AweSums

Marvels and Mysteries of Mathematics

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LECTURE 9

Peter Lynch
School of Mathematics & Statistics
University College Dublin

Evening Course, UCD, Autumn 2019

Outline

Outline

Meaning and Content of Mathematics

The word Mathematics comes from Greek $\mu\alpha\theta\eta\mu\alpha$ (máthéma), meaning "knowledge" or "study" or "learning".

It is the study of topics such as

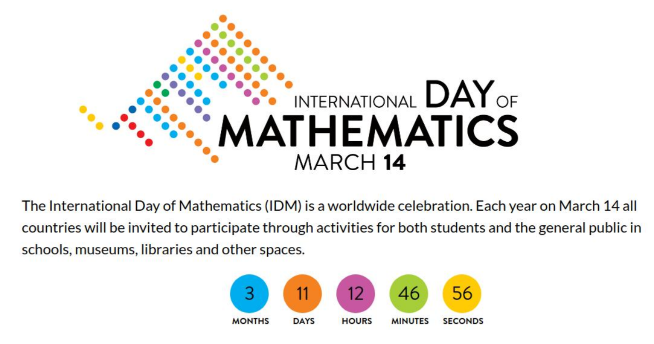
- Quantity (numbers)
- Structure (patterns)
- Space (geometry)
- Change (analysis).

See Wikipedia page on Mathematics

[Tobies & Neunzert (2012)]

International Day of Maths

- **Website:** https://www.idm314.org/



March 14 is 'so-called' Pi-Day

- International Day of Maths
- Archimedes post on blog
- Sums of squares for primes

Outline

Prime & Composite Numbers

A prime number is a number that cannot be broken into a product of smaller numbers.

The first few primes are 2, 3, 5, 7, 11, 13, 17 and 19.

There are 25 primes less than 100.

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There are 25 primes less than 100.

Numbers that are not prime are called **composite.** They can be expressed as **products of primes.**

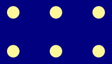
Thus, $6 = 2 \times 3$ is a composite number.

The number 1 is neither prime nor composite.

The Atoms of the Number System

A line of six spots

can be arranged in a rectangular array:



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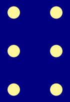

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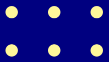

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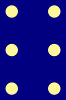

Note that

$$2 \times 3 = 3 \times 2$$

This is the **commutative law of multiplication.**

The Atoms of the Number System

The primes play a role in mathematics analogous to the elements of Mendeleev's Periodic Table.

Just as a chemical molecule can be constructed from the 100 or so fundamental elements, any whole number be constructed by combining prime numbers.

The primes 2, 3, 5 are the hydrogen, helium and lithium of the number system.

Some History

In 1792 Carl Friedrich Gauss, then only 15 years old, found that the proportion of primes less that n decreased approximately as $1/\log n$.

Around 1795 Adrien-Marie Legendre noticed a similar logarithmic pattern of the primes, but it was to take another century before a proof emerged.

In a letter written in 1823 the Norwegian mathematician Niels Henrik Abel described the distribution of primes as *the most remarkable result in all of mathematics.*

Percentage of Primes Less than *N*

Table : Percentage of Primes less than *N*

100	25	25.0%
1,000	168	16.8%
1,000,000	78,498	7.8%
1,000,000,000	50,847,534	5.1%
1,000,000,000,000	37,607,912,018	3.8%

We can see that the percentage of primes is falling off with increasing size.

But the rate of decrease is very slow.

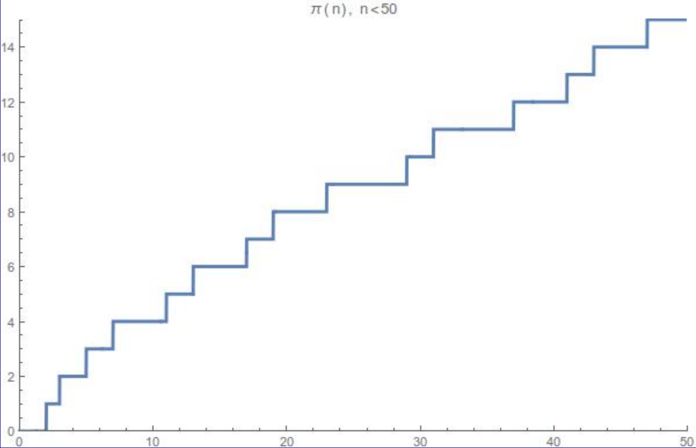

Figure : The prime counting function $\pi(n)$ for $0 \leq n \leq 50$.

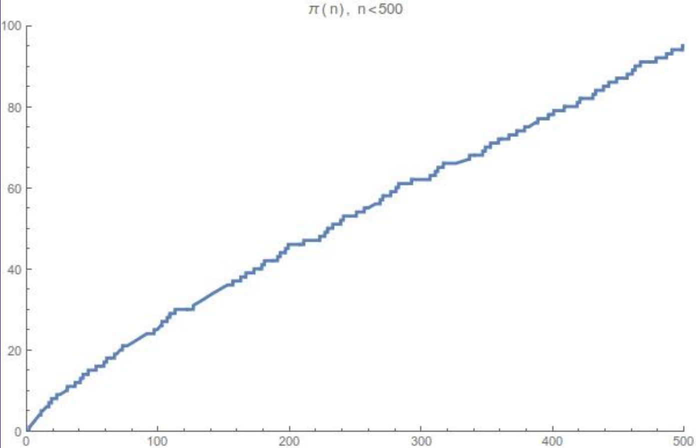

Figure : The prime counting function $\pi(n)$ for $0 \leq n \leq 500$.

Is There a Pattern in the Primes?

It is a simple matter to make a list of
all the primes less that 100 or 1000.

It becomes clear very soon that
no clear pattern is emerging.

The primes appear to be scattered at random.

Figure : Prime numbers up to 100

Is There a Pattern in the Primes?

Do the primes settle down as *n* becomes larger?

Between 9,999,900 and 10,000,000 (100 numbers) there are 9 primes.

Between 10,000,000 and 10,000,100 (100 numbers) there are just 2 primes.

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(100 numbers) there are 9 primes.

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(100 numbers) there are just 2 primes.

What kind of function could generate this behaviour?

We just do not know.

Is There a Pattern in the Primes?

The gaps between primes are very irregular.

- Can we find a pattern in the primes?
- Can we find a formula that generates primes?
- How can we determine the hundreth prime?
- What is the thousanth? The millionth?

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WolframAlpha is a Computational Knowledge Engine.

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Wolfram Alpha is based on Wolfram's flagship product Mathematica, a computational platform or toolkit that encompasses computer algebra, symbolic and numerical computation, visualization, and statistics.

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It is freely available through a web browser.

Euler’s Formula for Primes

No mathematician has ever found a *useful* formula that generates all the prime numbers.

Euler found a beautiful little formula:

$$n^2 - n + 41$$

This gives prime numbers for n between 1 and 40.

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Euler found a beautiful little formula:

$$n^2 - n + 41$$

This gives prime numbers for n between 1 and 40.

But for $n = 41$ we get

$$41^2 - 41 + 41 = 41 \times 41$$

a composite number.

The Infinitude of Primes

Euclid proved that there is no finite limit to the number of primes.

His proof is a masterpiece of symplicity.

(See Dunham book or Wikipedia: *Euclid's Theorem.*)

Some Unsolved Problems

There appear to be an infinite number of prime pairs

$$(2n-1, 2n+1)$$

There are also gaps of arbitrary length:

for example, there are 13 consecutive composite numbers between 114 and 126.

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There are also gaps of arbitrary length:

for example, there are 13 consecutive composite numbers between 114 and 126.

We can find gaps as large as we like:

Show that $N! + 1$ is followed by a sequence of $N - 1$ composite numbers.

Primes have been used as markers of civilization.

In the novel Cosmos, by Carl Sagan,
the heroine detects a signal:

- First 2 pulses
- Then 3 pulses
- Then 5 pulses
- ...
- Then 907 pulses.

In each case, a prime number of pulses.

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- **Then 5 pulses**
- **. . .**
- **Then 907 pulses.**

In each case, a prime number of pulses.

Could this be due to any natural phenomenon?
Is it evidence of extra-terrestrial intelligence?

Which Primes are Sums of Squares?

```
(* PRINT THE FIRST 100 PRIME NUMBERS *)

primes = {};
For[i = 1, i < 100, i ++, AppendTo[primes, Prime[i]]]
Print["PRIMES"]
primes

PRIMES

Out[51]= {2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43,
 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, 101,
 103, 107, 109, 113, 127, 131, 137, 139, 149, 151,
 157, 163, 167, 173, 179, 181, 191, 193, 197, 199,
 211, 223, 227, 229, 233, 239, 241, 251, 257, 263,
 269, 271, 277, 281, 283, 293, 307, 311, 313, 317,
 331, 337, 347, 349, 353, 359, 367, 373, 379, 383,
 389, 397, 401, 409, 419, 421, 431, 433, 439, 443,
 449, 457, 461, 463, 467, 479, 487, 491, 499, 503,
 509, 521, 523}

(* PRINT THE FIRST 100 SQUARE NUMBERS *)

squares = {};
```

Which Primes are Sums of Squares?

```
509, 521, 523}
(* PRINT THE FIRST 100 SQUARE NUMBERS *)
squares = {};
For[i = 1, i < 25, i++, AppendTo[squares, i^2]]
Print["SQUARES"]
squares
SQUARES
Out[63]= {1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121,
 144, 169, 196, 225, 256, 289, 324, 361, 400,
 441, 484, 529, 576}
Prime[1 000 000 000]
Out[69]= 22 801 763 489
```

Which Primes are Sums of Squares?

A Theorem of Fermat states that:

A prime number n may be expressed as a sum of squares if and only if

$$p \equiv 1 (\mathrm{mod}) 4$$

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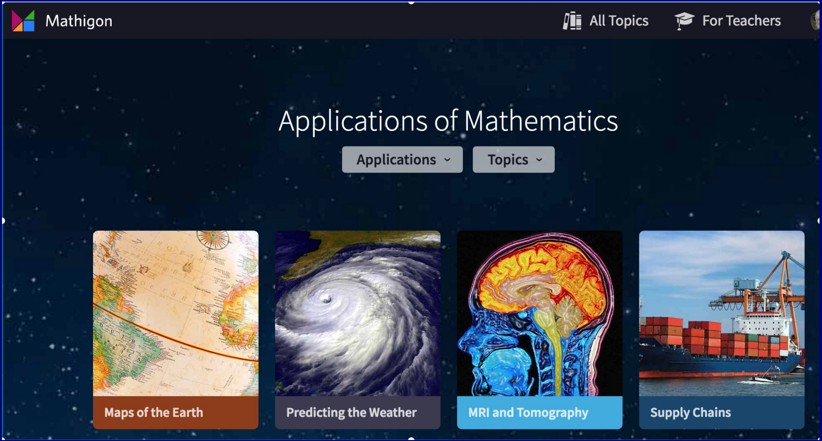
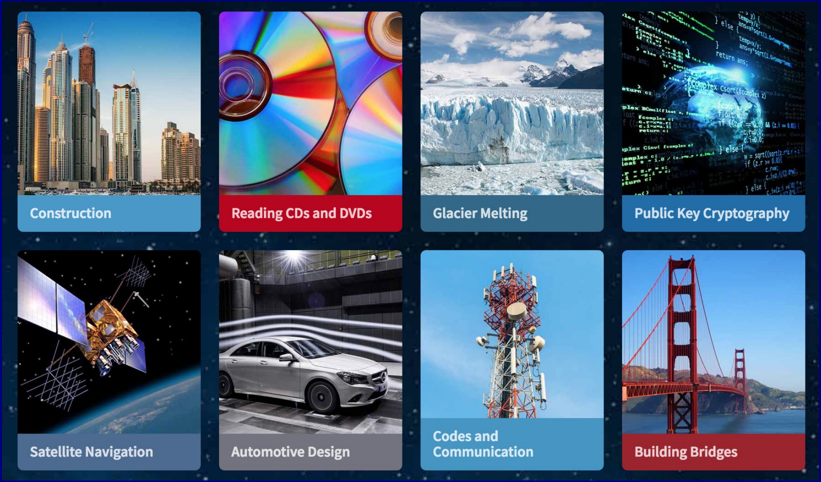
In plain language, if n divided by 4 has remainder 1.

Outline

Applications on mathigon.org




Maps of the Earth
Predicting the Weather
MRI and Tomography
Supply Chains
Finance and Banking
Internet and Phones
Cosmology
Computers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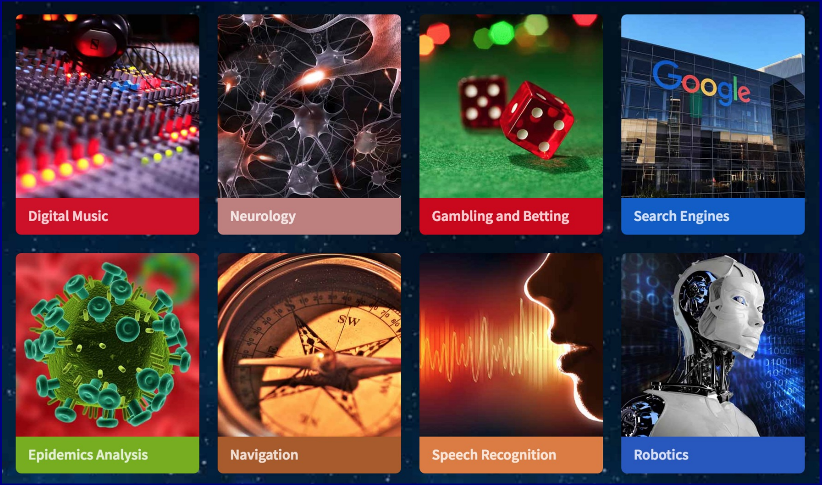
Reading CDs and DVDs
Glacier Melting
Public Key Cryptography
Satellite Navigation
Automotive Design
Codes and Communication
Building Bridges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Neurology
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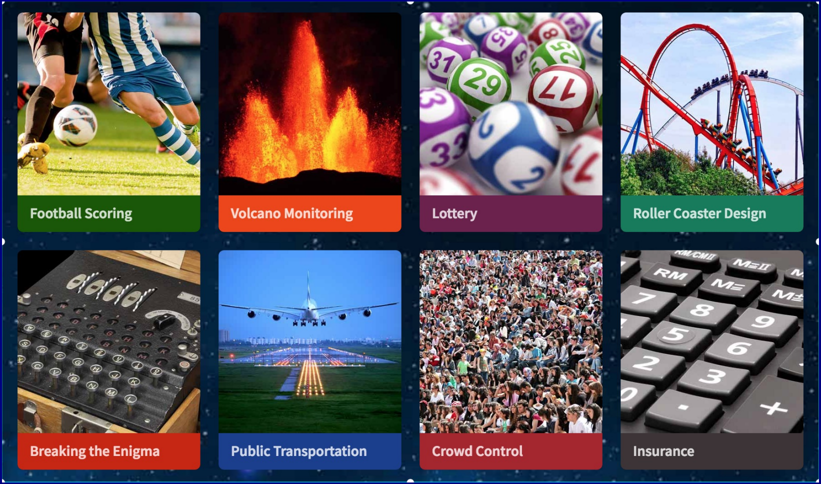
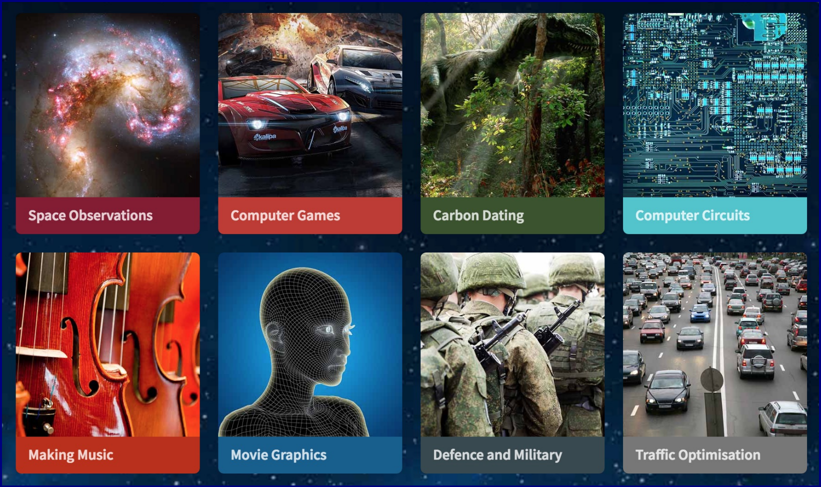
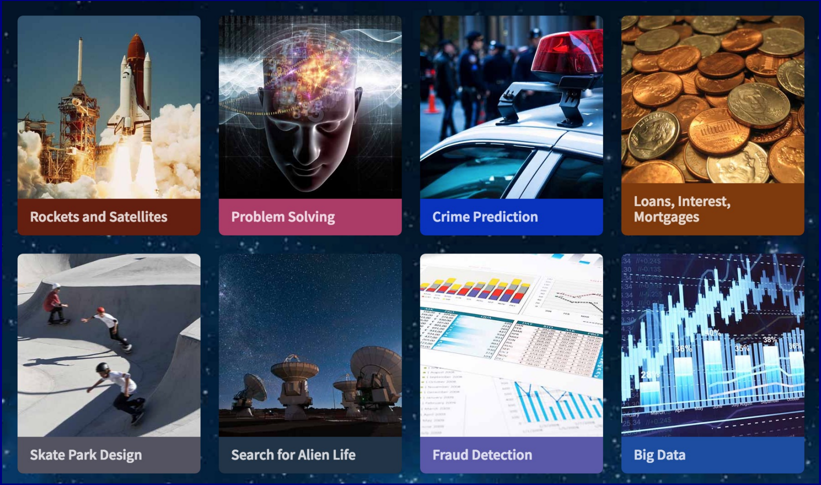
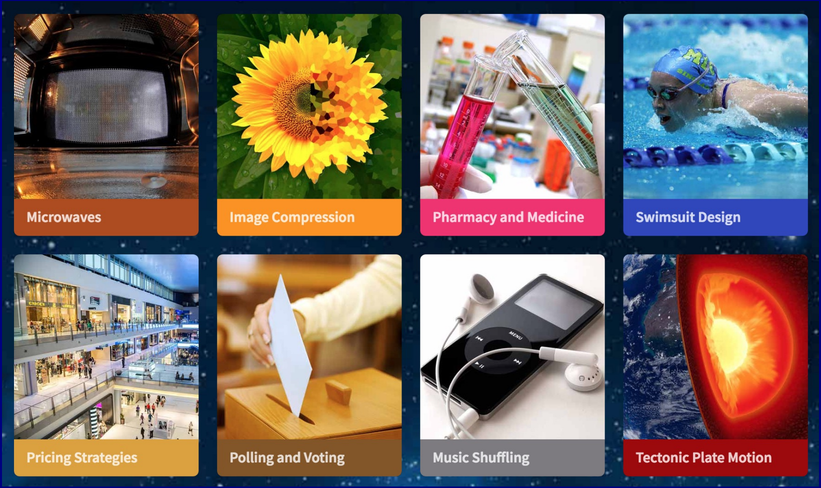
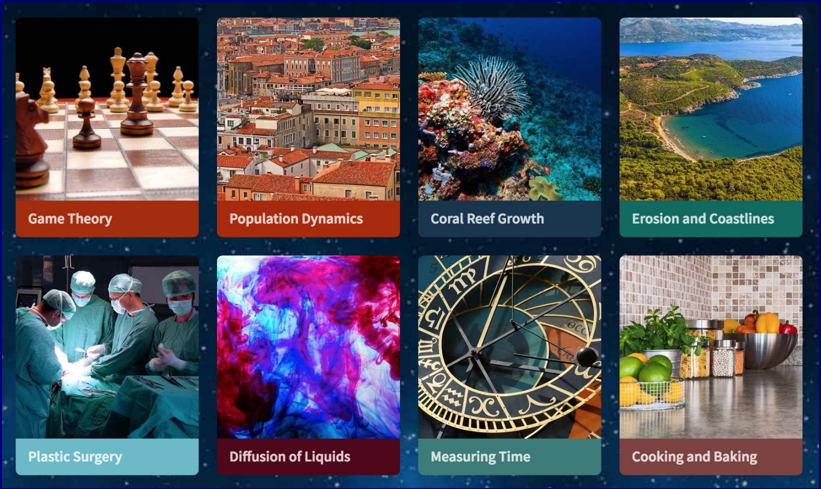
Google
Search Engines
Epidemics Analysis
SW
Navigation
Speech Recognition
Robotics


Football Scoring
Volcano Monitoring
Lottery
Roller Coaster Design
Breaking the Enigma
Public Transportation
Crowd Control
Insurance


Space Observations
Computer Games
Carbon Dating
Computer Circuits
Making Music
Movie Graphics
Defence and Military
Traffic Optimisation


Rockets and Satellites
Problem Solving
Crime Prediction
Loans, Interest, Mortgages
Skate Park Design
Search for Alien Life
Fraud Detection
Big Data


Microwaves
Image Compression
Pharmacy and Medicine
Swimsuit Design
Pricing Strategies
Polling and Voting
Music Shuffling
Tectonic Plate Motion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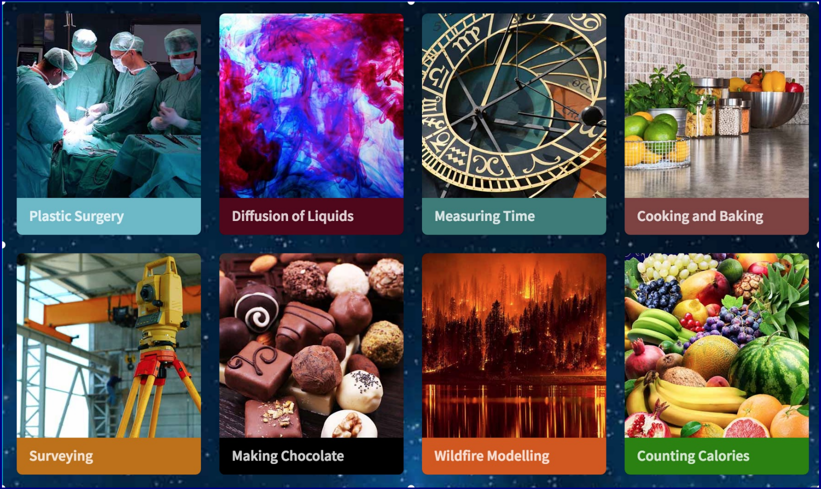
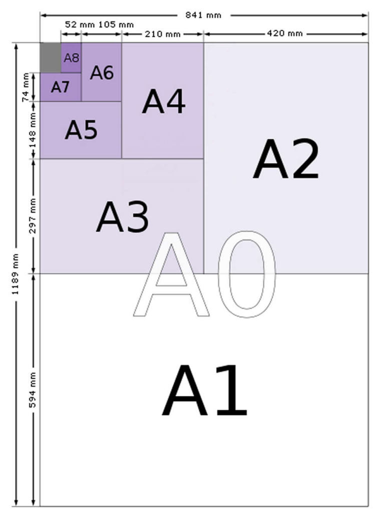
Population Dynamics
Coral Reef Growth
Erosion and Coastlines
Plastic Surgery
Diffusion of Liquids
Measuring Time
Cooking and Baking


Plastic Surgery
Diffusion of Liquids
Measuring Time
Cooking and Baking
Surveying
Making Chocolate
Wildfire Modelling
Counting Calories

Outline

Standard Paper Sizes



Standard sizes of A-series paper.

The ratio of heights to widths is always $\sqrt{2}$.

Making a Square

The standard sizes of paper are designed so that each has the same shape (or aspect ratio), and the largest, A0, has an area of one square metre.

PUZZLE:
Is it possible to form a square out of sheets of A4 sized paper (without them overlapping)?

Outline

Distraction: A Curious Year, AD 1089

What is so special about the year 1089?

- Palmyra destroyed by an earthquake.
- First Cistercian monastery, Cîteaux Abbey, founded in southern France.
- The Council of Melfi issues decrees against simony and clerical marriage.

Such vital information is obtained from Wikipedia.

Distraction: A Curious Number

Think of a three-digit number, for example 275.

Calculate the difference between this number and the number formed by reversing digits:

$$572 - 275 = 297$$

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Calculate the difference between this number and the number formed by reversing digits:

$$572 - 275 = 297$$

Now repeat the process, this time adding numbers:

$$297 + 792 = 1089$$

What is so special about the number 1089?

Distraction: A Curious Number

This “trick” nearly always works.

But it can fail in some cases.

Can you find the conditions for success?

See the Wikipedia page “1089 (number)”.

Outline

Topology: a Major Branch of Mathematics

Topology is all about continuity and connectivity.

Here are some of the topics in Topology:

- The Bridges of Königsberg
- Doughnuts and Coffee-cups
- Knots and Links
- Nodes and Edges: Graphs
- The Möbius Band

In this lecture, we look at Knots and Links.

Pretzel Puzzle

Look at the two "pretzels" here:

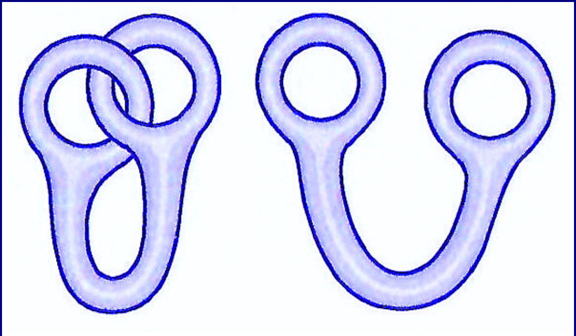


Figure : Two "Pretzels". Are they equivalent?

Knot Theory

A **knot** is an embedding of the unit circle S^1 into three-dimensional space R^3.

Two knots are equivalent if one can be distorted into the other without breaking it.

A knot is a mapping of the unit circle into three-space.

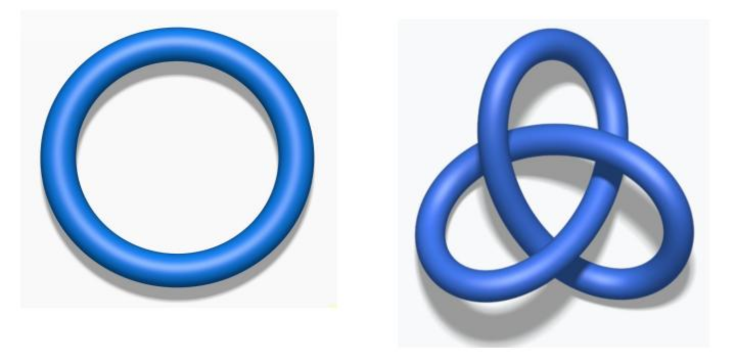


Figure : Left: Unknot. Right: Trefoil.

These two knots aren't equivalent: we can't distort the circle into the trefoil without breaking it.

Knots that are Mirror Images

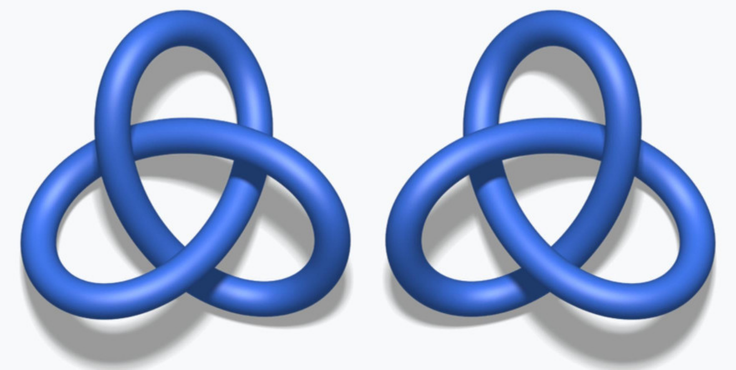


Figure : Levo and Dextro Trefoils.

These knots are not equivalent. We cannot change one into the other without breaking it.

The Simplest Knots and Links

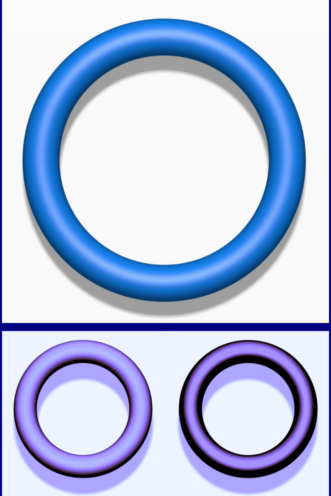


Figure : Top: The Unknot. Bottom: The Unlink.

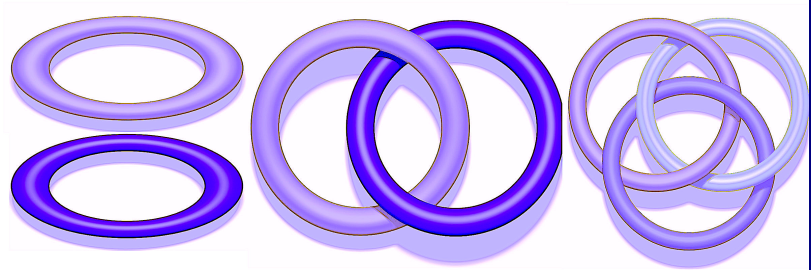


Figure : Unlink, Hopf Link and Borromean Rings.

The Hopf Link

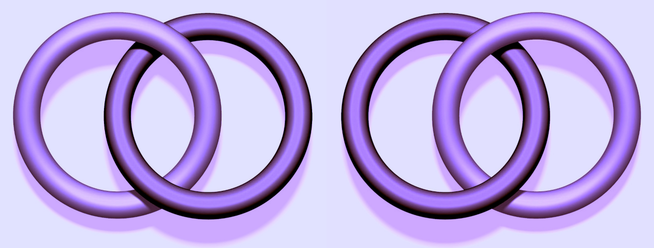


Figure : The Hopf Link and its mirror image. Equivalent?

Rings of Borromeo

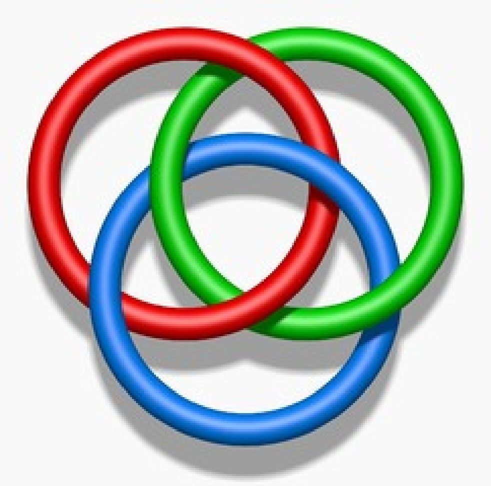


Figure : No two rings are linked! Are the three?

Genus of a Surface

The genus of a topological surface is,
in simple terms, the number of holes in it.

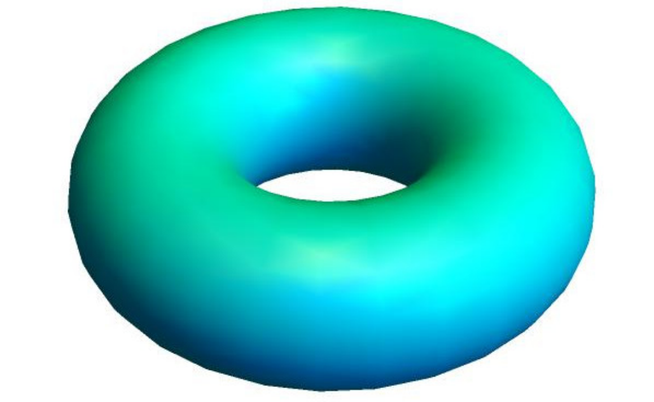
A sphere has no holes, so has genus 0.

A donut has one hole, so has genus 1.

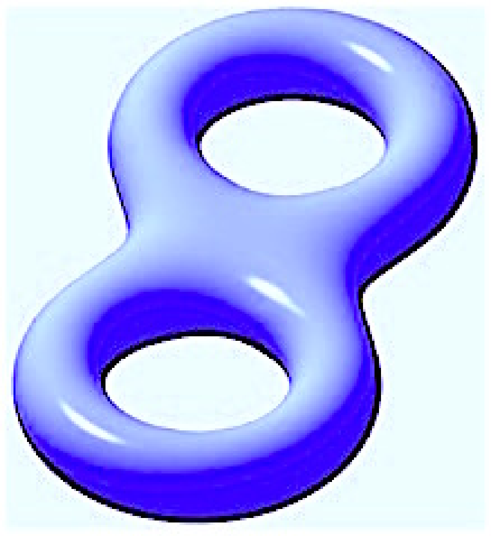
Surfaces can have any number of holes; any genus.

The Sphere, of Genus 0

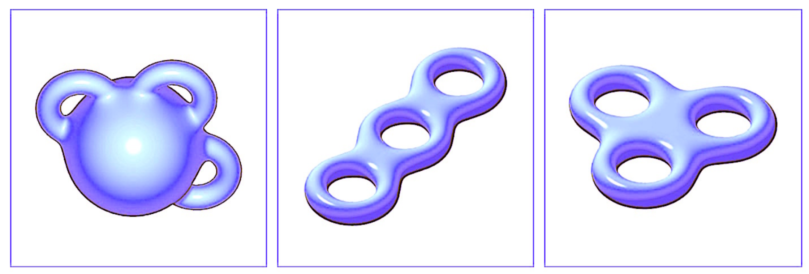
The Torus, of Genus 1



The Double Torus, of Genus 2



Some Surfaces of Genus 3



Topologists have classified all surfaces in 3-space.

Link between Number Theory and Physics

Forty years ago, physics and and topology had little or nothing to do with one another.

In the 1980s, mathematicians and physicists found ways to use physics to study the properties of shapes.

The field has never looked back.

http://www.quantamagazine.org/secret-link-uncovered-between-pure-math-and-physics-20171201/

Triple Torus

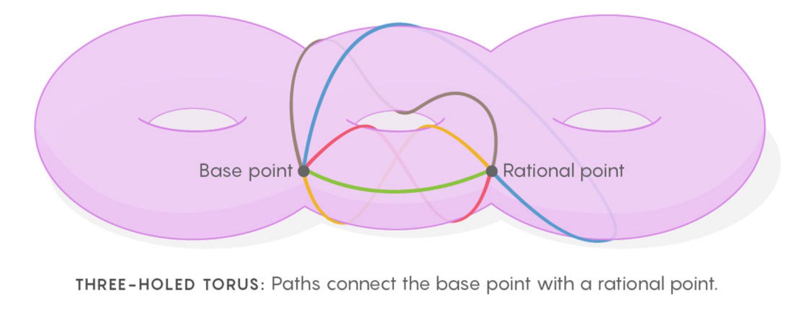


Figure : Rational solutions of $x^4 + y^4 = 1$ are on this surface

Pretzel Puzzle

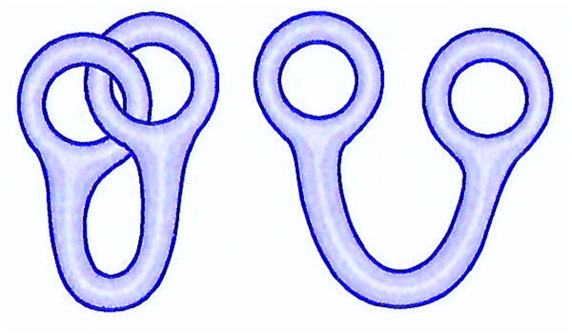


Figure : Two “Pretzels”. Are they equivalent?

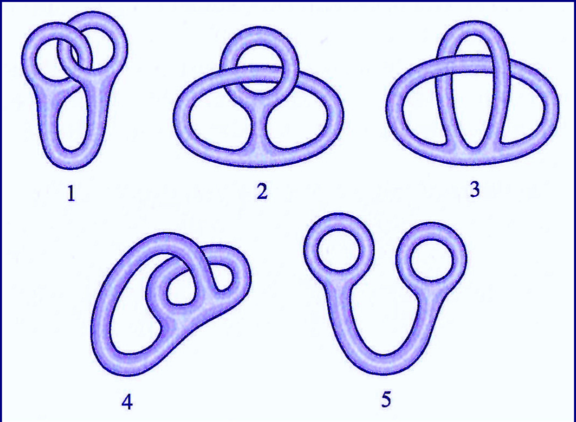


Figure : Equivalence!

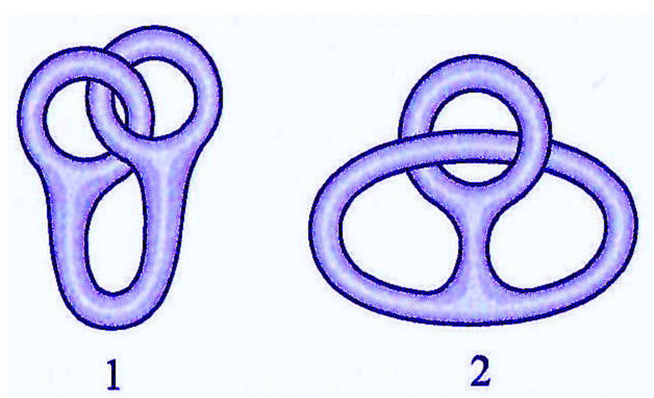


Figure : Make the left-hand loop bigger.

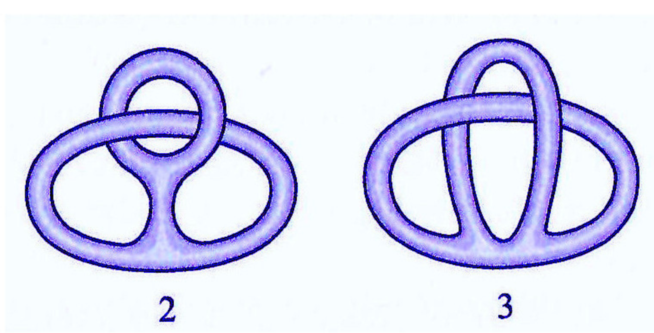


Figure : Make the other loop bigger.

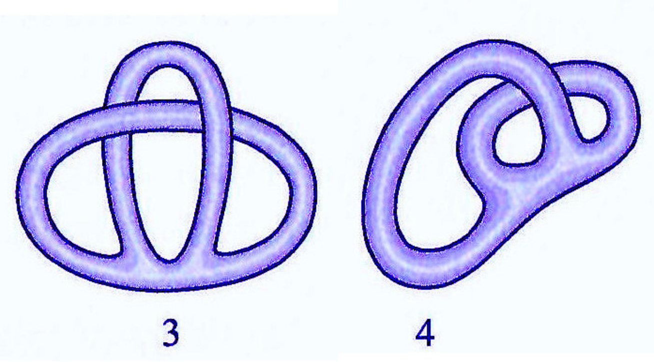


Figure : Pull the top loop away to the side.

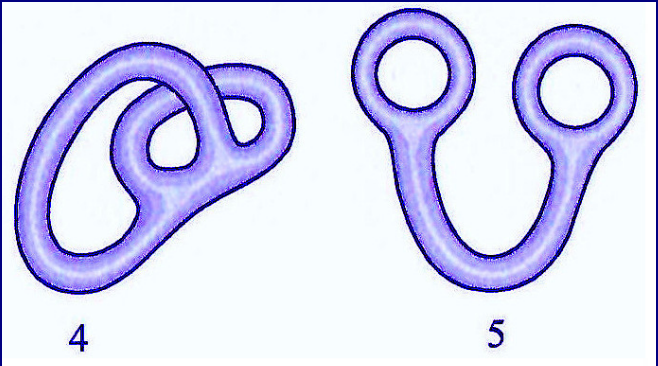


Figure : Smoothly distort to the final form.

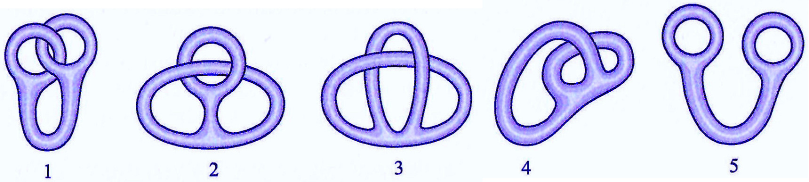


Figure : Combining all the distortions. Equivalence!

Another Surprising Result

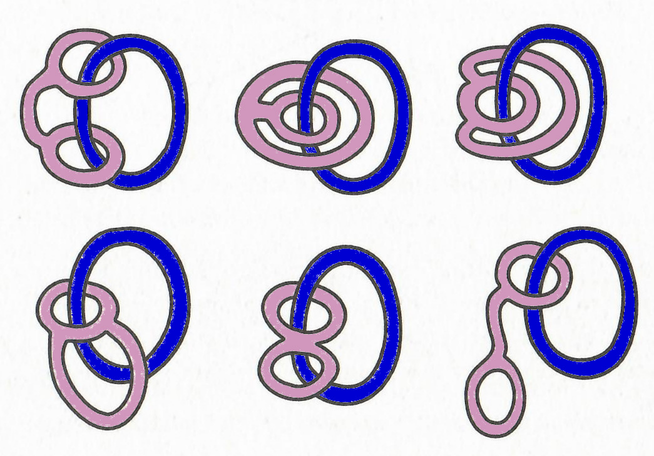


Figure : We can unlink one of the hand-cuffs.

Outline

Source of Some Puzzles

Mathematical Lateral Thinking Puzzles
by
Paul Slone & Des MacHale

Slicing a Cake with One Cut

Bake a cake that you can slice into 6 equal pieces with one cut?

Hint: The cake can be any shape you like

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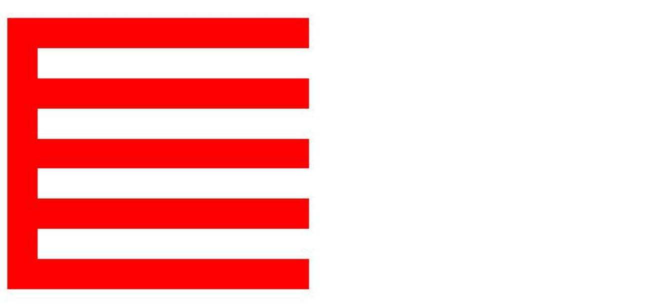
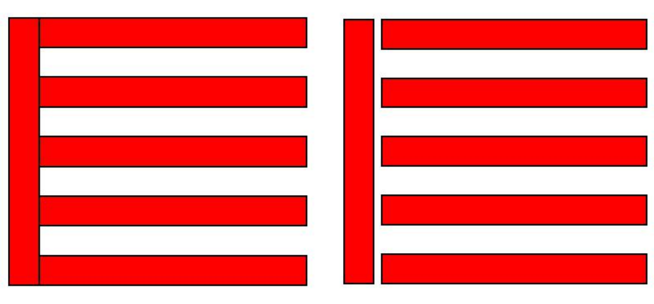


Slicing a Cake with One Cut

Bake a cake that you can slice into 6 equal pieces with one cut?

Hint: The cake can be any shape you like



Student Solution: Snake Cake

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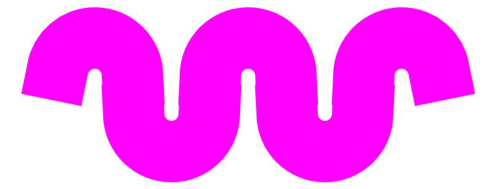

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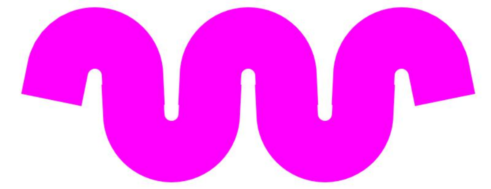

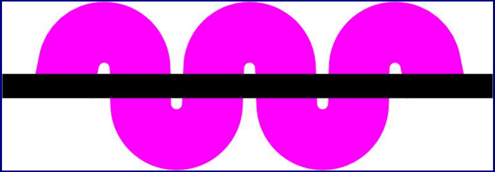

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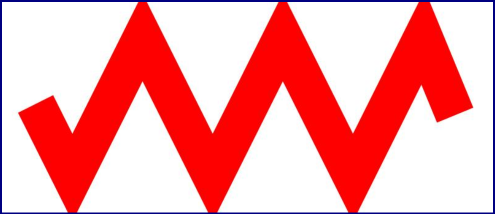

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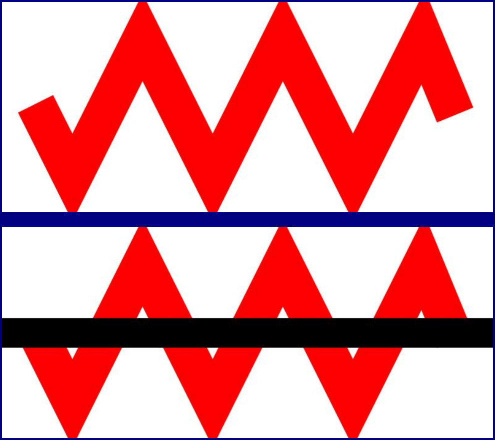
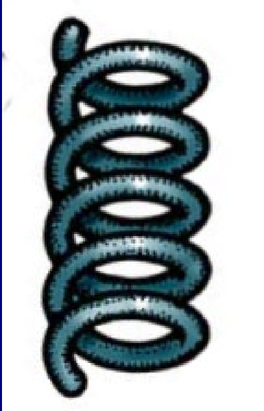


A Three-dimensional Cake

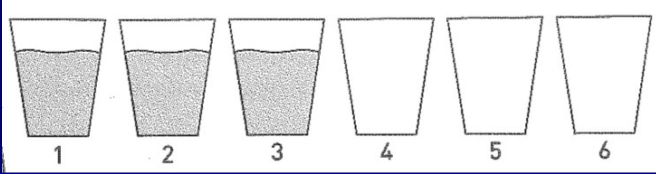


Cake in the form of a helix.

This is like twist . . .

. . . pastry twisted round a stick and cooked over a camp-fire.

Rearrange Six Glasses



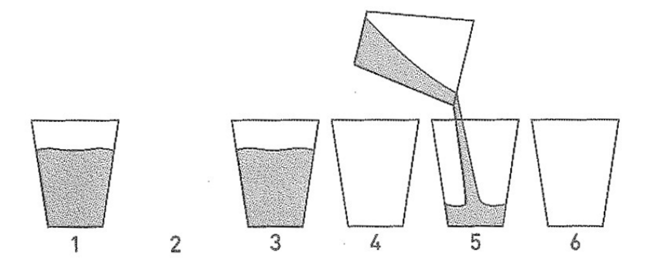
There are six glasses in a row.

Glasses 1, 2 and 3 are full.
Glasses 4, 5 and 6 are empty.

How can you arrange for the full and empty glasses to alternate, moving only one glass?

Rearrange Six Glasses

First, pour water from Glass 2 into glass 5:



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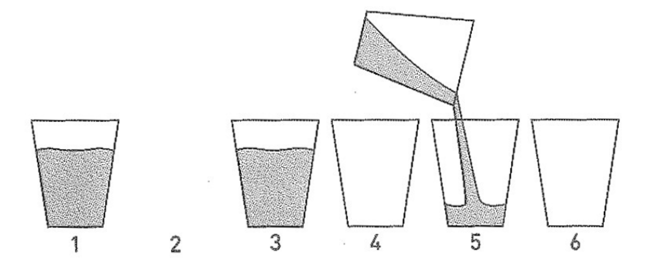
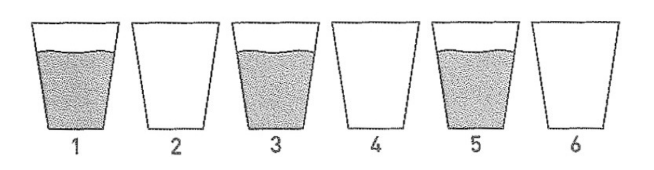


Then, place Glass 2 in its original position:



Outline

David Hilbert (1862–1943)

David Hilbert, from a contemporary postcard.

Hilbert's Problems

In August 1900, David Hilbert addresed the International Congress of Mathematicians in the Sorbonne in Paris:

> ***"Who of us would not be glad to lift the veil behind which the future lies hidden; to cast a glance at the next advances of our science and at the secrets of its development during future centuries?"***

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Hilbert presented 23 problems that challenged mathematicians through the twentieth century.

Hilbert’s Problems

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 37, Number 4, Pages 407–436
S 0273-0979(00)00881-8
Article electronically published on June 26, 2000

MATHEMATICAL PROBLEMS

DAVID HILBERT

Lecture delivered before the International Congress of Mathematicians at Paris in 1900.

Hilbert’s eighth problem concerned itself with what is called the Riemann Hypothesis (RH).

RH is generally regarded as the deepest and most important unproven mathematical problem.

Anyone who can prove it is assured of lasting fame.

Why is RH Important?

A large number of mathematical theorems (1000's) depend for their validity on the RH.

Were RH to turn out to be false, many of these mathematical arguments would simply collapse.

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Were RH to turn out to be false, many of these mathematical arguments would simply collapse.

In 2000, industrialist Landon Clay donated $7M, with $1M for each of 7 problems in mathematics.

The Riemann hypothesis is one of these problems.

http://www.claymath.org/millennium-problems

Why is RH Important?

Whoever proves Riemann's hypothesis will have completed thousands of theorems that start like this:

"Assuming that the Riemann hypothesis is true . . . ".

He or she will be assured of lasting fame.

Those who establish fundamental mathematical results probably come closer to immortality than almost anyone else.

Outline

What is Randomness?

Randomness is a slippery concept,
defying precise definition.

Toss a coin and get a sequence like 1001110100.

Some uses of Random Numbers:

- Computer simulations of fluid flow.
- Interactions of subatomic particles.
- Evolution of galaxies.

Tossing coins is impractical.
We need more effective methods.

Defining Randomness?

Richard von Mises (1919):

A binary sequence is random if the proportion of zeros and ones approaches 50% and if this is also true for any sub-sequence.

Consider (0101010101).

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Consider (0101010101).

Andrey Kolmogorov defined the complexity of a binary sequence as the length of a computer program or algorithm that generates it.

The phrase a sequence of one million 1s completely defines a sequence.

Non-random sequences are compressible.
Randomness and incompressibility are equivalent.

Pseudo-random versus Truly Random

Pseudo-random number generators are algorithms that use mathematical formulae to produce sequences of numbers.

The sequences appear completely random and satisfy various statistical conditions for randomness.

Pseudo-random numbers are valuable for many applications but they have serious difficiencies.

Truly Random Number Generators

True random number generators extract randomness from physical phenomena that are completely unpredictable.

Atmospheric noise is the **static generated by lightning** [globally there are 40 flashes/sec]. It can be detected by an ordinary radio.

Truly Random Number Generators

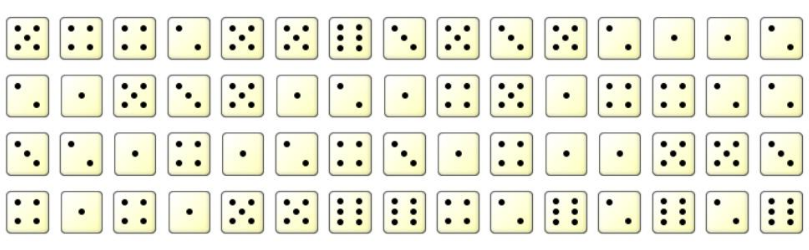
Atmospheric noise passes all the statistical checks for randomness.

Dr Mads Haahr of Trinity College, Dublin uses atmospheric noise to produce random numbers.

Results available on on the website: `random.org`.

20 Random Coin Tosses

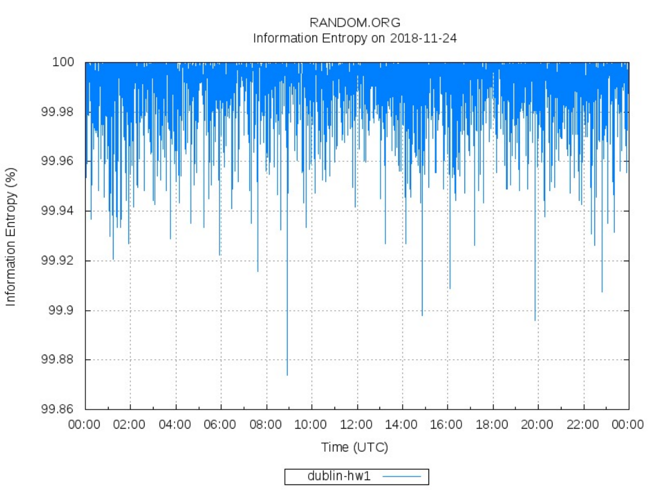
60 Dice Rolls



100 Random Numbers in [0,99]

17	60	57	66	4	71	59	36	8	49
87	64	94	82	6	38	14	87	76	72
97	38	44	59	56	24	20	6	24	97
0	40	14	77	18	98	41	39	6	79
21	59	49	86	91	81	65	64	3	11
92	17	65	6	37	98	84	17	70	93
60	52	1	98	20	2	65	9	57	3
48	86	27	3	71	51	57	56	2	2
13	14	73	65	11	32	17	7	91	37
3	8	10	67	0	72	0	42	15	24

Quality of Random Numbers



PRNG versus TRNG

Characteristic	Pseudo-Random Number Generators	True Random Number Generators
Efficiency	Excellent	Poor
Determinism	Determinstic	Nondeterministic
Periodicity	Periodic	Aperiodic

Outline

Numerical Weather Prediction

Outline of a talk on NWP
given at UCC, March 2018.

`~/Dropbox/TALKS/NWP-UCC/NWP-UCC.pdf`

`https://maths.ucd.ie/~plynch/Talks/`

Thank you