βήτα	[b]	[v]
Γ γ	gamma, γάμμα	[g], [ŋ][7]	[ɣ] ~ [ʝ], [ŋ][8] ~ [ɲ][9]
Δ δ	delta, δέλτα	[d]	[ð]
Ε ε	epsilon, έψιλον	[e]	[e]
Ζ ζ	zeta, ζήτα	[zd][A]	[z]
Η η	eta, ήτα	[ɛː]	[i]
Θ θ	theta, θήτα	[tʰ]	[θ]
Ι ι	iota, ιώτα	[i] [iː]	[i], [ʝ],[10] [ɲ][11]
Κ κ	kappa, κάππα	[k]	[k] ~ [c]
Λ λ	lambda, λάμδα	[l]	[l]
Μ μ	mu, μυ	[m]	[m]

Letter	Name	Sound	
		Ancient[5]	Modern[6]
Ν ν	nu, νυ	[n]	[n]
Ξ ξ	xi, ξι	[ks]	[ks]
Ο ο	omicron, όμικρον	[o]	[o]
Π π	pi, πι	[p]	[p]
Ρ ρ	rho, ρώ	[r]	[r]
Σ σ/ς[13]	sigma, σίγμα	[s]	[s]
Τ τ	tau, ταυ	[t]	[t]
Υ υ	upsilon, ύψιλον	[y] [yː]	[i]
Φ φ	phi, φι	[pʰ]	[f]
Χ χ	chi, χι	[kʰ]	[x] ~ [ç]
Ψ ψ	psi, ψι	[ps]	[ps]
Ω ω	omega, ωμέγα	[ɔː]	[o]

Figure : The Greek Alphabet (from Wikipedia)

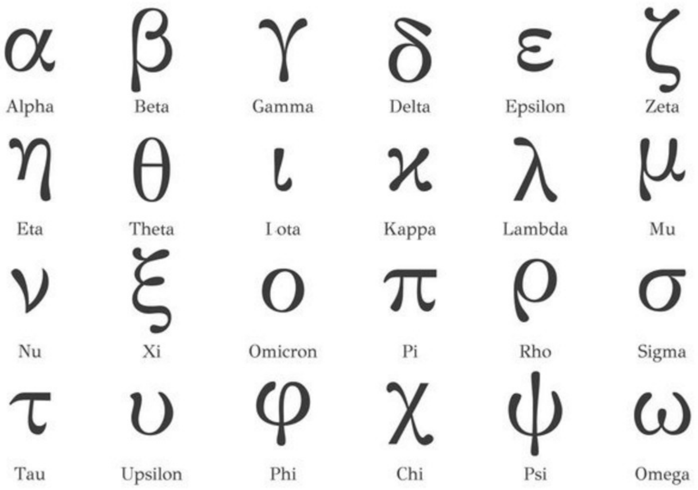


Figure : 24 beautiful letters

The First Six Letters

The first group of six letters.

$$\alpha \qquad \beta \qquad \gamma \qquad \delta \qquad \epsilon \qquad \zeta$$

$$A \qquad B \qquad \Gamma \qquad \Delta \qquad E \qquad Z$$

The Next Six Letters

The second group of six letters.

η	θ	ι	κ	λ	μ
H	Θ	I	K	Λ	M

The Next Six Letters

The third group of six letters.

$$\nu \quad \xi \quad o \quad \pi \quad \rho \quad \sigma$$

$$\mathrm{N} \quad \Xi \quad \mathrm{O} \quad \Pi \quad \mathrm{P} \quad \Sigma$$

The Last Six Letters

The final group of six letters.

τ	υ	ϕ	χ	ψ	ω
T	Υ	Φ	X	Ψ	Ω


Αεροδρόμιο
Airport
Ξενοφώντος
Ομόνοια
Μητροπόλεως
Ερμού
Πειραιώς
Β. Σοφίας


Κέντρο
Center
Λιμάνι
Port

Μουσειο Ακρόπολης
Acropolis Museum

Αθήνα
Δελφοί
400M

ΤΑΞΙ

A Few Greek Words (for practice)

$\kappa\lambda\iota\mu\alpha\xi$
$\delta\rho\alpha\mu\alpha$
$\nu\epsilon\kappa\tau\alpha\rho$
$\kappa\omega\lambda o\nu$

$\kappa o\sigma\mu o\varsigma$
$\mu\alpha\theta\eta\mu\alpha$
$\beta\iota\beta\lambda\iota o$
$\iota\delta\epsilon\alpha$

Climax: $\kappa\lambda\iota\mu\alpha\xi$
Drama: $\delta\rho\alpha\mu\alpha$
Nectar: $\nu\epsilon\kappa\tau\alpha\rho$
Colon: $\kappa\omega\lambda o\nu$

Cosmos: $\kappa o\sigma\mu o\varsigma$
Maths: $\mu\alpha\theta\eta\mu\alpha$
Book: $\beta\iota\beta\lambda\iota o$
Idea: $\iota\delta\epsilon\alpha$

A Few Greek Words (for practice)

$\kappa\omega\mu\alpha$
$\psi\upsilon\kappa\eta$
$\kappa\rho\iota\sigma\iota\varsigma$

$\alpha\nu\alpha\theta\epsilon\mu\alpha$
$\alpha\mu\beta\rho o\sigma\iota\alpha$
$\kappa\alpha\tau\alpha\sigma\tau\rho o\phi\eta$

Coma: $\kappa\omega\mu\alpha$
Psyche: $\psi\upsilon\kappa\eta$
Crisis: $\kappa\rho\iota\sigma\iota\varsigma$

Anathema: $\alpha\nu\alpha\theta\epsilon\mu\alpha$
Ambrosia: $\alpha\mu\beta\rho o\sigma\iota\alpha$
Catastrophe: $\kappa\alpha\tau\alpha\sigma\tau\rho o\phi\eta$

Outline

Picturing the Difference of Squares

How do we calculate

$$a^2 - b^2 \,?$$

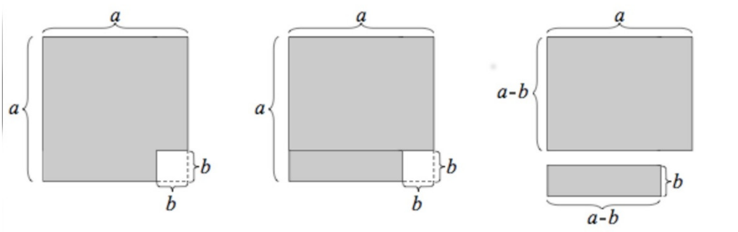
In school we may learn that

$$a^2 - b^2 = (a + b) * (a - b)$$

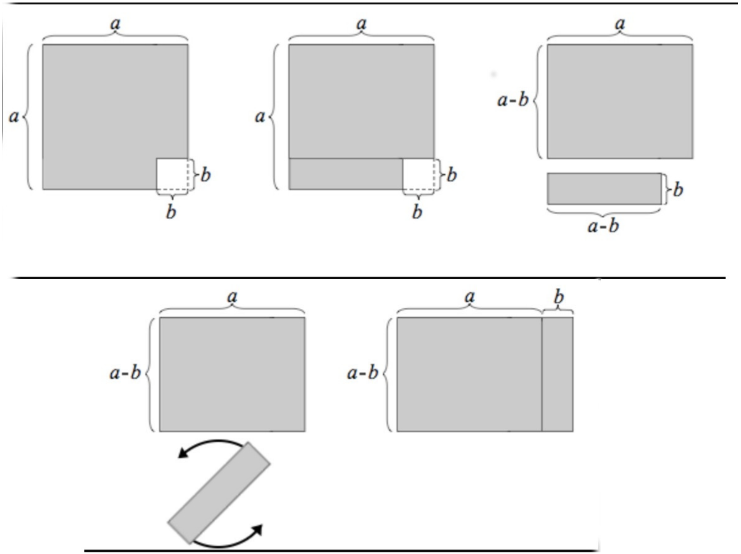
But can we make this understandable?

Yes: Using pictures.

A Pictorial Proof of $a^2 - b^2$



A Pictorial Proof of $a^2 - b^2$



Outline

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Enigmas of Infinity

Zeno of Elea devised several paradoxes involving infinity.

He argued that *one cannot travel from A to B:* to do so, one must first travel half the distance, then half of the remaining half, then half the remainder, and so on.

He concluded that motion is logically impossible.

Zeno was misled by his belief that the sum of finite quantities must grow without limit as more are added.

Enigmas of Infinity

Systematic mathematical study of infinite sets began around 1875 when *Georg Cantor* developed a theory of transfinite numbers.

He reasoned that the method of comparing the sizes of finite sets could be carried over to infinite ones.

If two finite sets, for example the cards in a deck and the weeks in a year, can be matched up one to one they must have the same number of elements.

Bijections

Mathematicians call a 1:1 correspondence a bijection.

Cantor used this approach to compare infinite sets: if there is a bijection between them, two sets are said to be *the same size.*

Cantor built an entire theory of infinity on this idea.

Hilbert's Hotel

We will look at a fantasy devised by David Hilbert.

We could call it a Gedankenexperiment

It was introduced in 1924 in a lecture
Über das Unendliche.

Hilbert's Hotel

Hilbert's Grand Hotel

Leading German mathematician David Hilbert constructed *an amusing metaphor* to illustrate the surprising and counter-intuitive properties of infinity.

He imagined a hotel with an infinite number of rooms.

Even with the hotel full, there is always room to accommodate an extra guest.

Simply move guest 1 to room 2, guest 2 to room 3 and so on, thereby vacating the first room.

Indeed, *an infinite number of new arrivals* could be accommodated: for all rooms n, move the guest in room n to room $2n$, and magically all the odd-numbered room become vacant.

Indeed, a countably infinite number of busses, each with a countably infinite number of passengers, can be accommodated.

Video:
https://www.youtube.com/watch?v=Uj3_KqkI9Zo&t=191s

http://world.mathigon.org/resources/Infinity/Miss_Marple.mp4

Outline

William Rowan Hamilton

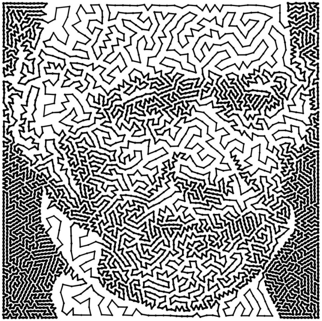
William Rowan Hamilton was Ireland's most renowned mathematician.

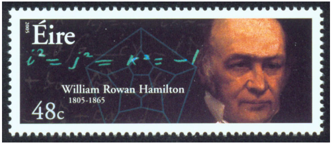
He made fundamental contributions to mathematics and physics.

His discoveries include

- **Least Action Principle**
- **Canonical equations of dynamics**
- **Quaternions**

Conical refraction / Bridge / Stamps





Éire
$i^2 = j^2 = k^2 = -1$
William Rowan Hamilton
1805-1865
48c

Here as he walked by
on the 16th of October 1843
Sir William Rowan Hamilton
in a flash of genius discovered
the fundamental formula for
quaternion multiplication
$i^2 = j^2 = k^2 = ijk = -1$
& cut it on a stone of this bridge

Hamilton's Icosian Game

Hamilton invented a game, called *The Icosian Game.*

It involves finding a path along a graph that visits every vertex and returns to the starting point.

Such a solution is called a Hamiltonian Cycle.

Dodecahedron

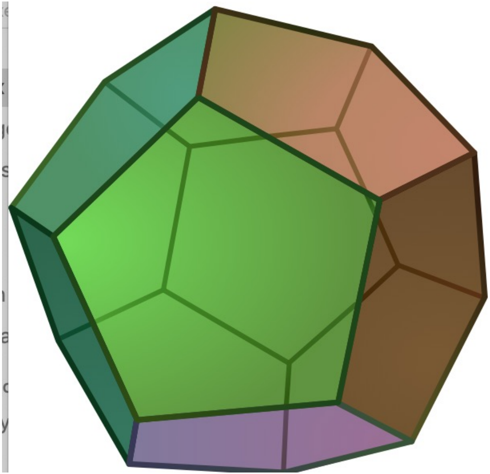


Figure : See Wikipedia page Dodecahedron.

Icosian Grid with Irish Locations

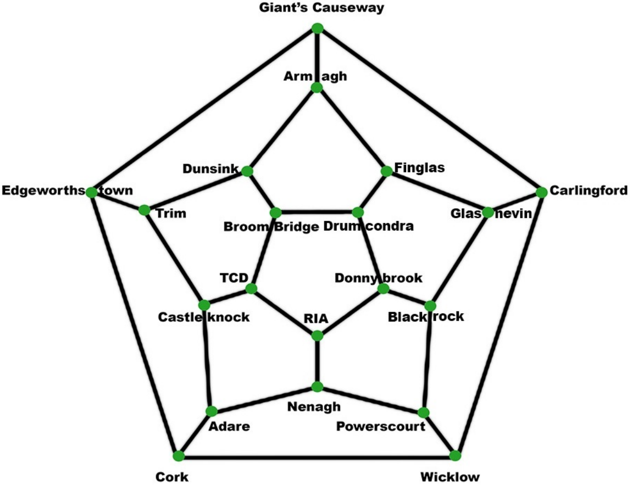


Figure : Credit Colm Mulcahy

Outline

Distraction: A Curious Year, AD 1089

What is so special about the year 1089?

- Palmyra destroyed by an earthquake.
- First Cistercian monastery, Cîteaux Abbey, founded in southern France.
- The Council of Melfi issues decrees against simony and clerical marriage.

Such vital information is obtained from Wikipedia.

Distraction: A Curious Number

Think of a three-digit number, for example 275.

Calculate the difference between this number and the number formed by reversing digits:

$$572 - 275 = 297$$

Now repeat the process, this time adding numbers:

$$297 + 792 = 1089$$

What is so special about the number 1089?

Distraction: A Curious Number

This “trick” nearly always works.

But it can fail in some cases.

Can you find the conditions for success?

See the Wikipedia page “1089 (number)”.

Outline

Topology: a Major Branch of Mathematics

Topology is all about continuity and connectivity.

We will look at a few examples of connectivity.

- A Circle
- A Square
- A Triangle

What makes them different?
What makes them the same?

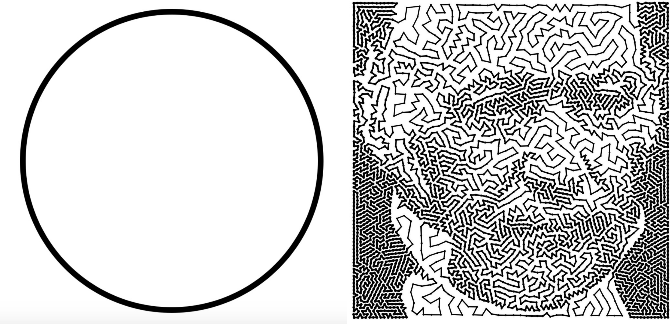


Figure : Topographically equivalent curves in the plane

Jordan Curves are topologically equivalent to a circle.

They are also called simple closed curves and are important for the *Travelling Salesman Problem.*

Topology: a Major Branch of Mathematics

Topology is all about continuity and connectivity.

We will look at a few aspects of Topology.

- **The Bridges of Königsberg**
- **Doughnuts and Coffee-cups**
- **Knots and Links**
- **Nodes and Edges: Graphs**
- **The Möbius Band**

Let us start with the *London Underground Map.*

The London Underground Map

Figure : Topographical map of the Underground

The London Underground Map

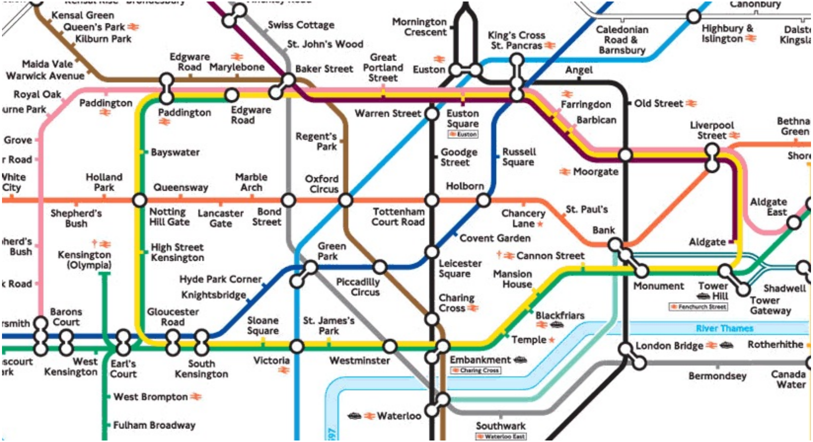


Figure : Topological map of the Underground

The London Underground Map

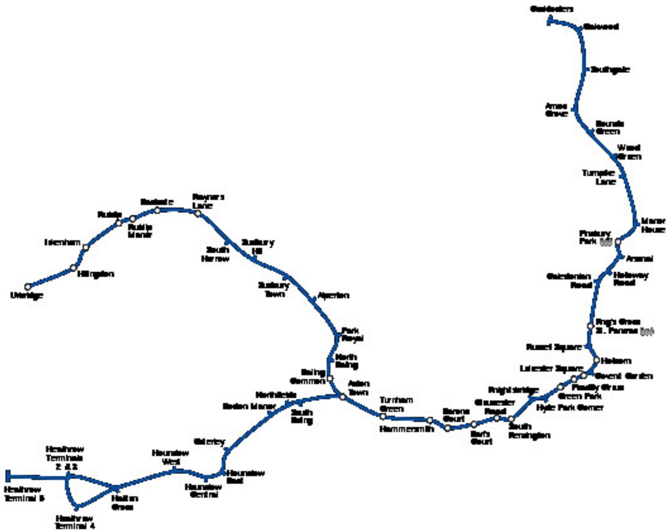
Properties of a simple closed loop:

- No branches. No travel options.
- Start anywhere: end up there again.
- Definite direction (CW or CCW).
- An Inside and an Outside.

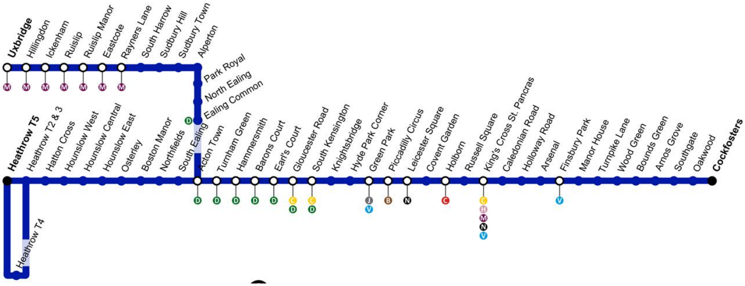
It is topologically equivalent to a circle.

Draw a (complicated) simple loop.

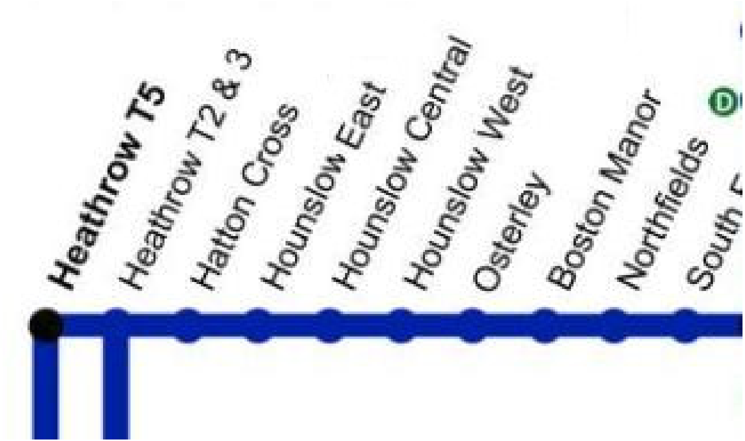
Piccadilly Line, Topographic



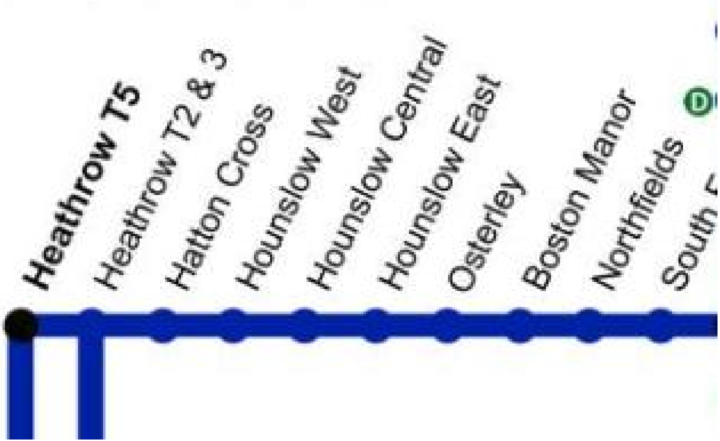
Piccadilly Line, Topological



Piccadilly Line, Detail

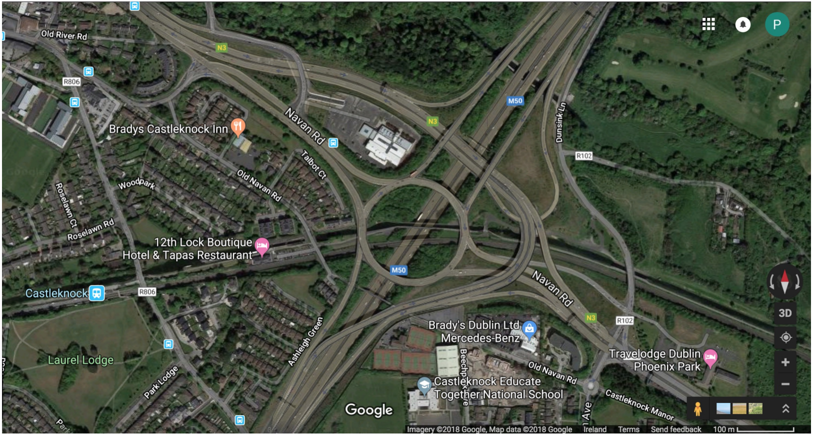


Piccadilly Line, Detail

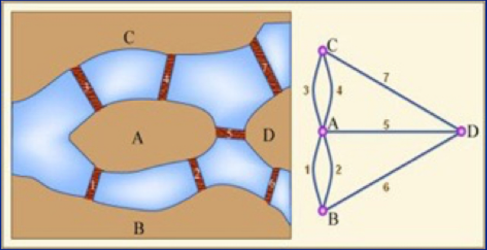


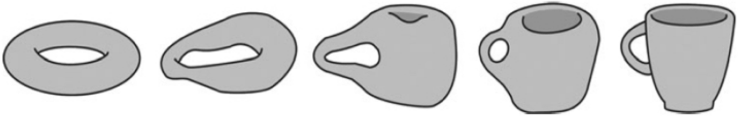
Spaghetti Junction on M50

Spaghetti Junction on M50



Topology
is often called
Rubber Sheet Geometry





Definition of a Topologist

Continuous distortion without tearing or glueing.

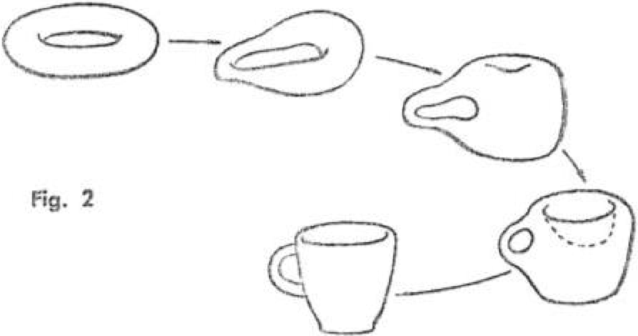


Fig. 2

Figure : "A topologist is someone who doesn't know the difference between a doughnut and a coffee-cup." [Joke!]

Topological Invariance

Topology is about *Continuity* and *connectedness.*

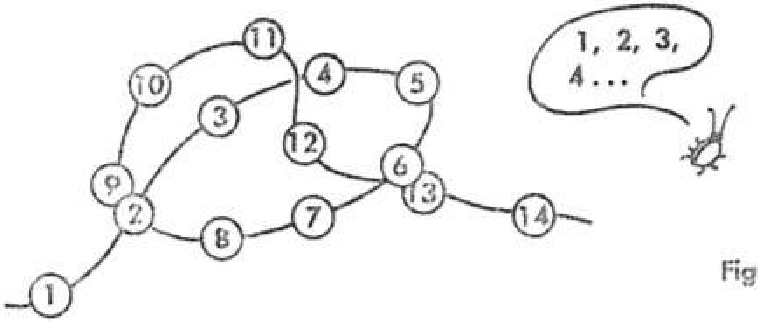
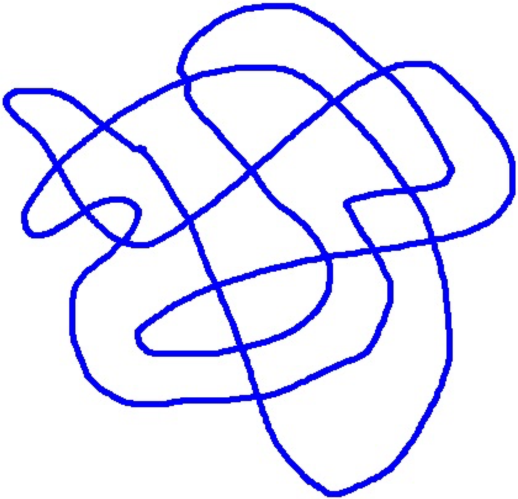


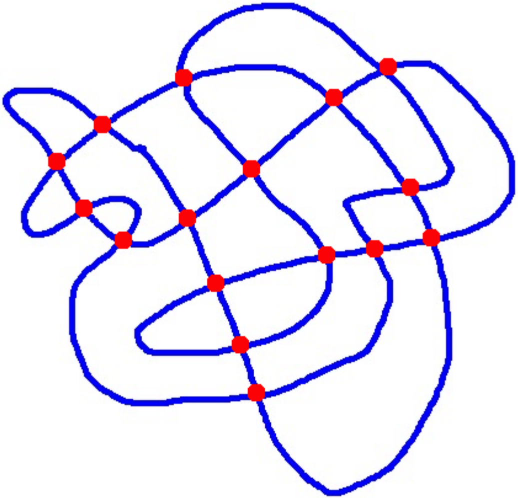
Fig. 1

Figure : Order of points unchanged under distortion. A bug sees only the order of the points, not the shape of the curve.

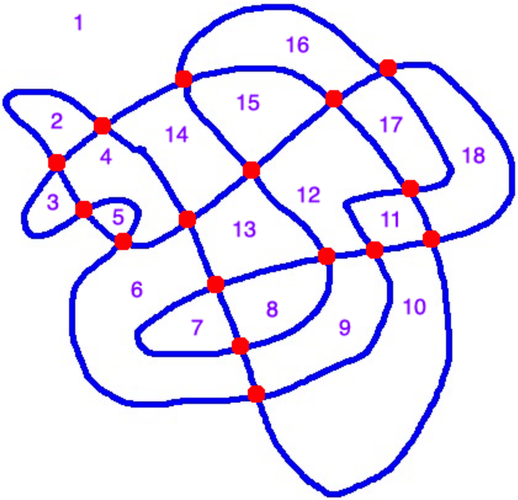
The Doodle Theorem



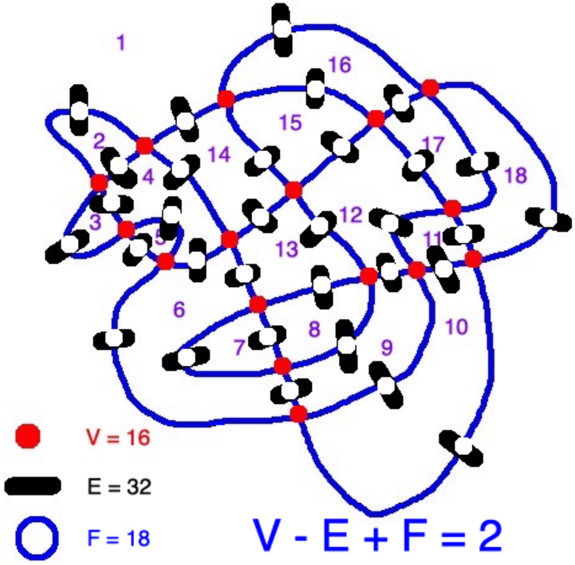
The Doodle Theorem



The Doodle Theorem

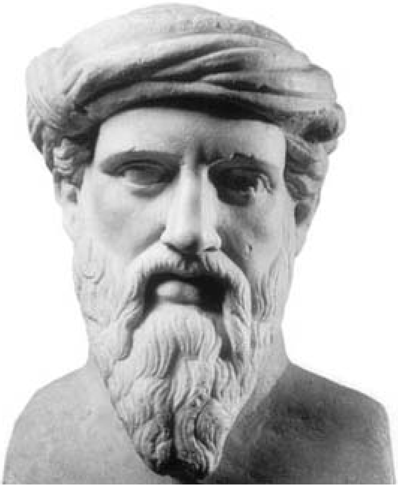


The Doodle Theorem



$$V - E + F = 2$$

Outline



The Thallasic Age

The period from 800 BC to AD 800.

$$\Theta\alpha\lambda\alpha\sigma\sigma\alpha \text{ — the Sea.}$$

- The first Olympic Games in 776 BC
- Homer and Hesiod lived around 700 BC
- Greek mathematics began to thrive
- First two major figures: Thales and Pythagoras.

Pythagoras (c. 570–495 BC)

Pythagoras was

- Born on the island of Samos (off Turkey).
- Philosopher, mystic, prophet and religious leader.
- Contemporary with Confucius and Lao-Tzu.

Words philosophy (love of learning) and mathematics (that which is learned) attributed to Pythagoras.

May have been first person to imagine that natural phenomena can be understood through mathematics.

Pythagoras (c. 570–495 BC)

- **No contemporary documents**
- **Myth, legend and tradition**
- **Second or third hand accounts**
  often written centuries later
- **Aristotle's biography no longer extant.**

Hardly any statement about Pythagoras uncontested.

Difficult to separate history from myth and legend.

Pythagoras (c. 570–495 BC)

- Travelled to Egypt, Babylon and perhaps India
- Mathematics, astronomy and religious lore
- Theorem on right-angled triangles
- Result known to Babylonians 1000 years earlier
- No record of a proof by Pythagoras survives.

The Pythagoreans

Around 530 BC Pythagoras moved to Croton in Magna Graecia (now Southern Italy).

He established an organization or school (philosophical / religious / political).

Both men and women were members of "The Pythagoreans"

Adherents were very secretive:
Bound by an oath of allegiance

Led lives of temperance; observed strict moral codes.

Pythagorean Women

“Women were given equal opportunity to study as Pythagoreans, and learned practical domestic skills in addition to philosophy.

“Women were held to be different from men, sometimes in positive ways.

“The priestess, philosopher and mathematician Themistoclea is regarded as Pythagoras’ teacher; Theano, Damo and Melissa as female disciples.”

From the Wikipedia article: *The Pythagoreans.*

Pythagorean Quotes

- "I was *Euphorbus* at the siege of Troy."
- "In anger, refrain from both speech and action."
- "Educate the children and it won't
  be necessary to punish the men."
- "Abstain from beans!"

- "There is geometry in the humming of the strings,
  There is music in the spacing of the spheres."
- "Number rules the universe."

Harmony & Discord

By tradition, Pythagoras discovered the principles of *musical harmony.*

Stringed instruments produce harmonious sounds when string lengths are ratios of small numbers.

Extended this idea to the heavens: planets emit sounds according to their speed of movement

Concept of the "harmony of the spheres".

Johannes Kepler: Harmonices Mundi

All is Number

The motto of the Pythagoreans: *All is Number.*

All natural phenomena in the universe can be expressed using whole numbers or ratios of them.

For the Pythagoreans, numbers were the essence and source of all things.

Modern physics holds that, at its deepest level, the universe is mathematical in nature.

This view is a topic of current serious discussion (*The Mathematical Universe*, by Max Tegmark).

Thank you