AweSums

Marvels and Mysteries of Mathematics

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LECTURE 8

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Evening Course, UCD, Autumn 2019

Outline

Outline

Meaning and Content of Mathematics

The word Mathematics comes from Greek $\mu\alpha\theta\eta\mu\alpha$ (máthéma), meaning "knowledge" or "study" or "learning".

It is the study of topics such as

- **Quantity (numbers)**
- **Structure (patterns)**
- **Space (geometry)**
- **Change (analysis).**

Outline

Pascal's Triangle

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

Combinatorial Symbol

$$\binom{n}{r} \quad \text{“}n \text{ choose } r\text{”}$$

This symbol represents the number of combinations of r objects selected from a set of n objects.

Pascal's Triangle: Combinations

$$\binom{0}{0}$$

$$\binom{1}{0} \quad \binom{1}{1}$$

$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$

$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$

$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$

Pascal's Triangle

Pascal's triangle is a triangular array of the binomial coefficients.

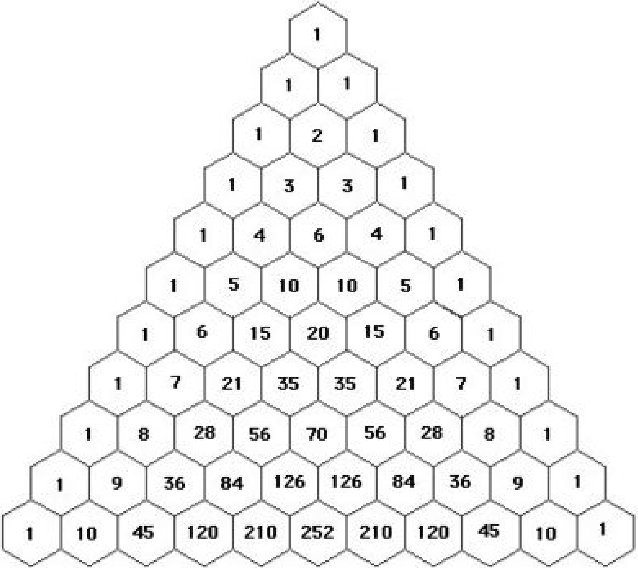
It is named after French mathematician *Blaise Pascal.*

It was studied centuries before him in:

- **India (Pingala, C2BC)**
- **Persia (Omar Khayyam, C11AD)**
- **China (Yang Hui, C13AD).**

Pascal's *Traité du triangle arithmétique* (Treatise on Arithmetical Triangle) was published in 1665.

Draw Pascal's triangle on the board.


1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1

Pascal's Triangle

The rows of Pascal's triangle are numbered starting with row n = 0 at the top (0-th row).

The entries in each row are numbered from the left beginning with k = 0.

The triangle is easily constructed:

- **A single entry 1 in row 0.**
- **Add numbers above for each new row.**

The entry in the nth row and k-th column of Pascal's triangle is denoted $\binom{n}{k}$.

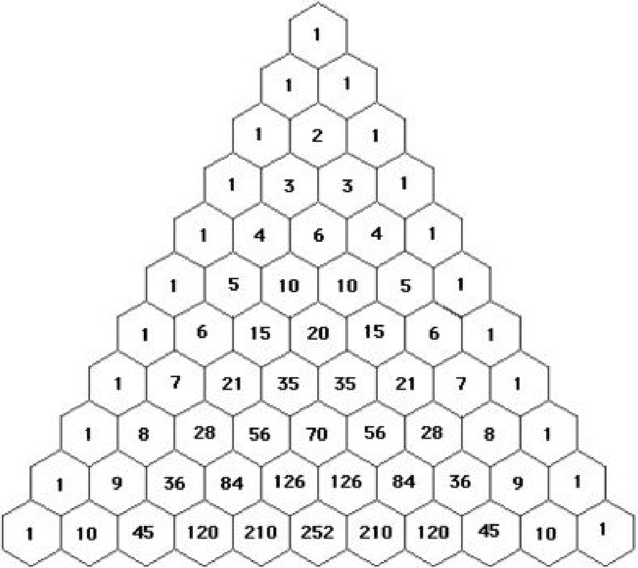
The entry in the topmost row is $\binom{0}{0} = 1$.

Pascal's Identity

The construction of the triangle may be written:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

This relationship is known as Pascal's Identity.


1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1

Pascal’s Triangle & Fibonacci Numbers.

1							
1	1						
1	2	1					
1	3	3	1				
1	4	6	4	1			
1	5	10	10	5	1		
1	6	15	20	15	6	1	
1	7	21	35	35	21	7	1

Figure : Pascal’s Triangle and Fibonacci Numbers

Where are the Fibonacci Numbers hiding here?

Sierpinski's Gasket

Sierpinski Gasket: usual construction by trisection.

Sierpinski's Gasket is constructed by starting with an equilateral triangle, and successively removing the central triangle at each scale.

Sierpinski's Gasket at Stage 6

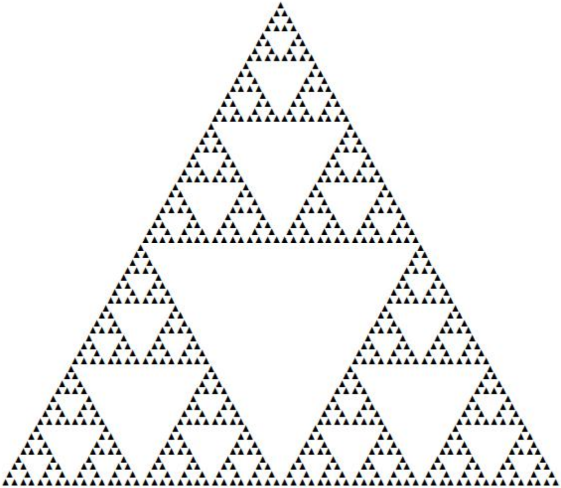


Figure : Result after 6 subdivisions

Sierpinski's Gasket in Pascal's Triangle

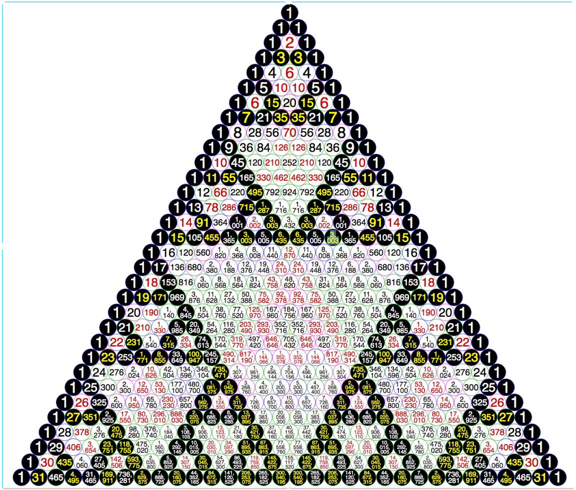


Figure : Odd numbers are in black

Remember Walking in Manhattan?

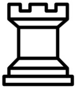
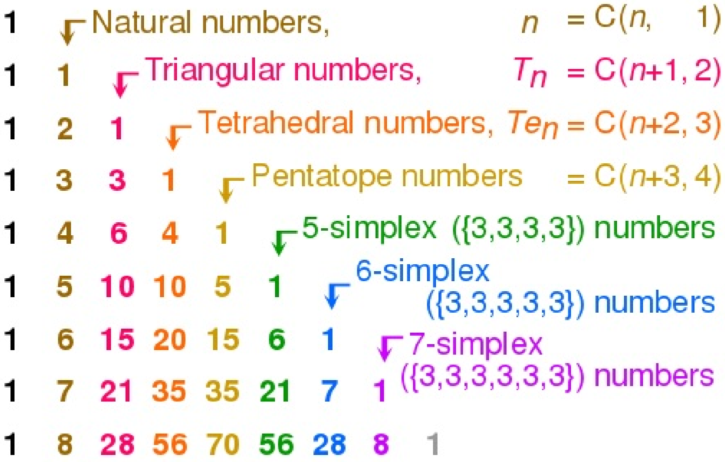
	1	1	1
1	2	3	4
1	3	6	10
1	4	10	20

Figure : Number of routes for a rook in chess.

Geometric Numbers in Pascal's Triangle

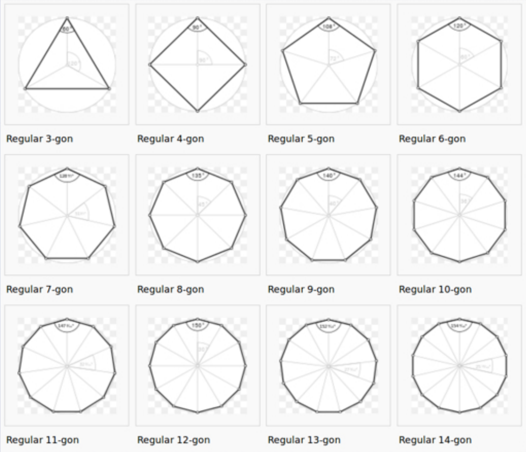


Outline

Euler's polyhedron formula.

Carving up the globe.

Regular Polygons



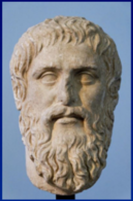
The Platonic Solids
(polyhedra)

Tetrahedron (four faces)	Cube or hexahedron (six faces)	Octahedron (eight faces)	Dodecahedron (twelve faces)	Icosahedron (twenty faces)
				

These five regular polyhedra were discovered in ancient Greece, perhaps by Pythagoras.

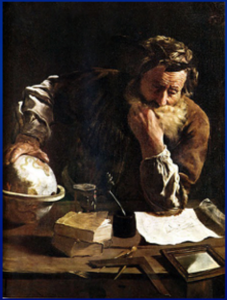
Plato used them as models of the universe.

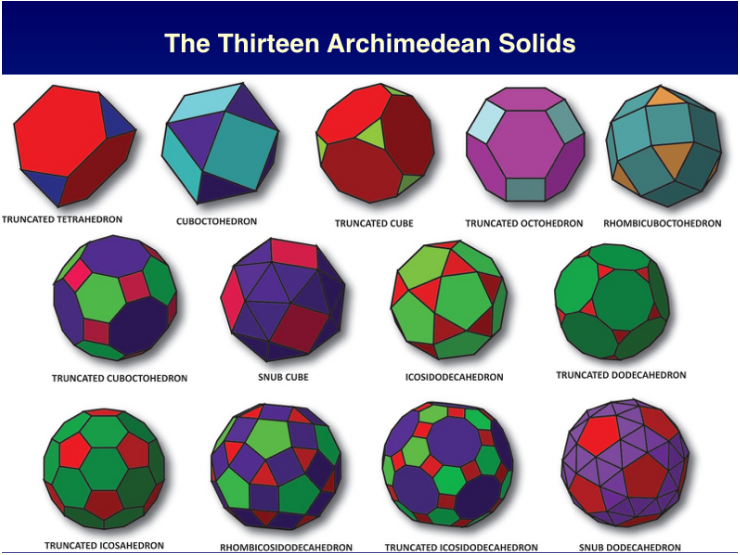
They are analysed in Book XIII of Euclid's Elements.



There are only five **Platonic** solids.

But **Archimedes** found, using different types of polygons, that he could construct 13 new solids.





Check V - E + F for the Truncated Cube

Euler's Polyhedron Formula

The great Swiss mathematician, **Leonard Euler**, noticed that, for all (convex) polyhedra,

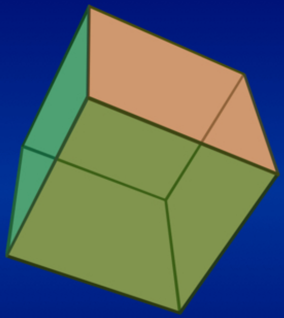
$$V - E + F = 2$$

where

- V = Number of vertices
- E = Number of edges
- F = Number of faces

Mnemonic: Very Easy Formula

For example, a Cube

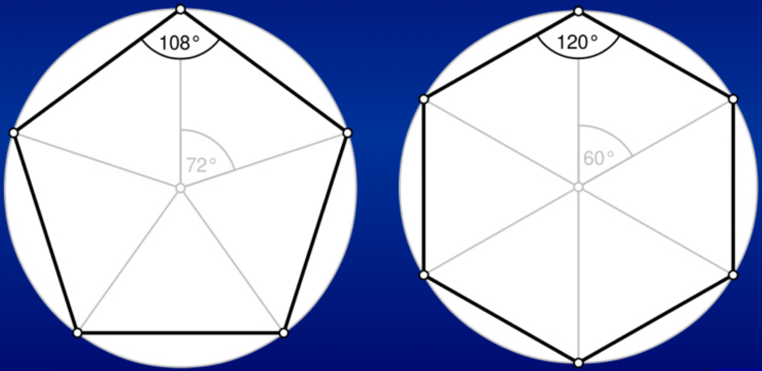


Number of vertices: V = 8
Number of edges: E = 12
Number of faces: F = 6

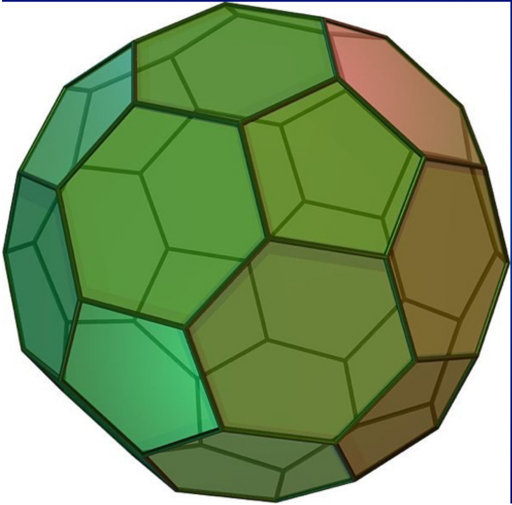
$$(V - E + F) = (8 - 12 + 6) = 2$$

Mnemonic: Very Easy Formula

Pentagons and Hexagons



The Truncated Icosahedron

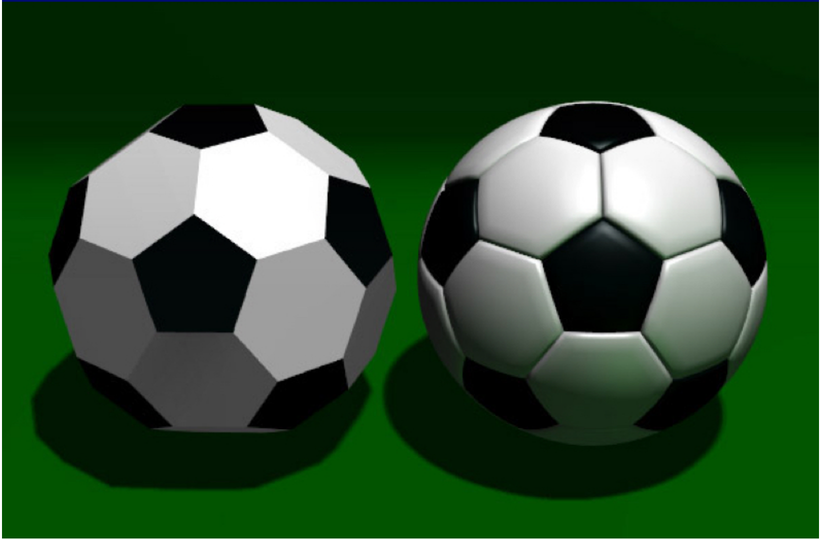


An Archimedean solid
with
pentagonal and
hexagonal faces.

The Truncated Icosahedron

Whare have you seen this before?

The Truncated Icosahedron



The "Buckyball", introduced at the the 1970 World Cup Finals in Mexico.

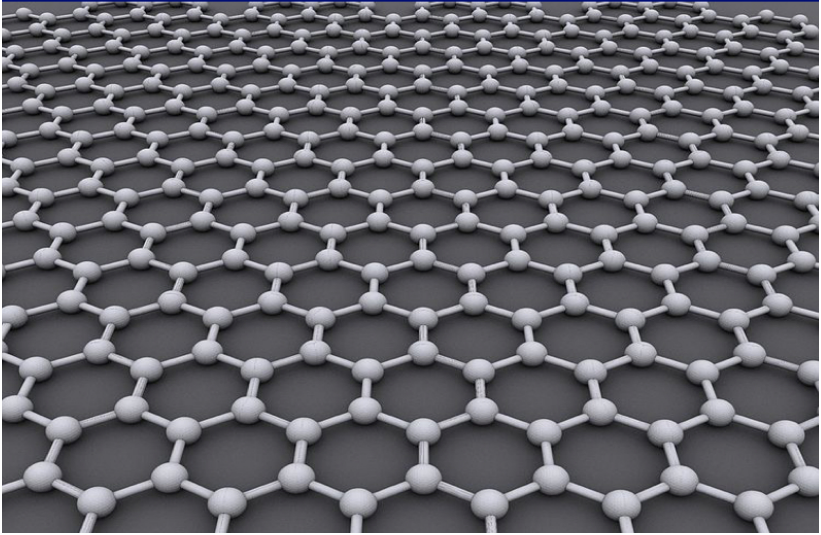
It has 32 panels: 20 hexagons and 12 pentagons.

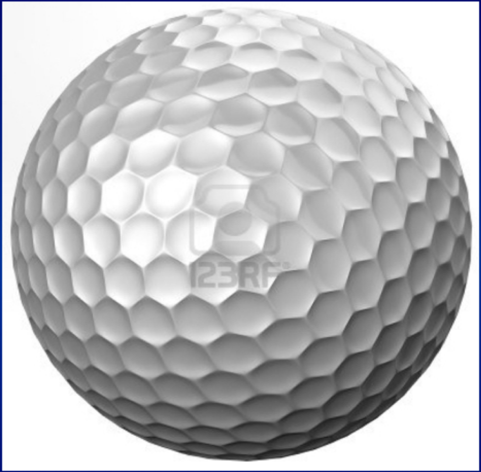

A Geodesic Dome designed by the American architect
Richard Buckminster "Bucky" Fuller.

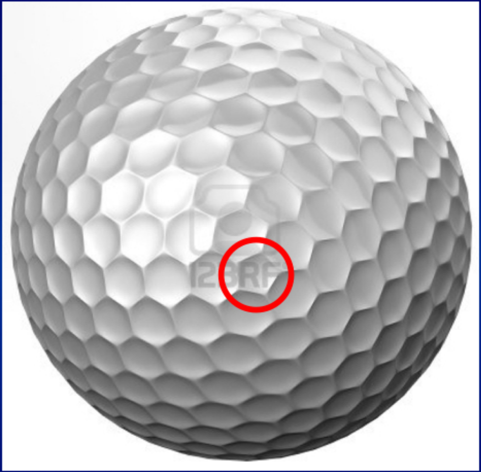
Buckminsterfullerene is a molecule with formula C_{60}

It was first synthesized in 1985.

Graphene
A hexagonal pattern of carbon one atom thick



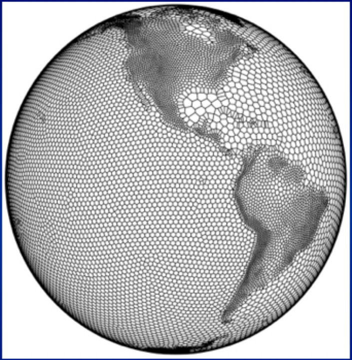




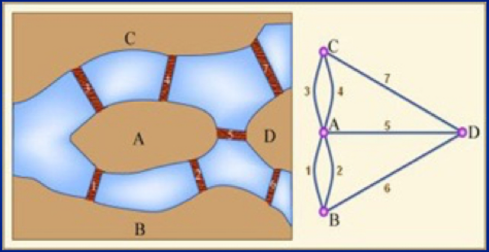
Euler's Polyhedron Formula

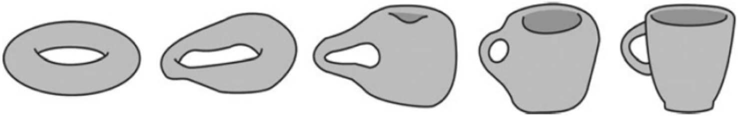
$$V - E + F = 2$$

still holds.

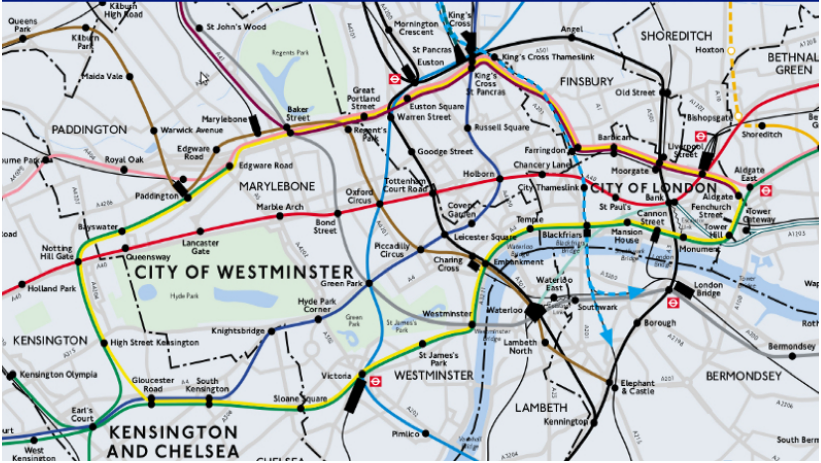


Topology
is often called
Rubber Sheet Geometry

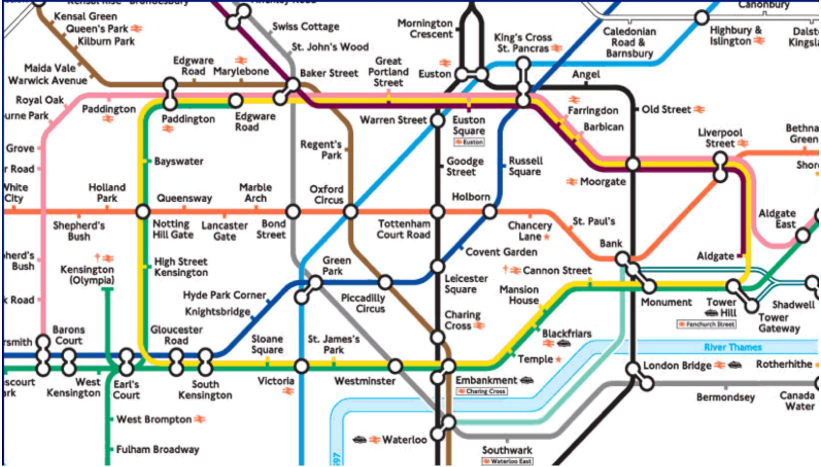




Topology and the London Underground
Topographical Map



Topology and the London Underground
Topological Map



Outline

Distraction 6A: Slicing a Pizza (Again)

Cut the pizza using only straight cuts.

There should be exactly one piece of pepperoni on each slice of pizza.

Minimum number of cuts?

Abstract Formulation

Last Week's Problem:
Plane cut by n lines. How many regions are formed?

Cuts	Segments (1D)	Regions (2D)	Solids (3D)
0	1	1	1
1	2	2	2
2	3	4	4
3	4	7	8
4	5	11	15
5	6	16	26
6	7	22	42

What is the pattern here?

Cutting Lines, Planes and Spaces

Cuts	Segments (1D)	Regions (2D)	Solids (3D)
0	1	1	1
1	2	2	2
2	3	4	4
3	4	7	8
4	5	11	15
5	6	16	26
6	7	22	42

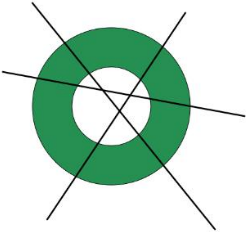
There is a pattern here.
It is reminiscent of Pascal's Triangle.

Distraction 6A: Doughnut-Slicing Problem

BRILLIANT   Courses   **Practice**   Community

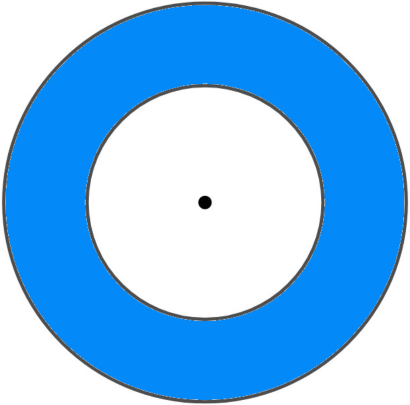
Cutting an Annulus

Discrete Mathematics Level 2

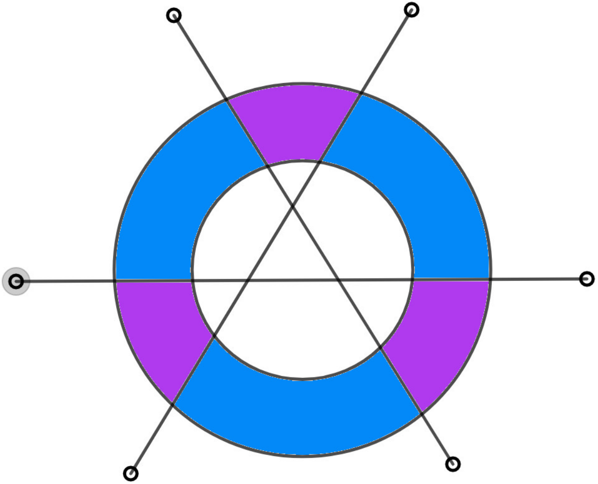


What is the maximum number of pieces into which a ring can be cut by 3 straight lines?
In the image above, the ring is cut into 6 pieces by 3 lines.

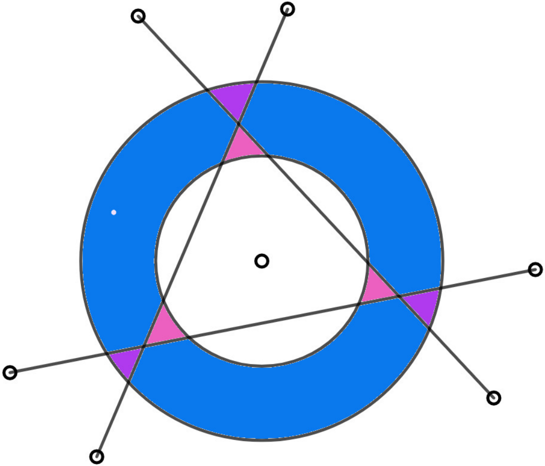
Distraction 6A: Slicing a (Flat) Doughnut



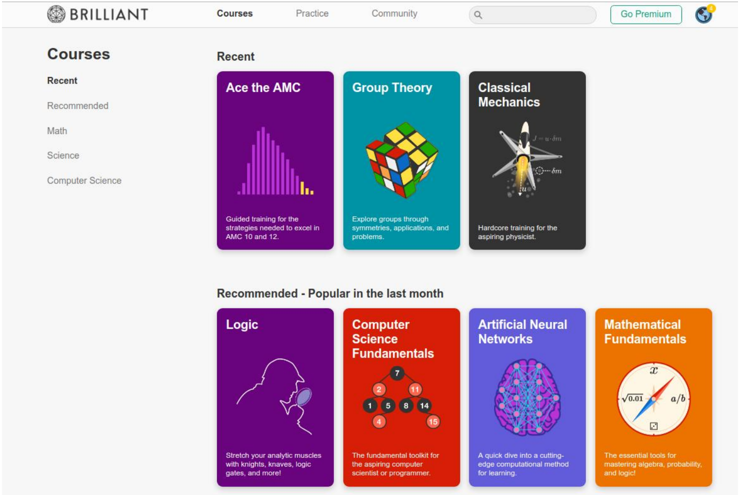
Distraction 6A: Slicing a (Flat) Doughnut



Distraction 6A: Slicing a (Flat) Doughnut



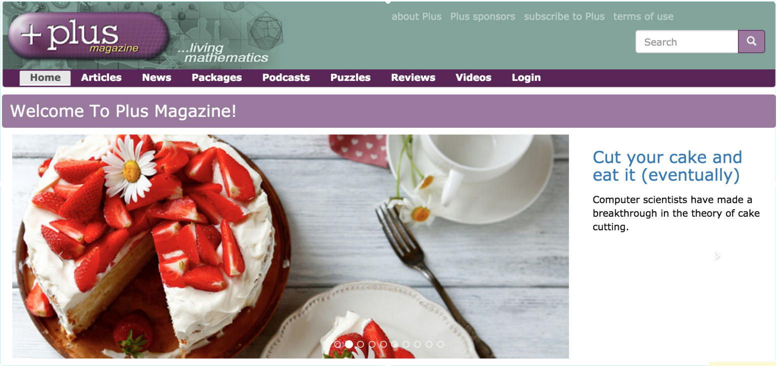
Distraction 6A: Brilliant Website



https://brilliant.org/

Outline

Distraction 7: Plus Magazine



PLUS: The Mathematics e-zine

`https://plus.maths.org/`

Outline

The Scientific Revolution

INTRODUCTION

This week, we will look at developments in the sixteenth and seventeenth centuries.

Figure from mathigon.org

The Heliocentric Model

In 1543, *Nicolaus Copernicus* (1473–1543) published *"On the Revolutions of the Celestial Spheres"*.

He explained that the Sun is at the centre of the universe and that the Earth and planets move around it in circular orbits.

Danish astronomer *Tycho Brahe* (1546–1601) made very accurate observations of the movements of the planets, and developed his own model of the solar system.

Johannes Kepler (1571–1630)

Johannes Kepler (1571–1630) succeeded Brahe as imperial mathematician.

After many years of struggling, Kepler succeeded in formulating his three Laws of Planetary Motion.

Kepler's Laws describe the solar system much as we know it to be true today.

Kepler's Laws

- **The planets move on elliptical orbits, with the Sun at one of the two foci.**
  This explains why the Sun appears larger at

  some times of the year and smaller at others.
- **A line joining the planet and the Sun sweeps out equal areas in equal times.**
  This means that a planet moves faster when

  close to the Sun, and slower when further away.
- **The square of the orbital period is proportional to the cube of the mean radius of the orbit.**
  This law allows us to find the orbital time

  of a planet if we know the size of the orbit.

Jovian Year from Kepler's Third Law

- **Distance from Sun to Earth: 1.0 AU**
- **Distance from Sun to Jupiter: 5.2 AU**
-
- **Rotation Period of Earth: 1 Year**
- **Rotation Period of Jupiter: To Be Found**

$$\frac{{P_J}^2}{{P_E}^2} = \frac{{R_J}^3}{{R_E}^3}$$

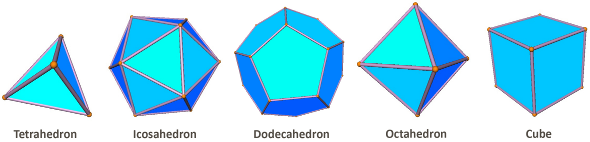
$${P_J}^2 = {R_J}^3$$

$$P_J = {R_J}^{\frac{3}{2}} \qquad P_J = (5.2)^{\frac{3}{2}} \approx 12 \textbf{ Years}$$

The *Mysterium Cosmographicum*

There were six known planets in Kepler's time: Mercury, Venus, Earth, Mars, Jupiter, Saturn.

There are precisely five platonic solids:



This gave Kepler an extraordinary idea!

`https://thatsmaths.com/2016/10/13/`

`\keplers-magnificent-mysterium-cosmographicum/`

Galileo Galelii (1564–1630)

Galileo introduced the *telescope* to astronomy, and made some dramatic discoveries.

He observed the four large moons of Jupiter *revolving around that planet.*

He established the laws of inertia that underlie Newton's dynamical laws.

Four Remarkable Scientists

Figure from mathigon.org

Isaac Newton (1642–1727)

In 1687, Isaac Newton published the Principia Mathematica. He established the mathematical foundations of dynamics.

Between any two masses there is a force:

$$F = \frac{GMm}{r^2}$$

This is the force of gravity and gravity is what makes the planets move around the Sun.

Newton's Laws imply and explain Kepler's laws.

Outline

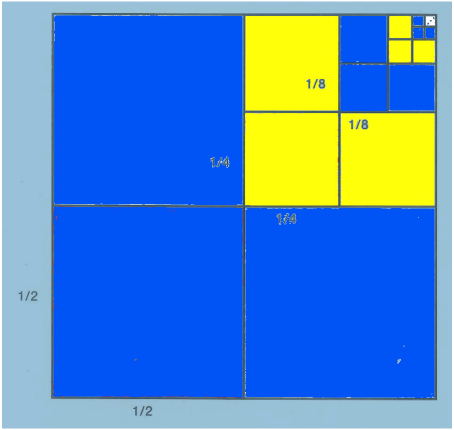
Distraction 8: Sum by Inspection

Can you guess the sum of this series:

$$\left(\frac{1}{2}\right)^2 + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{8}\right)^2 + \left(\frac{1}{16}\right)^2 + \cdots$$

Distraction 8: Sum by Inspection

We will find the shaded area without calculation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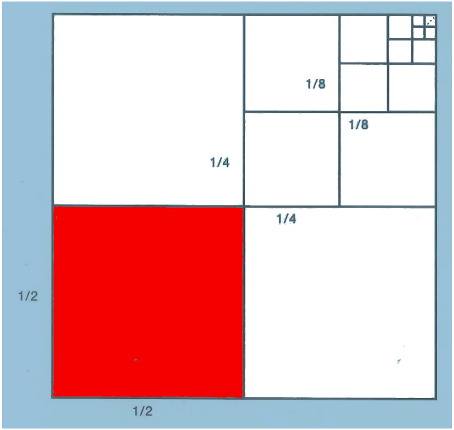
1/2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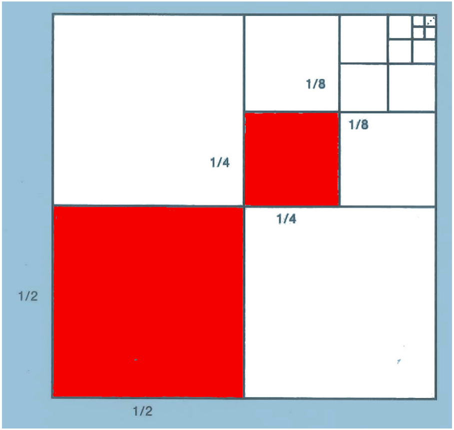
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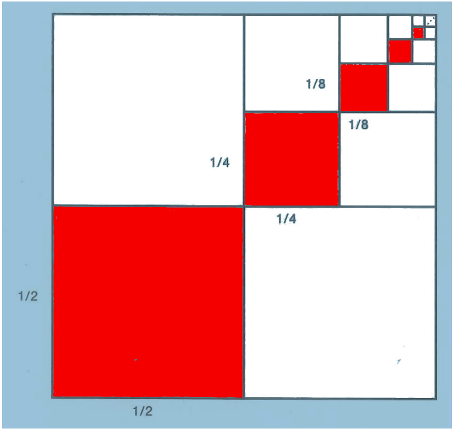
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Proof by Inspection

Look at the figure in two different ways

At each scale, we have three squares the same size, and we keep one of them (red) and omit the others.

So, the area of the shaded squares is $\frac{1}{3}$.

However, it is also given by the series

$$\left(\frac{1}{2}\right)^2 + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{8}\right)^2 + \left(\frac{1}{16}\right)^2 + \cdots$$

Therefore we can sum the series:

$$\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \cdots = \frac{1}{3}$$

Outline

Carl Friedrich Gauss (1777–1855)

Carl Friedrich Gauss (1777–1855)

A German mathematician who made profound contributions to many fields of mathematics:

- Number theory
- Algebra
- Statistics
- Analysis
- Differential geometry
- Geodesy & Geophysics
- Mechanics & Electrostatics
- Astronomy

One of the greatest mathematicians of all time.

Gauss Outsmarts his Teacher

Gauss was a genius. He was known as

The Prince of Mathematicians.

When very young, Gauss outsmarted his teacher.

I can now reveal a fact unknown to historians:

The teacher got his own back. Ho! ho! ho!

Gauss Outsmarts his Teacher

Gauss's school teacher tasked the class:

- **Add up all the whole numbers from 1 to 100.**

Gauss solved the problem in a flash.

He wrote the correct answer,

$$5,050$$

on his slate and handed it to the teacher.

How did Gauss do it?

First, Gauss wrote the numbers in a row:

$$1 \quad 2 \quad 3 \quad \ldots \quad 98 \quad 99 \quad 100$$

Next he wrote them again, *in reverse order:*

$$\begin{array}{ccccccc} 1 & 2 & 3 & \ldots & 98 & 99 & 100 \\ 100 & 99 & 98 & \ldots & 3 & 2 & 1 \end{array}$$

Then he added the two rows, column by column:

$$\begin{array}{ccccccc} 1 & 2 & 3 & \ldots & 98 & 99 & 100 \\ 100 & 99 & 98 & \ldots & 3 & 2 & 1 \\ --- & --- & --- & --- & --- & --- & --- \\ 101 & 101 & 101 & \ldots & 101 & 101 & 101 \end{array}$$

Clearly, the total for the two rows is 10,100.

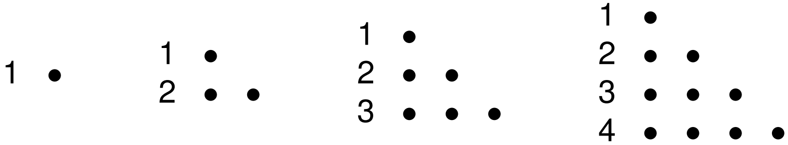
But every number from 1 to 100 is counted twice.

$$\therefore \ \mathbf{1 + 2 + 3 + \cdots + 98 + 99 + 100 = 5{,}050}$$

Triangular Numbers

Gauss had calculated the 100-th triangular number.

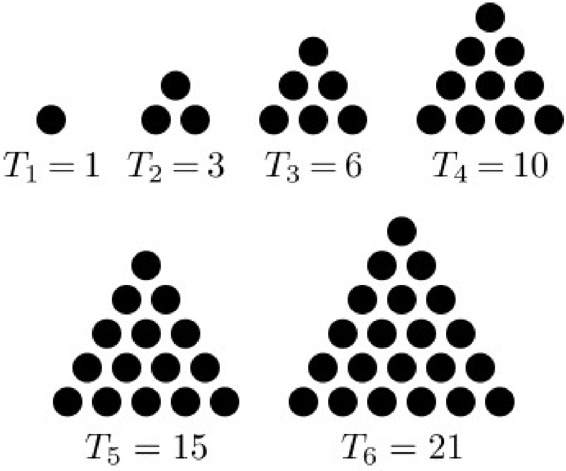
Let us take a geometrical look at the sums of the first few natural numbers:



We see that the sums can be arranged as triangles.

Triangular Numbers

The first few *triangular numbers* are $\{1, 3, 6, 10, 15, 21\}$.



Let's look at the 10th triangular number.

For $n = 10$ the pattern is:

1	•
2	• •
3	• • •
4	• • • •
5	• • • • •
6	• • • • • •
7	• • • • • • •
8	• • • • • • • •
9	• • • • • • • • •
10	• • • • • • • • • •

How do we compute its value? Gauss's method!

It is easy to show that the n-th triangular number is

$$T_n = (1 + 2 + 3 + \cdots + n) = \tfrac{1}{2}n(n+1)$$

We do just as Gauss did, and list the numbers twice:

$$\begin{array}{cccccc}
1 & 2 & 3 & \dots & n-1 & n \\
n & n-1 & n-2 & \dots & 2 & 1 \\
--- & --- & --- & \dots & --- & --- \\
n+1 & n+1 & n+1 & \dots & n+1 & n+1
\end{array}$$

There are n columns, each with total $n+1$.

So the grand total is $n \times (n+1)$.

Each number has been counted twice, so

$$T_n = \tfrac{1}{2}n(n+1)$$

Let's check this for Gauss's problem of $n = 100$:

$$T_{100} = 1 + 2 + 3 + \cdots + 100 = \frac{100 \times 101}{2} = 5,050$$

Gauss's approach was to look at the problem from a new angle.

Such *lateral thinking* is very common in mathematics:

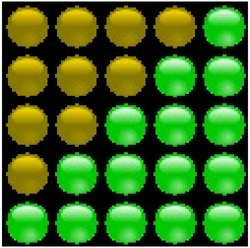
Problems that look difficult can sometimes be solved easily when tackled from a different angle.

Two Triangles Make a Square

A nice property of *consecutive* triangular numbers:

$$\begin{array}{ccc} & & \\ \star & & \\ \star & \star & \\ \star & \star & \star \end{array} \quad + \quad \begin{array}{cccc} \bullet & & & \\ \bullet & \bullet & & \\ \bullet & \bullet & \bullet & \\ \bullet & \bullet & \bullet & \bullet \end{array} \quad = \quad \begin{array}{cccc} \bullet & \star & \star & \star \\ \bullet & \bullet & \star & \star \\ \bullet & \bullet & \bullet & \star \\ \bullet & \bullet & \bullet & \bullet \end{array}$$

$$T_3 + T_4 = 6 + 10 = 16 = 4^2$$



Triangular Numbers

We have seen, by means of geometry that the sum of two consecutive triangular numbers is a square.

Now let us prove this algebraically:

$$\begin{aligned} T_n + T_{n+1} &= \tfrac{1}{2}n(n+1) + \tfrac{1}{2}(n+1)(n+2) \\ &= \tfrac{1}{2}(n+1)[n+(n+2)] \\ &= \tfrac{1}{2}(n+1)[2(n+1)] \\ &= (n+1)^2 \end{aligned}$$

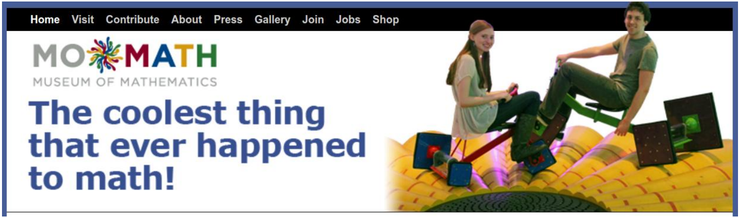
The result is *a perfect square.*

Puzzle

What is the sum of all the numbers from 1 up to 100 and back down again?

The answer is in the video coming up now.

A Video from the Museum of Mathematics



VIDEO: Beautiful Maths, available at

`http://momath.org/home/beautifulmath/`

Video by James Tanton

Gauss Outsmarted by his Teacher

The teacher thought that he would have a half-hour of peace and quiet while the pupils grappled with the problem of adding up the first 100 numbers.

He was annoyed when Gauss came up almost immediately with the correct answer 5,050.

So, he said:

"Oh, you zink you are zo zmart!
Zo, multiply ze first 100 numbers."

EXERCISE: Zink about that!

A Lateral Thinking Puzzle

- Jill is 23 years younger than her father.
- What age was she when she was half his age?

Let Jill's age be J. Let her father's age be F.

$$F - J = 23$$

Hint: Be Smart
There is no need for tricky algebra.

Thank you