



# Outline

**Introduction**

**Symmetries of Triangle and Square**

**Möbius Band I**

**Cookie Row**

**Moessner's Magic**

**The Golden Ratio**

**Hilbert's Problems**

**Random Number Generators**

**The Sieve of Eratosthenes**

**Numerical Weather Prediction**



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# Meaning and Content of Mathematics

The word **Mathematics** comes from Greek *μαθημα* (*máthéma*), meaning “knowledge” or “study” or “learning”.

It is the study of topics such as

- ▶ Quantity (numbers)
- ▶ Structure (patterns)
- ▶ Space (geometry)
- ▶ Change (analysis).



***Reminder: A square from A4 paper sheets.***

**PUZZLE:**

**Is it possible to form a square out of sheets of A4 sized paper (without them overlapping)?**

**Remember: Ratio of width to height is  $1 : \sqrt{2}$ .**



# A Square from A4 Paper Sheets

Let dimensions be: Width = 1 unit. Height =  $\sqrt{2}$  units.

Suppose there are  $a$  short sides and  $b$  long sides along the *lower horizontal edge* of the big square.

Then the length of the horizontal edge is

$$H = a.1 + b.\sqrt{2}$$

Suppose there are  $c$  short sides and  $d$  long sides along the *left vertical edge* of the big square.

So the length of the vertical edge is

$$V = c.1 + d\sqrt{2}$$



Since the region is square,  $V = H$  and we must have

$$a.1 + b.\sqrt{2} = c.1 + d\sqrt{2}$$

Therefore

$$\begin{aligned} a + b\sqrt{2} &= c + d\sqrt{2} \\ a - c &= (d - b)\sqrt{2} \\ \left(\frac{a - c}{d - b}\right) &= \sqrt{2} \end{aligned}$$

But the left side is a ratio of two whole numbers, whereas the right side is irrational.

This is impossible. There is no solution!

*Reductio ad absurdum.*



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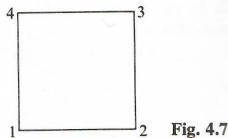
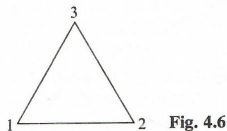
Numerical Weather Prediction





# Symmetries of the Triangle and Square: The Dihedral Groups $D_3$ and $D_4$

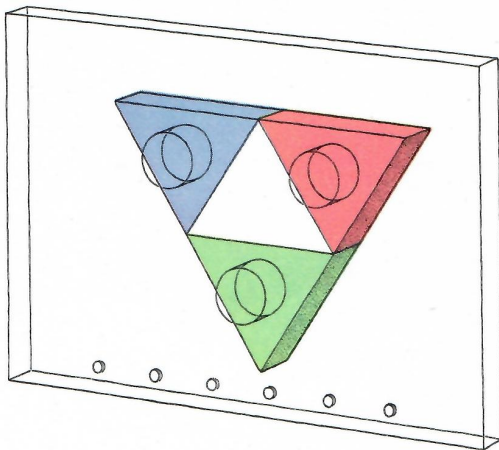
Let's look at symmetries of the triangle and square.



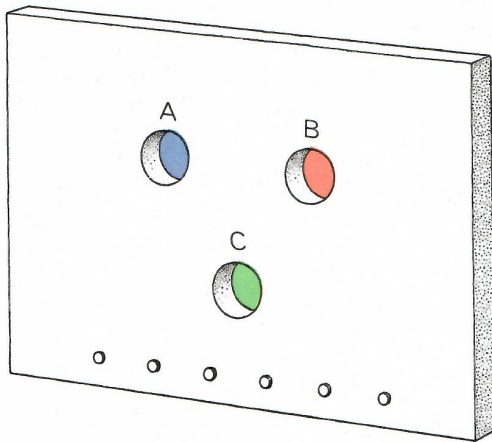
They correspond to the dihedral groups  $D_3$  and  $D_4$ .









d



a

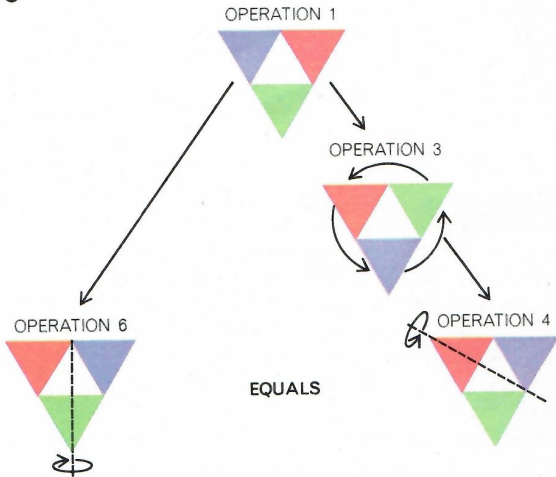


b

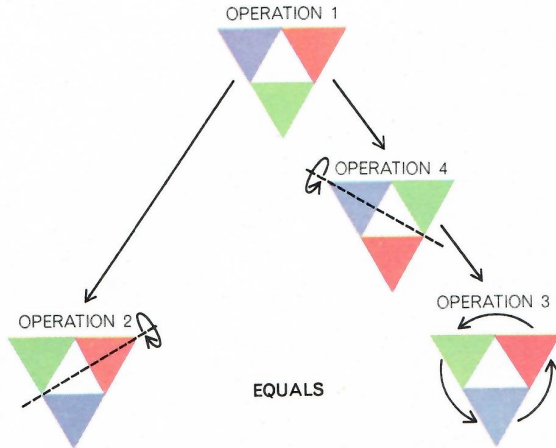
OPERATION	RESULT
1. NO CHANGE:	
2. SWITCH A AND C:	
3. REPLACE A BY B, B BY C, C BY A:	
4. SWITCH C AND B:	
5. REPLACE A BY C, B BY A, C BY B:	
6. SWITCH A AND B:	



e



*f*



C

		FIRST OPERATION					
		1	2	3	4	5	6
SECOND OPERATION	1						
	2						
	3						
	4						
	5						
	6						

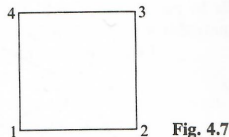
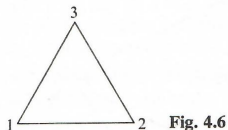


## Skip to end of Section: Counting Symmetries





# Symbols for Transformations of Triangle



$$\rho_0 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \quad \mu_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix},$$

$$\rho_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \quad \mu_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix},$$

$$\rho_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, \quad \mu_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}.$$



# The Third Dihedral Group $D_3$

	$\rho_0$	$\rho_1$	$\rho_2$	$\mu_1$	$\mu_2$	$\mu_3$
$\rho_0$	$\rho_0$	$\rho_1$	$\rho_2$	$\mu_1$	$\mu_2$	$\mu_3$
$\rho_1$	$\rho_1$	$\rho_2$	$\rho_0$	$\mu_2$	$\mu_3$	$\mu_1$
$\rho_2$	$\rho_2$	$\rho_0$	$\rho_1$	$\mu_3$	$\mu_1$	$\mu_2$
$\mu_1$	$\mu_1$	$\mu_3$	$\mu_2$	$\rho_0$	$\rho_2$	$\rho_1$
$\mu_2$	$\mu_2$	$\mu_1$	$\mu_3$	$\rho_1$	$\rho_0$	$\rho_2$
$\mu_3$	$\mu_3$	$\mu_2$	$\mu_1$	$\rho_2$	$\rho_1$	$\rho_0$

Fig. 4.5



# Subgroup $Z_3$ of Third Dihedral Group $D_3$

	$\rho_0$	$\rho_1$	$\rho_2$
$\rho_0$	$\rho_0$	$\rho_1$	$\rho_2$
$\rho_1$	$\rho_1$	$\rho_2$	$\rho_0$
$\rho_2$	$\rho_2$	$\rho_0$	$\rho_1$

Fig. 4.5



# The Third Dihedral Group $D_3$

	$\rho_0$	$\rho_1$	$\rho_2$	$\mu_1$	$\mu_2$	$\mu_3$
$\rho_0$	$\rho_0$	$\rho_1$	$\rho_2$	$\mu_1$	$\mu_2$	$\mu_3$
$\rho_1$	$\rho_1$	$\rho_2$	$\rho_0$	$\mu_2$	$\mu_3$	$\mu_1$
$\rho_2$	$\rho_2$	$\rho_0$	$\rho_1$	$\mu_3$	$\mu_1$	$\mu_2$
$\mu_1$	$\mu_1$	$\mu_3$	$\mu_2$	$\rho_0$	$\rho_2$	$\rho_1$
$\mu_2$	$\mu_2$	$\mu_1$	$\mu_3$	$\rho_1$	$\rho_0$	$\rho_2$
$\mu_3$	$\mu_3$	$\mu_2$	$\mu_1$	$\rho_2$	$\rho_1$	$\rho_0$

Fig. 4.5



# Subgroup $Z_2$ of Third Dihedral Group $D_3$

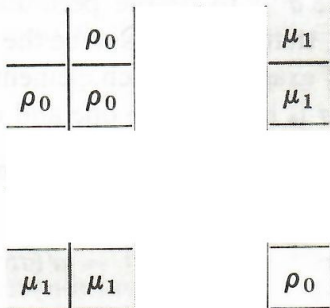
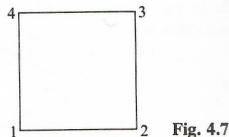
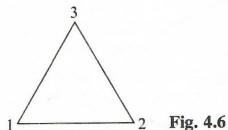


Fig. 4.5



# Symbols for Transformations of Square



$$\rho_0 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}, \quad \mu_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix},$$

$$\rho_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}, \quad \mu_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix},$$

$$\rho_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}, \quad \delta_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix},$$

$$\rho_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}, \quad \delta_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix}.$$



# The Fourth Dihedral Group $D_4$

	$\rho_0$	$\rho_1$	$\rho_2$	$\rho_3$	$\mu_1$	$\mu_2$	$\delta_1$	$\delta_2$
$\rho_0$	$\rho_0$	$\rho_1$	$\rho_2$	$\rho_3$	$\mu_1$	$\mu_2$	$\delta_1$	$\delta_2$
$\rho_1$	$\rho_1$	$\rho_2$	$\rho_3$	$\rho_0$	$\delta_2$	$\delta_1$	$\mu_1$	$\mu_2$
$\rho_2$	$\rho_2$	$\rho_3$	$\rho_0$	$\rho_1$	$\mu_2$	$\mu_1$	$\delta_2$	$\delta_1$
$\rho_3$	$\rho_3$	$\rho_0$	$\rho_1$	$\rho_2$	$\delta_1$	$\delta_2$	$\mu_2$	$\mu_1$
$\mu_1$	$\mu_1$	$\delta_1$	$\mu_2$	$\delta_2$	$\rho_0$	$\rho_2$	$\rho_1$	$\rho_3$
$\mu_2$	$\mu_2$	$\delta_2$	$\mu_1$	$\delta_1$	$\rho_2$	$\rho_0$	$\rho_3$	$\rho_1$
$\delta_1$	$\delta_1$	$\mu_2$	$\delta_2$	$\mu_1$	$\rho_3$	$\rho_1$	$\rho_0$	$\rho_2$
$\delta_2$	$\delta_2$	$\mu_1$	$\delta_1$	$\mu_2$	$\rho_1$	$\rho_3$	$\rho_2$	$\rho_0$

Fig. 4.8



# Counting Symmetries of the Square

## Counting Symmetries

Can you find all the symmetries of the familiar square?

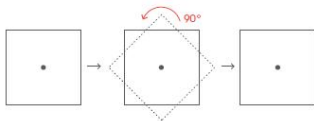
### WHAT IS SYMMETRY?

Symmetries are transformations of an object that preserve its size and shape and whose result is indistinguishable from the original.



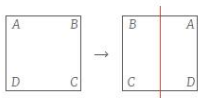
For example, a line cuts a square into two equal parts, each one the mirror image of the other. This is called **line symmetry**.

The square also has **rotational symmetry**. After rotating a square counterclockwise about its center point (the intersection of its diagonals) 90 degrees, it looks the same as before.



### HOW MANY SYMMETRIES DOES A SQUARE HAVE?

Hint: Label the corners A, B, C, and D to specify each symmetry of the square by some arrangement of the four letters.



As an example, reflect the square across a vertical line through its center and watch where the labels go.



We can denote the resulting line symmetry as BADC.

See worksheet: [CountingSymmetriesWorksheet.pdf](#)





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# The Möbius Band



**You may be familiar with the Möbius strip or Möbius band. It has one side and one edge.**

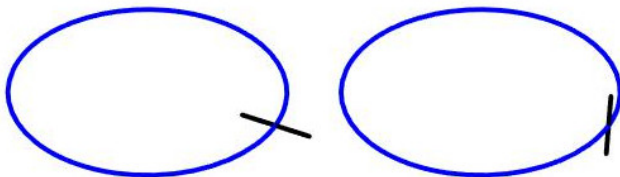
**It was discovered independently by August Möbius and Johann Listing in the same year, 1858.**



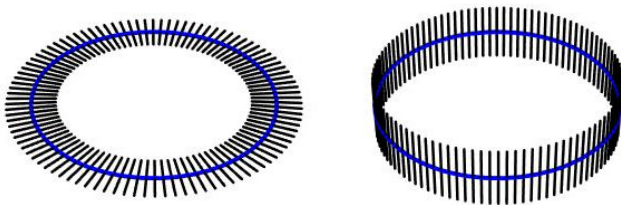
# Building the Band

It is easy to make a Möbius band from a paper strip.

For a geometrical construction, we start with a circle and a small line segment with centre on this circle.

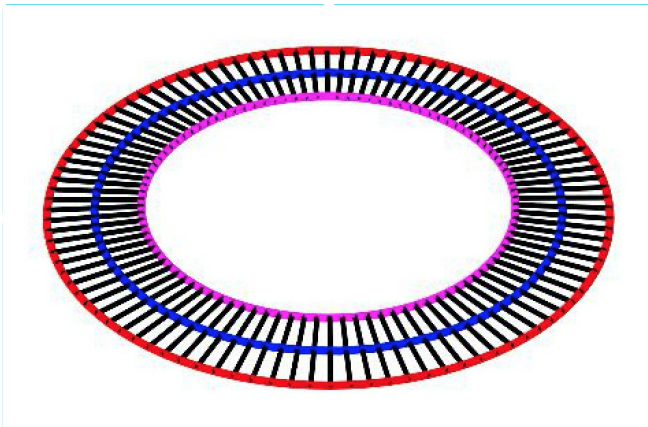


**Now move the line segment around the circle:**



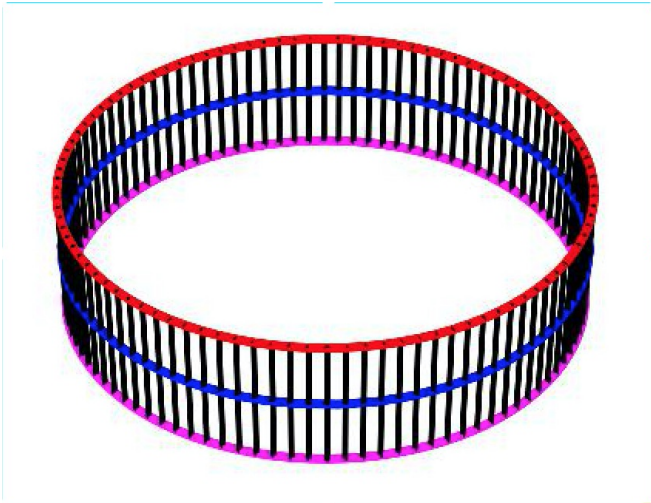
**To show the boundary of the surface, we color one end of the line segment red and the other magenta.**





**Figure :** The boundary comprises two unlinked circles



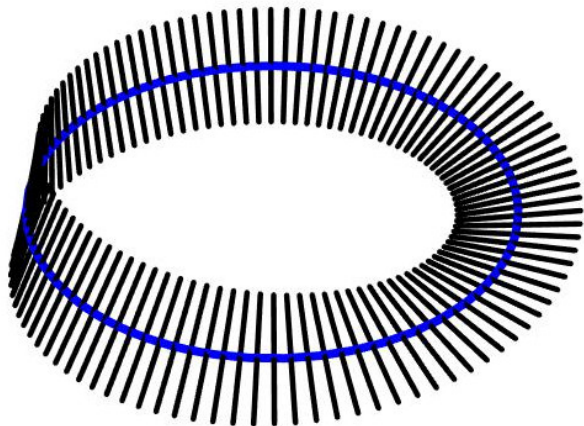


**Figure :** The boundary comprises two unlinked circles



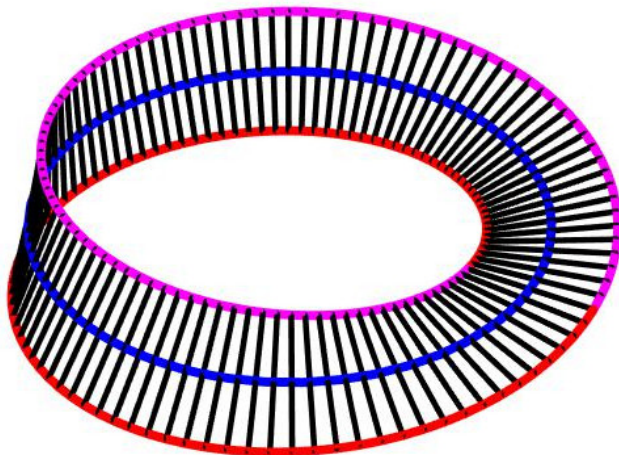
# The Möbius Band

Now, as the line moves, we give it a half-twist:



# The Möbius Band

The two boundary curves now join up to become one:





# The Möbius Band

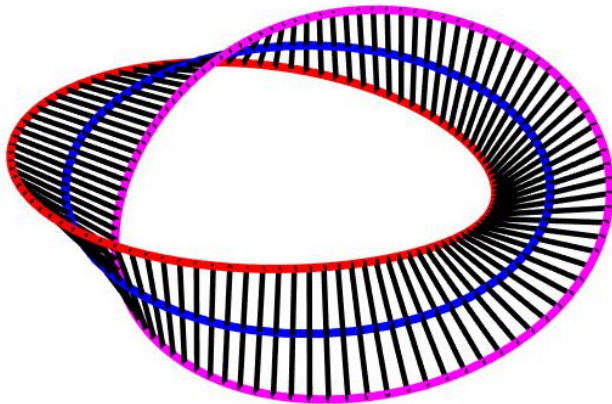
**The Möbius Band has only one side.**

**It is possible to get from any point on the surface to any other point *without crossing the edge*.**

**The surface also has just one edge.**



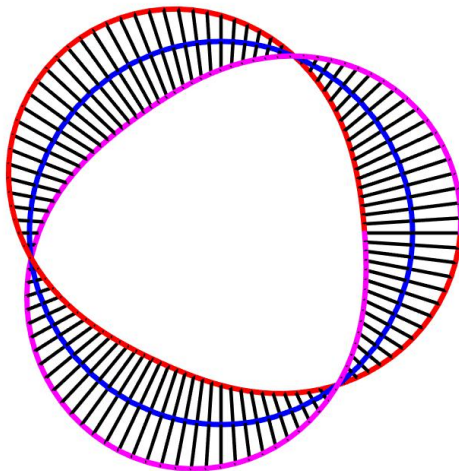
# Band with a Full Twist



**Figure :** The boundary comprises two linked circles

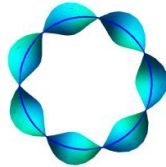
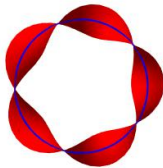


# Band with Three Half-twists

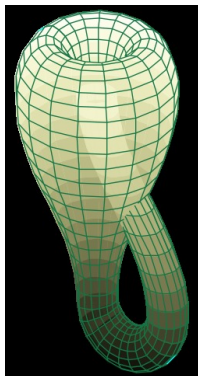


**Figure :** The boundary is a knot, a trefoil curve





# Two Möbius Bands make a Klein Bottle



**A mathematician named Klein  
Thought the Möbius band was divine.  
Said he: “If you glue  
The edges of two,  
You’ll get a weird bottle like mine.”**



# Equations for the Möbius Band

**The process of moving the line segment around the circle leads us to the equations for the Möbius band.**

**In cylindrical polar coordinates the circle is**

$$(r, \theta, z) = (a, \theta, 0).$$

**The tip of the segment, relative to its centre, is**

$$(r, \theta, z) = (b \cos \phi, 0, b \sin \phi)$$

**where  $b = \frac{1}{2}\ell$  is half the segment length and  $\phi = \alpha\theta$ , with  $\alpha$  determining the amount of twist.**

**The tip of the line has  $(r, z) = (a + b \cos \alpha\theta, b \sin \alpha\theta)$ .**



# Equations for the Möbius Band

In cartesian coordinates, the equations become

$$x = (a + b \cos \alpha \theta) \cos \theta$$

$$y = (a + b \cos \alpha \theta) \sin \theta$$

$$z = (b \sin \alpha \theta)$$

These are the parametric equations for the twisted bands, with  $\theta \in [0, 2\pi]$  and  $b \in [-\ell, \ell]$ .

For the Möbius band,  $\alpha = \frac{1}{2}$ .



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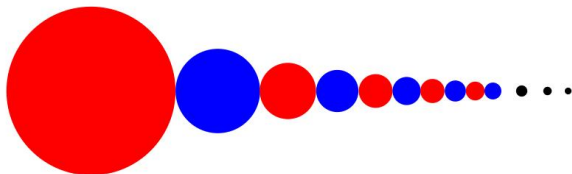




# A Surprising Result

Let us consider an infinite row of cookies each smaller than the previous one.

Assume that the radius of the  $n$ -th cookie is  $1/n$ . Then the surface area is  $\pi/n^2$ .



# A Surprising Result

The total length of the row of cookies is

$$2 \sum_{n=1}^{\infty} \frac{1}{n}$$

This is the divergent harmonic series.

The total surface area of the cookies is

$$\sum_{n=1}^{\infty} \pi \times \left(\frac{1}{n}\right)^2 = \pi \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^3}{6}$$

This series is known as the **Basel series**, and it is **convergent**, with sum  $\pi^2/6$ .



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# Alfred Moessner's Conjecture

*Aus den Sitzungsberichten der Bayerischen Akademie der Wissenschaften  
Mathematisch-naturwissenschaftliche Klasse 1951 Nr. 3*

## Eine Bemerkung über die Potenzen der natürlichen Zahlen

Von Alfred Moessner in Gunzenhausen

Vorgelegt von Herrn O. Perron am 2. März 1951

## A Remark on the Powers of the Natural Numbers



# Moessner's Construction: $n=2$

We start with the sequence of natural numbers:

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 ...

Now we delete *every second number* and calculate the sequence of partial sums:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1		4		9		16		25		36		49		64	

The result is the sequence of perfect squares:

$1^2$   $2^2$   $3^2$   $4^2$   $5^2$   $6^2$   $7^2$   $8^2$  ...



# Moessner's Construction: $n=3$

Now we delete *every third number* and calculate the sequence of partial sums.

Then we delete *every second number* and calculate the sequence of partial sums:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	3		7	12		19	27		37	48		61	75		91
1		8				27			64			125			216

The result is the sequence of perfect cubes:

$$1^3 \quad 2^3 \quad 3^3 \quad 4^3 \quad 5^3 \quad 6^3 \quad \dots$$



# Moessner's Construction: $n=4$

The Moessner Construction also works for larger  $n$ :

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	3	6		11	17	24		33	43	54		67	81	96	
1	4			15	32			65	108			175	256		
1				16				81				256			

The result is the sequence of fourth powers:

$$1^4 \quad 2^4 \quad 3^4 \quad 4^4 \quad \dots$$



# Moessner's Constructions

**Remark:**

**Using Moessner's construction, we can generate a table of squares, cubes or higher powers.**

**The only arithmetical operations used are *counting* and *addition*!**

**Are there any other sequences generated in this way?**





# Moessner's Construction for $n!$

We begin by striking out the *triangular numbers*,  
 $\{1, 3, 6, 10, 15, 21, \dots\}$  and form partial sums.

Next, we delete the final entry in each group and form partial sums. This process is repeated indefinitely:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
	2		6	11		18	26	35		46	58	71	85		101
			6			24	50			96	154	225			326
						24				120	274				600
										120					720

This yields the *factorial numbers*:

1! 2! 3! 4! 5! 6! ...



# Beautiful Math

The beauty of maths?  
*What do mathematicians think?*

**VIDEO: Beautiful Maths, available at**







<http://momath.org/home/beautifulmath/>

**Video by James Tanton**

Try to disregard the antipodean exuberance!



# Wikipedia Mathematics Portal

Topics in mathematics <span style="float: right;">edit</span>			
General	Foundations	Number theory	Discrete mathematics
<ul style="list-style-type: none"> <li>Mathematicians</li> <li>History of mathematics</li> <li>Philosophy of mathematics</li> <li>Mathematical notation</li> <li>Mathematical beauty</li> <li>Mathematics education</li> <li>Areas of mathematics</li> <li>Outline of mathematics</li> <li>List of mathematical symbols</li> <li>Wikipedia Books: Mathematics</li> </ul>	 <ul style="list-style-type: none"> <li>Foundations of mathematics</li> <li>Mathematical logic                             <ul style="list-style-type: none"> <li>Proof theory                                     <ul style="list-style-type: none"> <li>Gödel's incompleteness theorems</li> </ul> </li> <li>Model theory</li> <li>Recursion theory</li> </ul> </li> <li>Set theory (portal)                             <ul style="list-style-type: none"> <li>Naive set theory</li> <li>Axiomatic set theory</li> </ul> </li> <li>Category theory (portal)                             <ul style="list-style-type: none"> <li>Topos theory</li> </ul> </li> </ul>	 <ul style="list-style-type: none"> <li>Number theory (portal) <span style="float: right;">4</span></li> <li>Algebraic number theory</li> <li>Analytic number theory</li> <li>Arithmetic                             <ul style="list-style-type: none"> <li>Fundamental theorem of arithmetic</li> </ul> </li> <li>Numbers                             <ul style="list-style-type: none"> <li>Natural numbers</li> <li>Prime numbers</li> <li>Rational numbers</li> <li>Algebraic numbers</li> </ul> </li> </ul>	 <ul style="list-style-type: none"> <li>Discrete mathematics (portal)</li> <li>Combinatorics                             <ul style="list-style-type: none"> <li>Combinatorial geometry</li> <li>Coding theory</li> <li>Combinatorial design</li> <li>Enumerative combinatorics</li> <li>Combinatorial optimization</li> </ul> </li> <li>Graph theory</li> <li>Order theory                             <ul style="list-style-type: none"> <li>Lattice theory</li> </ul> </li> <li>Digital Signal Processing</li> </ul>
Algebra	Analysis	Geometry and topology	Applied mathematics
<ul style="list-style-type: none"> <li>Algebra (portal)</li> <li>Elementary algebra</li> <li>Abstract algebra                             <ul style="list-style-type: none"> <li>Group theory</li> <li>Ring theory</li> <li>Field theory</li> <li>Commutative algebra</li> </ul> </li> <li>Geometric algebra</li> <li>Linear algebra                             <ul style="list-style-type: none"> <li>Matrix theory</li> <li>Multilinear algebra</li> </ul> </li> <li>Universal algebra</li> <li>Fundamental theorem of algebra</li> </ul>	 <ul style="list-style-type: none"> <li>Analysis (portal)</li> <li>Calculus                             <ul style="list-style-type: none"> <li>Fundamental theorem of calculus</li> <li>Vector calculus</li> <li>Geometric calculus</li> </ul> </li> <li>Measure theory</li> <li>Real analysis</li> <li>Complex analysis                             <ul style="list-style-type: none"> <li>Differential equations                                     <ul style="list-style-type: none"> <li>Ordinary differential equations</li> <li>Partial differential equations</li> </ul> </li> <li>Integral equations</li> <li>Approximation theory</li> <li>Special functions</li> <li>Potential theory</li> <li>Harmonic analysis                                     <ul style="list-style-type: none"> <li>Fourier analysis</li> </ul> </li> <li>Functional analysis</li> <li>Operator theory</li> </ul> </li> </ul>	 <ul style="list-style-type: none"> <li>Geometry (portal)</li> <li>Euclidean geometry                             <ul style="list-style-type: none"> <li>Trigonometry</li> </ul> </li> <li>Analytic geometry</li> <li>Non-Euclidean geometry                             <ul style="list-style-type: none"> <li>Affine geometry</li> </ul> </li> <li>Projective geometry</li> <li>Convex geometry</li> <li>Discrete geometry</li> <li>Algebraic geometry                             <ul style="list-style-type: none"> <li>Differential geometry                                     <ul style="list-style-type: none"> <li>Riemannian geometry</li> <li>Lie groups</li> </ul> </li> </ul> </li> <li>Topology (portal)                             <ul style="list-style-type: none"> <li>General topology</li> <li>Algebraic topology</li> <li>Geometric topology</li> <li>Differential topology</li> </ul> </li> </ul>	 <ul style="list-style-type: none"> <li>Applied mathematics</li> <li>Mathematical modeling</li> <li>Mathematical physics</li> <li>Dynamical systems                             <ul style="list-style-type: none"> <li>Control theory                                     <ul style="list-style-type: none"> <li>Calculus of variations</li> </ul> </li> </ul> </li> <li>Optimization</li> <li>Mathematical economics                             <ul style="list-style-type: none"> <li>Game theory</li> <li>Mathematical finance</li> </ul> </li> <li>Statistics (portal)</li> <li>Probability theory                             <ul style="list-style-type: none"> <li>Stochastic processes</li> </ul> </li> <li>Numerical analysis</li> <li>Theoretical computer science                             <ul style="list-style-type: none"> <li>Computability theory</li> <li>Complexity theory</li> </ul> </li> <li>Cryptography (portal)</li> <li>Information theory</li> </ul>
Index of mathematics articles <span style="float: right;">edit</span>			



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Möbius Band I

Cookie Row

Moessner's Magic

**The Golden Ratio**

Hilbert's Problems

Random Number Generators

The Sieve of Eratosthenes

Numerical Weather Prediction



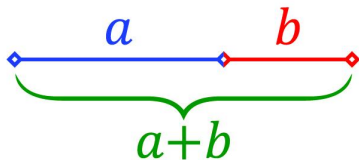
# Golden Rectangle in Your Pocket



Aspect ratio is about  $\phi = \frac{1+\sqrt{5}}{2} \approx 1.618$ .



# Geometric Ratio: $a + b$ is to $a$ as $a$ is to $b$ .



$$\left[ \frac{\text{Short Bit}}{\text{Long Bit}} \right] = \left[ \frac{\text{Long Bit}}{\text{Full Line}} \right] \quad \text{or} \quad \frac{b}{a} = \frac{a}{a+b}$$

Let the blue segment be  $a = 1$  and the whole line  $\phi$ .

Then  $b = \phi - 1$  and we have

$$\frac{\phi - 1}{1} = \frac{1}{\phi}$$



$$\phi - 1 = \frac{1}{\phi}$$

**This means  $\phi$  solves a quadratic equation:**

$$\phi^2 - \phi - 1 = 0$$

**Recall the two solutions of a quadratic equation**

$$ax^2 + bx + c = 0 \quad \text{are} \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

**In the present case, this means that the roots are**

$$\phi = \frac{1 \pm \sqrt{1 + 4}}{2}$$

**We take the *positive root*, giving**

$$\phi = \frac{1 + \sqrt{5}}{2} \approx 1.618$$

**This is the golden ratio.**



# Check the Solution

The quadratic equation is

$$\phi^2 - \phi - 1 = 0 \quad \text{or} \quad \phi^2 = \phi + 1$$

Suppose

$$\phi = \frac{1 + \sqrt{5}}{2}$$

Then

$$\phi + 1 = \frac{3 + \sqrt{5}}{2} \quad \text{and} \quad \phi^2 = \frac{3 + \sqrt{5}}{2}$$





# Golden Rectangle



Ratio of breath to height is  $\phi = \frac{1+\sqrt{5}}{2}$ .



# Golden Rectangle in Your Pocket



Aspect ratio is about  $\phi = \frac{1+\sqrt{5}}{2} \approx 1.618$ .



# Terminology

- ▶ **Golden Ratio. Golden Number. Golden Mean.**
- ▶ **Golden Proportion. Golden Cut.**
- ▶ **Golden Section. Medial Section.**
- ▶ ***Divine Proportion.* Divine Section.**
- ▶ **Extreme and Mean Ratio.**
- ▶ **Various Other Terms.**



# Fibonacci Numbers

The Fibonacci sequence is the sequence

$$\{0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \dots\}$$

where *each number is the sum of the previous two*.

The Fibonacci numbers obey a recurrence relation

$$F_{n+1} = F_n + F_{n-1}$$

with the *starting values*  $F_0 = 0$  and  $F_1 = 1$ .

**Can we solve this recurrence relation for all  $F_n$ ?**



# Fibonacci Numbers

The *recurrence relation* is

$$F_{n+1} = F_n + F_{n-1}$$

We assume that the solution is of the form  $F_n = k\chi^n$ , where we have to find  $\chi$  (this is called an *Ansatz*).

Substitute this solution into the recurrence relation:

$$k\chi^{n+1} = k\chi^n + k\chi^{n-1}$$

Divide by  $k\chi^{n-1}$  to get the quadratic equation

$$\chi^2 = \chi + 1 \quad \text{or} \quad \chi^2 - \chi - 1 = 0$$

This is the quadratic we got for the golden number.



# Fibonacci Numbers

We found that  $F_n = k\phi^n$  where  $\phi$  is a root of

$$\phi^2 - \phi - 1 = 0$$

The two roots are

$$\frac{1 + \sqrt{5}}{2} \quad \text{and} \quad \frac{1 - \sqrt{5}}{2}$$

Then the full solution for the Fibonacci numbers is

$$F_n = \frac{1}{\sqrt{5}} \left[ \frac{1 + \sqrt{5}}{2} \right]^n - \frac{1}{\sqrt{5}} \left[ \frac{1 - \sqrt{5}}{2} \right]^n$$

Check that the conditions  $F_0 = 0$  and  $F_1 = 1$  are true.



# Fibonacci Numbers

$$F_n = \frac{1}{\sqrt{5}} \left[ \frac{1 + \sqrt{5}}{2} \right]^n - \frac{1}{\sqrt{5}} \left[ \frac{1 - \sqrt{5}}{2} \right]^n$$

The first term in square brackets is greater than 1, so the powers *grow rapidly with n*.

The second term in square brackets is less than 1, so the powers *become small rapidly with n*.

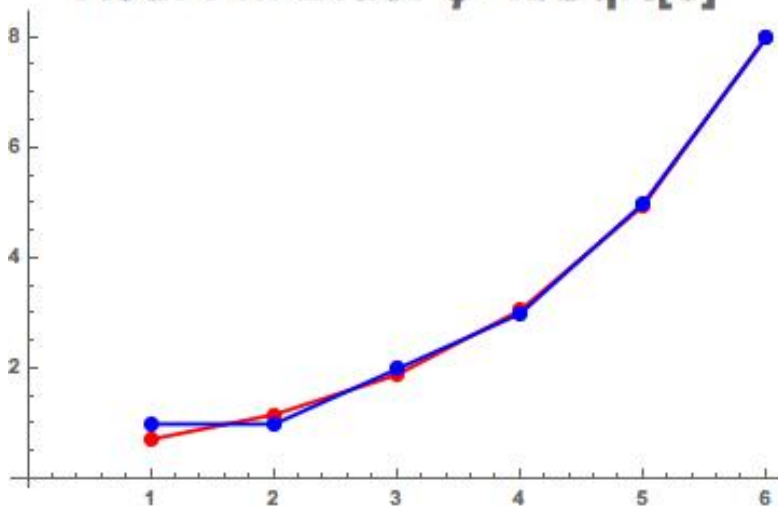
So, we ignore the second term and write

$$F_n \approx \frac{1}{\sqrt{5}} \left[ \frac{1 + \sqrt{5}}{2} \right]^n \quad \text{or} \quad F_n \approx \frac{\phi^n}{\sqrt{5}}$$



# Approximation to $F_n$

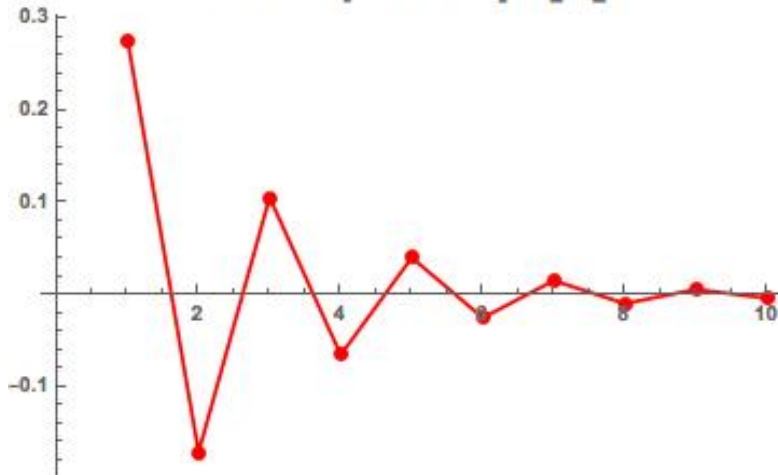
Red:  $F_n$ . Blue:  $\phi^n/\text{Sqrt}[5]$





# Oscillating Error of Approximation

$$F_n - \phi^n / \text{Sqrt}[5]$$



# Ratio $F_n/F_{n-1}$

$$F_n \approx \frac{\phi^n}{\sqrt{5}} \implies \frac{F_n}{F_{n-1}} \approx \phi$$

Let's consider the sequence of ratios of terms

$$\frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \frac{21}{13}, \frac{34}{21}, \dots$$

The ratios get closer and closer to  $\phi$ :

$$\frac{F_{n+1}}{F_n} \rightarrow \phi \quad \text{as } n \rightarrow \infty$$



# Continued Fraction for $\phi$

$$\phi^2 - \phi - 1 = 0 \implies \phi = 1 + \frac{1}{\phi}$$

Now use the equation to replace  $\phi$  on the right:

$$\phi = 1 + \frac{1}{\phi} = 1 + \frac{1}{1 + \frac{1}{\phi}} = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{\phi}}}$$

Eventually

$$\phi = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{\dots}}}}$$



# Continued Root for $\phi$

$$\phi^2 - \phi - 1 = 0 \implies \phi = \sqrt{1 + \phi}$$

Now use the equation to replace  $\phi$  on the right:

$$\phi = \sqrt{1 + \phi} = \sqrt{1 + \sqrt{1 + \phi}} = \sqrt{1 + \sqrt{1 + \sqrt{1 + \phi}}}$$

Eventually

$$\phi = \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \dots}}}}}$$



# Fibonacci Numbers in Nature

Look at post

**Sunflowers and Fibonacci: Models of Efficiency**  
on the *ThatsMaths* blog.



# Vi Hart's Videos

Vi Hart has many mathematical videos on YouTube.

- ▶ **On Fibonacci Numbers:** <https://www.youtube.com/watch?v=ahXIMUkSXX0>
- ▶ **On the Three Utilities Problem:** <https://www.youtube.com/watch?v=CruQy1WSfoU&feature=youtu.be>
- ▶ **On Continued Fractions:** <https://www.youtube.com/watch?v=a5z-OEIFw3s>



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**Hilbert's Problems**

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The Sieve of Eratosthenes

Numerical Weather Prediction







# Hilbert's Problems

In August 1900, David Hilbert addressed the *International Congress of Mathematicians* in the Sorbonne in Paris:

*“Who of us would not be glad to lift the veil behind which the future lies hidden; to cast a glance at the next advances of our science and at the secrets of its development during future centuries?”*

Hilbert presented 23 problems that challenged mathematicians through the twentieth century.



# Hilbert's Problems

BULLETIN (New Series) OF THE  
AMERICAN MATHEMATICAL SOCIETY  
Volume 37, Number 4, Pages 407–436  
S 0273-0979(00)00881-8  
Article electronically published on June 26, 2000

## MATHEMATICAL PROBLEMS

DAVID HILBERT

*Lecture delivered before the International Congress of Mathematicians at Paris in 1900.*

**Hilbert's eighth problem concerned itself with what is called the Riemann Hypothesis (RH).**

**RH is generally regarded as the deepest and most important unproven mathematical problem.**

**Anyone who can prove it is assured of lasting fame.**



# Why is RH Important?

**A large number of mathematical theorems (1000's) depend for their validity on the RH.**

**Were RH to turn out to be false, many of these mathematical arguments would simply collapse.**

**In 2000, industrialist Landon Clay donated \$7M, with \$1M for each of 7 problems in mathematics.**

**The Riemann hypothesis is one of these problems.**

<http://www.claymath.org/millennium-problems>



# Why is RH Important?

**Whoever proves Riemann's hypothesis will have completed thousands of theorems that start like this:**

***“Assuming that the Riemann hypothesis is true ...”.***

**He or she will be assured of lasting fame.**

**Those who establish fundamental mathematical results probably come closer to immortality than almost anyone else.**



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# What is Randomness?

Randomness is a *slippery concept*,  
defying precise definition.

Toss a coin and get a sequence like 1001110100.

Some uses of Random Numbers:

- ▶ Computer simulations of fluid flow.
- ▶ Interactions of subatomic particles.
- ▶ Evolution of galaxies.

Tossing coins is impractical.  
We need more effective methods.



# Defining Randomness?

*Richard von Mises (1919):*

A binary sequence is random if the proportion of zeros and ones approaches 50% and if this is also true for any sub-sequence.

Consider ( 0101010101 ).

*Andrey Kolmogorov* defined the complexity of a binary sequence as the length of a computer program or algorithm that generates it.

The phrase a sequence of one million 1s completely defines a sequence.

Non-random sequences are compressible.  
Randomness and incompressibility are equivalent.



# Pseudo-random versus Truly Random

**Pseudo-random number generators are algorithms that use mathematical formulae to produce sequences of numbers.**

**The sequences appear completely random and satisfy various statistical conditions for randomness.**

***Pseudo-random numbers* are valuable for many applications but they have serious deficiencies.**





# Truly Random Number Generators

***True random number generators extract randomness from physical phenomena that are completely unpredictable.***

**Atmospheric noise is the static generated by lightning [globally there are 40 flashes/sec]. It can be detected by an ordinary radio.**



# Truly Random Number Generators

**Atmospheric noise passes all the statistical checks for randomness.**

**Dr Mads Haahr of Trinity College, Dublin uses atmospheric noise to produce random numbers.**

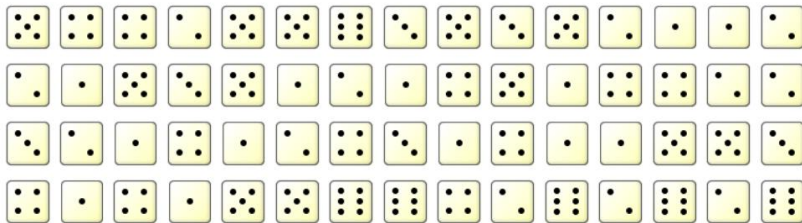
**Results available on on the website: [random.org](http://random.org).**



# 20 Random Coin Tosses



# 60 Dice Rolls

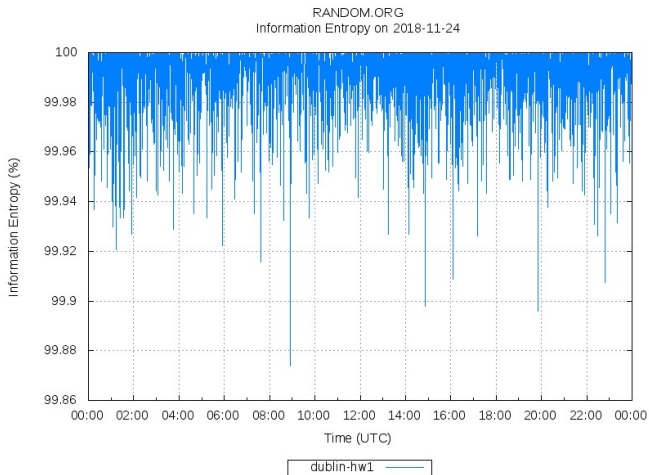


# 100 Random Numbers in [0,99]

17	60	57	66	4	71	59	36	8	49
87	64	94	82	6	38	14	87	76	72
97	38	44	59	56	24	20	6	24	97
0	40	14	77	18	98	41	39	6	79
21	59	49	86	91	81	65	64	3	11
92	17	65	6	37	98	84	17	70	93
60	52	1	98	20	2	65	9	57	3
48	86	27	3	71	51	57	56	2	2
13	14	73	65	11	32	17	7	91	37
3	8	10	67	0	72	0	42	15	24



# Quality of Random Numbers



# PRNG versus TRNG

Characteristic	Pseudo-Random Number Generators	True Random Number Generators
Efficiency	Excellent	Poor
Determinism	Deterministic	Nondeterministic
Periodicity	Periodic	Aperiodic



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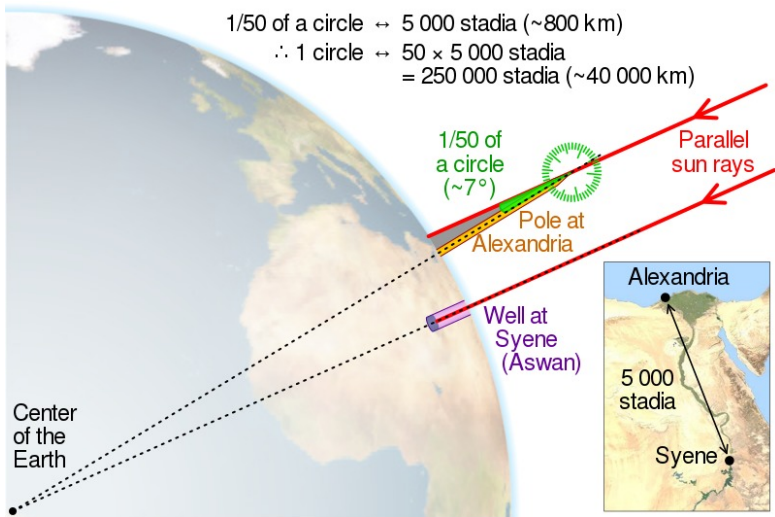
**The Sieve of Eratosthenes**

Numerical Weather Prediction





# Eratosthenes Measured the Earth



# The Sieve of Eratosthenes

**Eratosthenes was the Librarian in Alexandria when Archimedes flourished in Syracuse.**

**They were “pen-pals”.**

**Eratosthenes estimated size of the Earth.**

**He devised a systematic procedure for generating the prime numbers: the Sieve of Eratosthenes.**



# The Sieve of Eratosthenes

The idea:

- ▶ List all natural numbers up to  $n$ .
- ▶ Circle 2 and strike out all multiples of two.
- ▶ Move to the next number, 3.
- ▶ Circle it and strike out all multiples of 3.
- ▶ Continue till no more numbers can be struck out.

The numbers that have been circled are the prime numbers. Nothing else survives.

It is sufficient to go as far as  $\sqrt{n}$ .



# The Sieve of Eratosthenes

	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100



# The Sieve of Eratosthenes

	2	3		5		7		9	
11		13		15		17		19	
21		23		25		27		29	
31		33		35		37		39	
41		43		45		47		49	
51		53		55		57		59	
61		63		65		67		69	
71		73		75		77		79	
81		83		85		87		89	
91		93		95		97		99	



# The Sieve of Eratosthenes

	2	3		5		7			
11		13				17		19	
		23		25				29	
31				35		37			
41		43				47		49	
		53		55				59	
61				65		67			
71		73				77		79	
		83		85				89	
91				95		97			



# The Sieve of Eratosthenes

	2	3		5		7			
11		13				17		19	
		23						29	
31						37			
41		43				47		49	
		53						59	
61						67			
71		73				77		79	
		83						89	
91						97			



# The Sieve of Eratosthenes

	2	3		5		7			
11		13				17		19	
		23						29	
31						37			
41		43				47			
		53						59	
61						67			
71		73						79	
		83						89	
						97			





# The Sieve of Eratosthenes

	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100



# Is There a Pattern in the Primes?

It is a simple matter to make a list of all the primes less than 100 or 1000.

It becomes clear very soon that there is no clear pattern emerging.

The primes appear to be scattered at random.



Figure : Prime numbers up to 100



**The grand challenge is to find patterns  
in the sequence of prime numbers.**

**This is an enormously difficult problem  
that has taxed the imagination of the  
greatest mathematicians for centuries.**



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# Numerical Weather Prediction

**Outline of a talk on NWP  
given at UCC, March 2018.**

**~/Dropbox/TALKS/NWP-UCC/NWP-UCC.pdf**

**<https://maths.ucd.ie/~plynch/Talks/>  
See also HiRes Image on my website.**



**Thank you**

