AweSums

Marvels and Mysteries of Mathematics

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LECTURE 6

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Outline

Outline

Meaning and Content of Mathematics

The word Mathematics comes from Greek $\mu\alpha\theta\eta\mu\alpha$ (máthéma), meaning "knowledge" or "study" or "learning".

It is the study of topics such as

- Quantity (numbers)
- Structure (patterns)
- Space (geometry)
- Change (analysis).

Outline

Volume of a Sphere

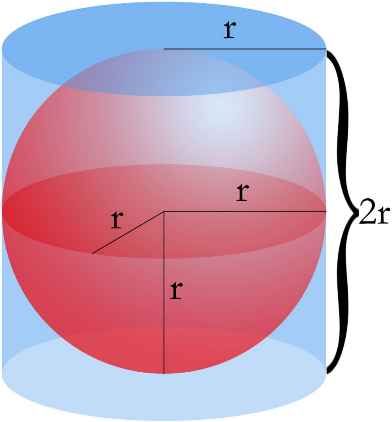


Figure : Archimedes found a formula for V_{SPHERE}

Who First Proved that C/D is Constant?

For *every circle,* the distance around it is just over three times the distance across it.

This has been “common knowledge” since the earliest times.

But mathematicians don’t trust common knowledge.

They demand proof.

Who was first to prove that the ratio of circumference C to diameter D has the same value for all circles?

What about Euclid?

You might expect to find a proof in Euclid's *Elements of Geometry*. *But Euclid couldn't prove it.*

Euclid's Prop. XII.2 says the areas of circles are to one another as the squares of their diameters:

$$\frac{A_1}{D_1^2} = \frac{A_2}{D_2^2} \,.$$

We would expect to find an analogous theorem: The circumferences of circles vary as their diameters:

$$\frac{C_1}{D_1} = \frac{C_2}{D_2}$$

but we do not find this anywhere in Euclid.

Archimedes Rules OK!

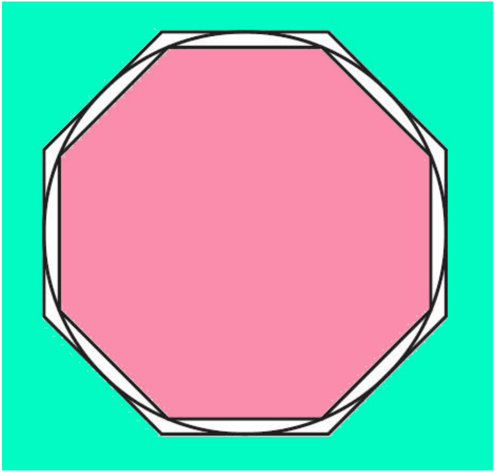
It required the genius of Archimedes to prove that C/D is the same for all circles.

He needed axioms beyond those of Euclid.

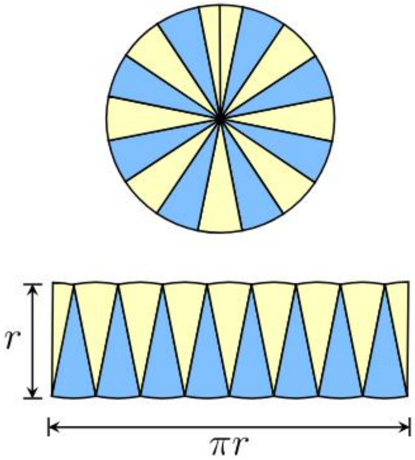
In his work *Measurement of a Circle,* Archimedes found the area of a circle.

It is equal to the area of a right-angled triangle with one leg equal to R and the other equal to C:

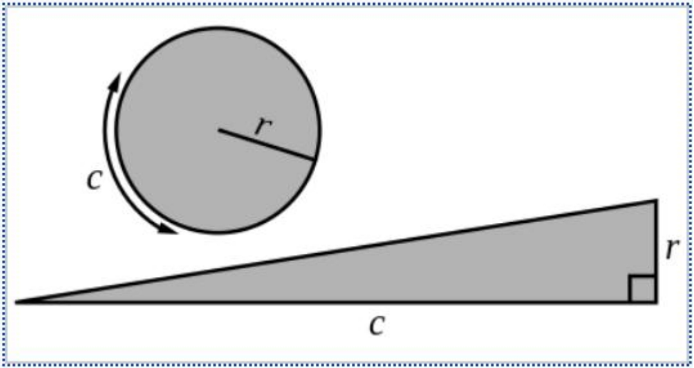
$$A = \tfrac{1}{2} RC \,.$$



Archimedes determined π accurately by *considering polygons within and around a circle.*



He determined the area of a circle by slicing it up into small triangles.



"Unzipping" the circle,
Archimedes obtained a triangle.

Lengths and Areas both involve π

Archimedes' theorem, together with Euclid's Proposition XII.2, implies that

$$\frac{C}{D} = \pi$$

is the same for every circle.

It also follows that the area constant is also π:

$$\frac{A}{R^2} = \frac{C}{2R} = \frac{C}{D} = \pi \, .$$

Sphere+Cone=Cylinder

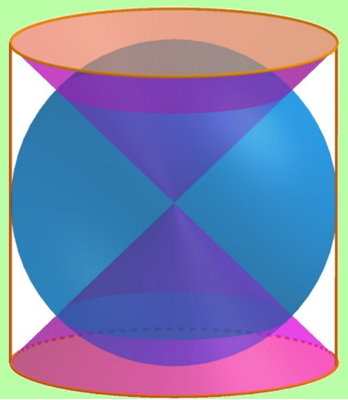


Figure : Volume: Sphere plus Cone equals Cylinder

On the Sphere and Cylinder

One of the most remarkable and important mathematical results obtained by Archimedes was the formula for the volume of a sphere.

Archimedes used a technique of sub-dividing the volume into slices and adding up, or integrating, the volumes of the slices.

This was essentially an application of the *integral calculus* formulated by Newton and Leibniz.

On the Sphere and Cylinder

Archimedes considered three volumes, a cylinder, cone and sphere, all on bases with the same area.

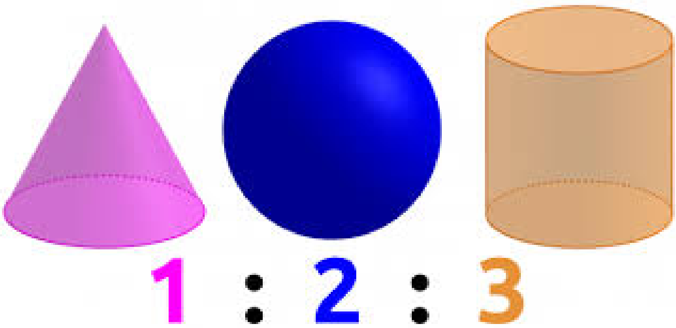


Figure : Cone, sphere and cylinder on the same base.

On the Sphere and Cylinder

Archimedes showed that the three volumes are in the ratio $1 : 2 : 3$.

Thus, in particular, the volume of the sphere is two thirds of the volume of the cylinder.

If we 'rearrange' the volume of the cone, **things become much clearer:**

We replace the cone by two cones, each of height r.

On the Sphere and Cylinder

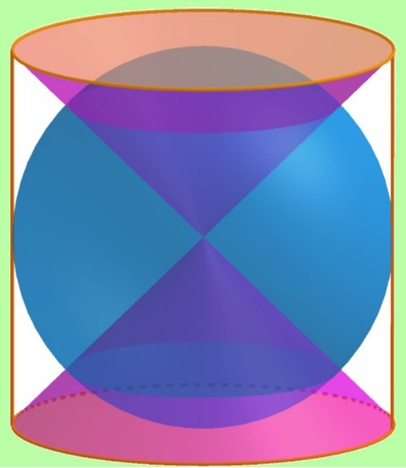


Figure : Cone, sphere and cylinder on the same base.

On the Sphere and Cylinder

We let z denote the vertical coordinate, and Δz be a small increment of height.

The cross-sections of the cone and sphere are

$$\begin{aligned} \Delta V_{\mathrm{CON}} &= \pi z^2 \Delta z \\ \Delta V_{\mathrm{SPH}} &= \pi(\sqrt{r^2 - z^2})^2 \Delta z = \pi(r^2 - z^2)\Delta z \,. \end{aligned}$$

Add to get the cross-sectional area of the cylinder:

$$\Delta V_{\mathrm{CON}} + \Delta V_{\mathrm{SPH}} = \Delta V_{\mathrm{CYL}} = \pi r^2 \Delta z \,,$$

This does not vary with height z.
It is the same as for the cylinder.

On the Sphere and Cylinder

REPLACE OR SUPPLEMEMT ABOVE SLIDE WITH A FIGURE

On the Sphere and Cylinder

Adding up the volumes of all slices:

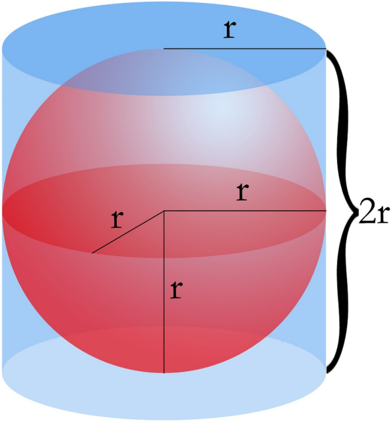
$$\Delta V_{\mathrm{CON}} + \Delta V_{\mathrm{SPH}} = \Delta V_{\mathrm{CYL}} = \pi r^2 H = 2\pi r^3 \,.$$

It is not quite so simple to show that

$$\begin{aligned} \Delta V_{\mathrm{CON}} &= \tfrac{1}{3}\Delta V_{\mathrm{CYL}} = \tfrac{1}{3}\pi r^2 H = \tfrac{2}{3}\pi r^3 \\ \Delta V_{\mathrm{SPH}} &= \tfrac{2}{3}\Delta V_{\mathrm{CYL}} = \tfrac{2}{3}\pi r^2 H = \tfrac{4}{3}\pi r^3 \,. \end{aligned}$$

However, this was well within the capability of the brilliant mathematician Archimedes.

On the Sphere and Cylinder



This result was carved on Archimedes' tomb.

Archimedes' Tomb as it appears today

Outline

What are Axioms?

How can we prove a theorem,
if we have nothing to start from?

We cannot prove something using nothing.
We need some starting point.

The basic building blocks are called Axioms.

Axioms are not proved, but are assumed true.

What are Axioms?

Axioms are important because the entire body of mathematics rests upon them.

If there are *too few axioms,* we can prove very little of interest from them.

If there are *too many axioms,* we can prove almost any result from them.

Consistency:
We must not have axioms that contradict each other.

What are Axioms?

Mathematicians assume that axioms are true without being able to prove them.

This is not problematic, because axioms *are normally intuitively obvious.*

There are usually only a few axioms.
For example, we may assume that

$$a \times b = b \times a$$

for any two numbers a and b.

But Hamilton found that for *quaternions,*

$$A \times B \neq B \times A\,.$$

Different sets of axioms lead to different kinds of mathematics.

Every area of mathematics has its own set of basic axioms.

When mathematicians have proven a theorem, they publish it for other mathematicians to check.

In principle, it is possible to break a proof into steps starting from the basic axioms.

Sometimes a mistake in the proof is found. Sometimes an error is not found for many years (e.g., an early "proof" of the *Four Colour Theorem.*)

Euclid's Axioms of Geomery

Euclid based his "Elements of Geometry"
on a set of five postulates or axioms:

"Let the following be postulated":

1. "To draw a straight line from any point to any point."
2. "To produce [extend] a finite straight line continuously in a straight line."
3. "To describe a circle with any centre and distance [radius]."
4. "That all right angles are equal to one another."
5. *The parallel postulate*: "That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles."

The fifth postulate, the parallel postulate, has been a great source of controversy and confusion.
This has led to *completely new areas of mathematics.*

Peano’s Axioms of Arithmetic

Giuseppi Peano constructed five axioms
to build up the set* $\mathbb{N}$ *of natural numbers:

$$\exists 0 : 0 \in \mathbb{N}$$

$$\forall n \in \mathbb{N} : \exists n' \in \mathbb{N}$$

$$\neg(\exists n \in \mathbb{N} : n' = 0)$$

$$\forall m, n \in \mathbb{N} : m' = n' \Rightarrow m = n$$

$$\forall A \subseteq \mathbb{N} : (0 \in A \wedge (n \in A \Rightarrow n' \in A)) \Rightarrow A = \mathbb{N}$$

The natural numbers may then be extended to the integers, rational numbers and real numbers.

Axioms of Set Theory

Set theory is the basic language of mathematics.

Many mathematical problems can be formulated in the language of set theory.

To prove them we need the *Set Theory Axioms.*

The most widely accepted axioms are the set of nine Zermelo-Fraenkel (ZF) axioms.

A tenth axiom, may also be assumed, the *Axiom of Choice.*

Zermelo-Fraenkel axioms

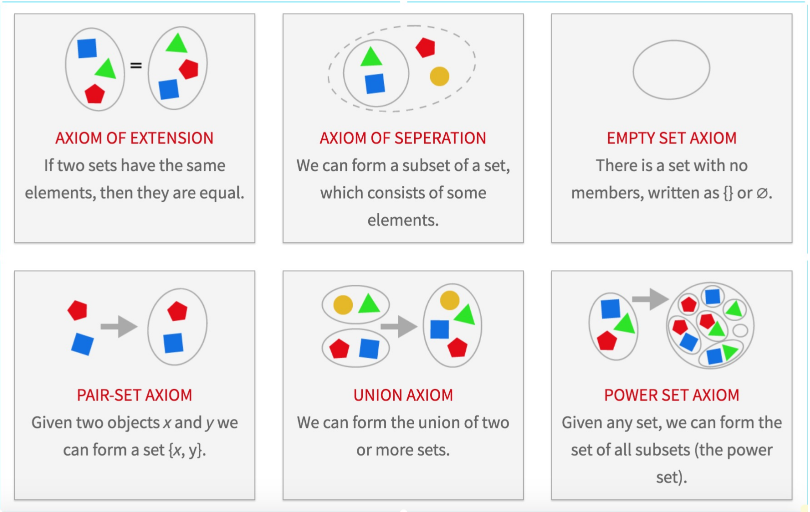


Image from Mathigon.org

Zermelo-Fraenkel axioms

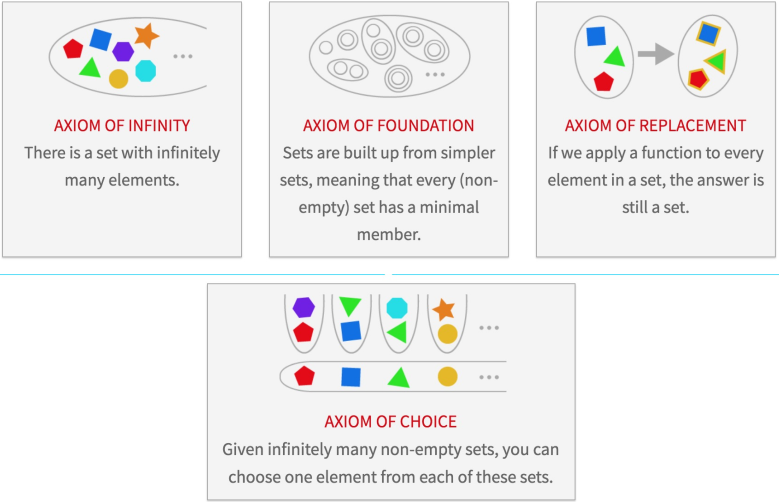


Image from Mathigon.org

Axiom of Choice

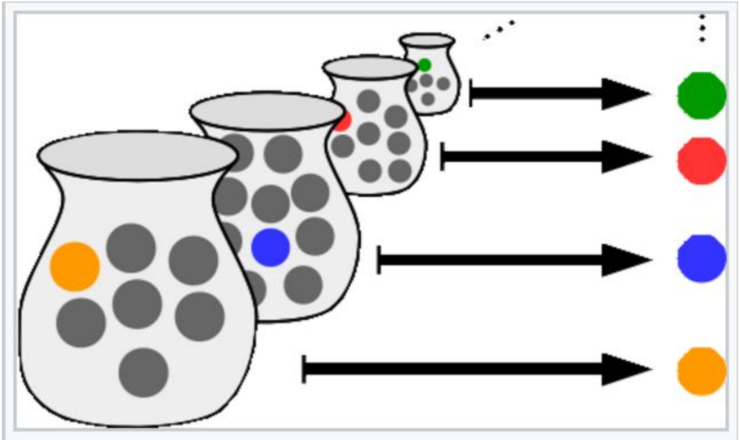


Image from Wikipedia

Axiom of Choice

The Axiom of Choice (AC) looks just as innocuous as the other nine axioms.
However it has unexpected consequences.

We can use AC to prove that it is possible to cut a sphere into five pieces and reassemble them into two spheres, each identical to the initial sphere.

This result is called the Banach-Tarski Theorem.

Banach-Tarski Theorem

The five pieces have fractal boundaries: they can't actually be made in practice.

Also, they are not *measurable:* they have no definite volume.

The Current Status

There is ongoing debate among logicians about whether or not to accept the Axiom of Choice.

Every collection of axioms forms a different "mathematical world". Different theorems may be true in different worlds.

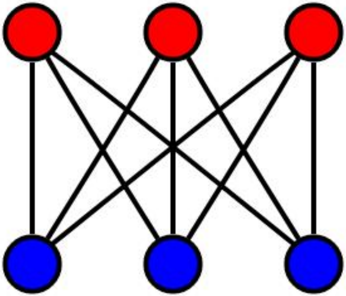
The question is:
Are we happy to live in a world where we can make two spheres from one.

See Wikipedia article: Axiom of Choice

Outline

Three Utilities Problem: Abstract

Is the complete 3×3 bipartite graph $K_{3,3}$ planar?



This is an abstract, jargon-filled question in topological graph theory.

We look at a simple, concrete version.

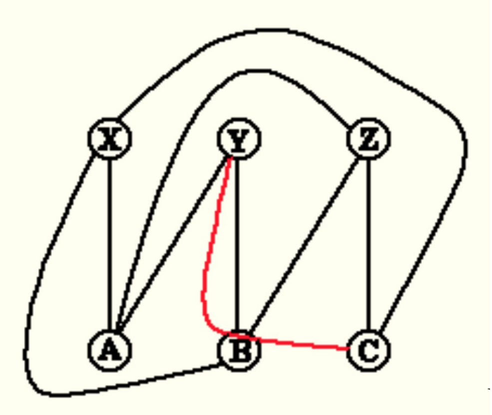
Three Utilities Problem: Concrete

We have to connect 3 utilities to 3 houses.

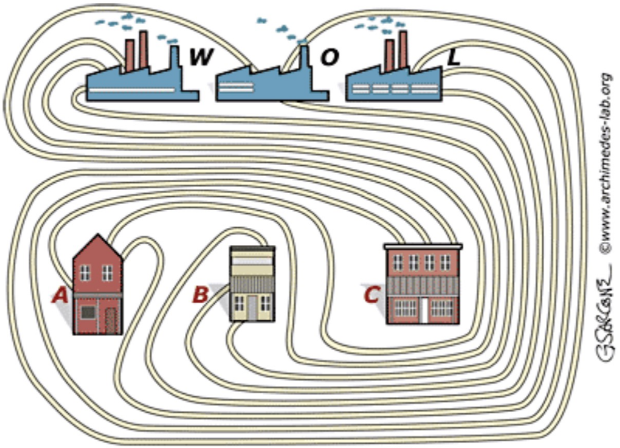
- Electricity
- Water
- Gas

The lines must not cross.

Three Utilities Problem: Have a Go

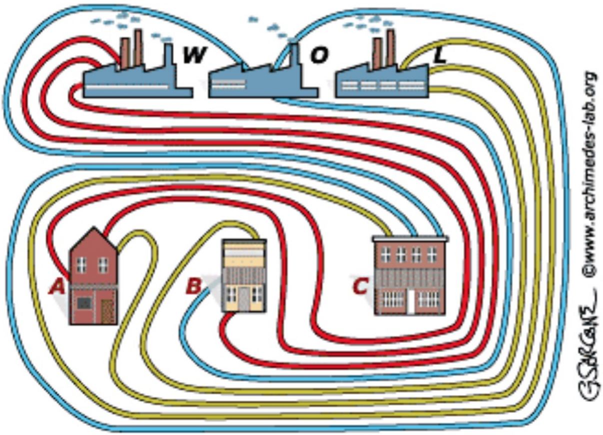


Three Utilities Problem: Solution!



http://www.archimedes-lab.org/How_to_Solve/Water_gas.html

Three Utilities Problem: No Solution!



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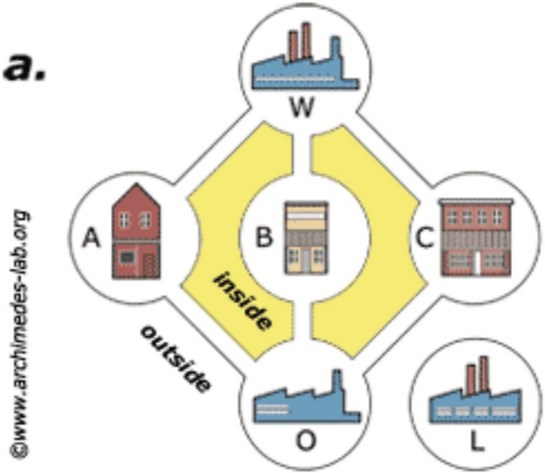
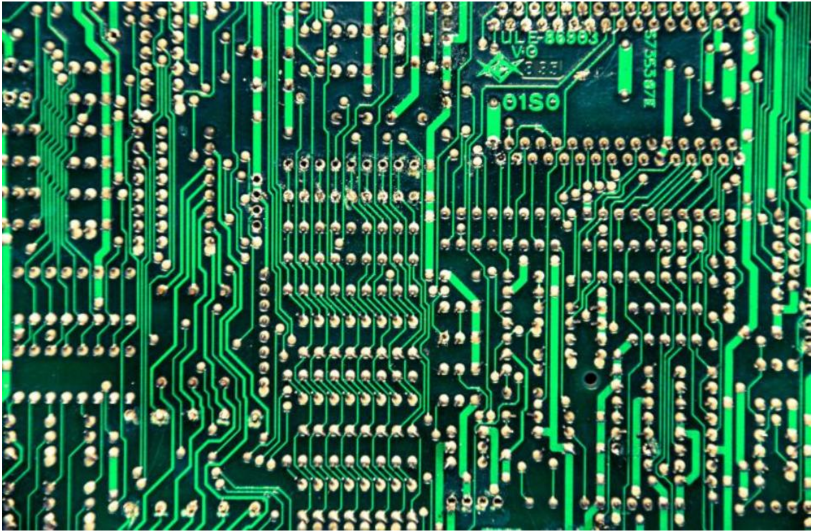
Three Utilities Problem

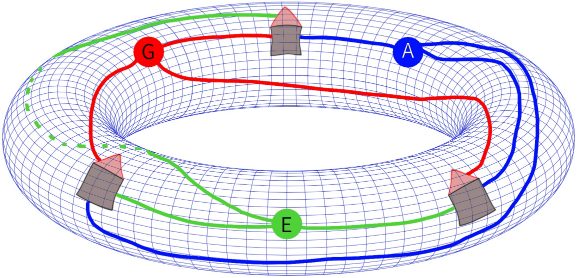


Three Utilities Problem: Application



Three Utilities Problem for Mugs

Three Utilities Problem on a Torus

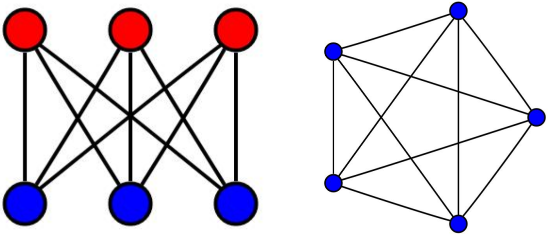


$K_{3,3}$ is a toroidal graph.

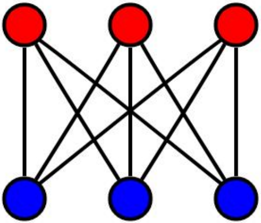
Vi Hart: `https://www.youtube.com/watch?v=CruQylWSfoU&feature=youtu.be&t=9`

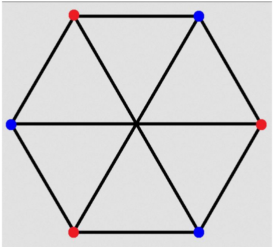
Three Utilities: Kuratowski's Theorem

If a graph contains $K_{3,3}$ or K_5 as a sub-graph, it is non-planar. If it does not contain either, it is planar.



Three Utilities: Equivalent Graphs





The two forms shown are equivalent.

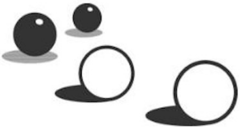
There are crossings in both.

Outline

Distraction 12: Conditional Probability

Conditional Probability

Conditional Probability: Level 3 Challenges



A box contains two white marbles and two black marbles. I pick a marble at random and set it aside. Then, I pick a second marble and notice that it is black.

Is it more likely that the first marble was white or black?

Distraction 12: Conditional Probability

Possibile outcomes of the experiment:

$$W_1 W_2 \qquad W_1 B_2 \qquad B_1 W_2 \qquad B_1 B_2$$

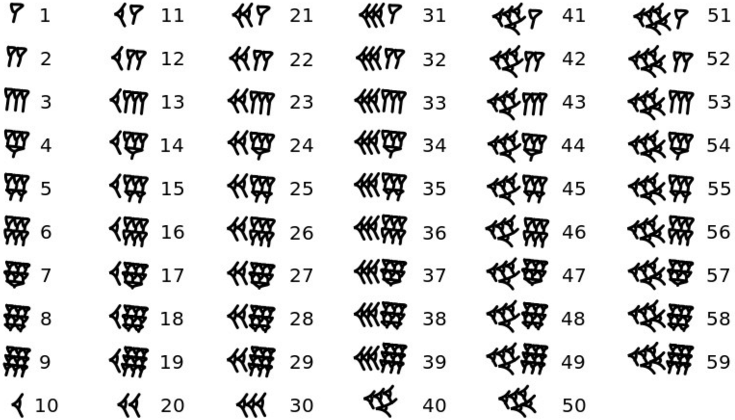
Are all four possibilities equally likely?

$$P(B_2) = P(W_1)P(B_2|W_1) + P(B_1)P(B_2|B_1)$$

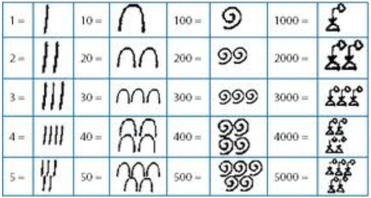
$$P(W_1) = \tfrac{1}{2} \quad P(B_1) = \tfrac{1}{2} \quad P(B_2|W_1) = \tfrac{2}{3} \quad P(B_2|B_1) = \tfrac{1}{3}$$

Outline

Babylonian Numerals



Ancient Egyptian Numerals



Ancient Hebrew and Greek Numerals

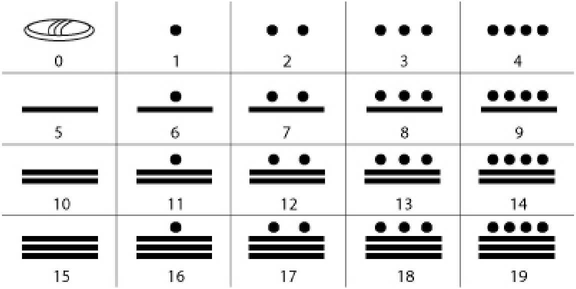
8	7	6	5	4	3	2	1
ח	ז	ו	ה	ד	ג	ב	א
Chet	Zayin	Vav	Hey	Dalet	Gimmel	Bet	Aleph

70	60	50	40	30	20	10	9
ע	ס	נ	מ	ל	כ	י	ט
Ayin	Samekh	Nun	Mem	Lamed	Kaf	Yod	Tet

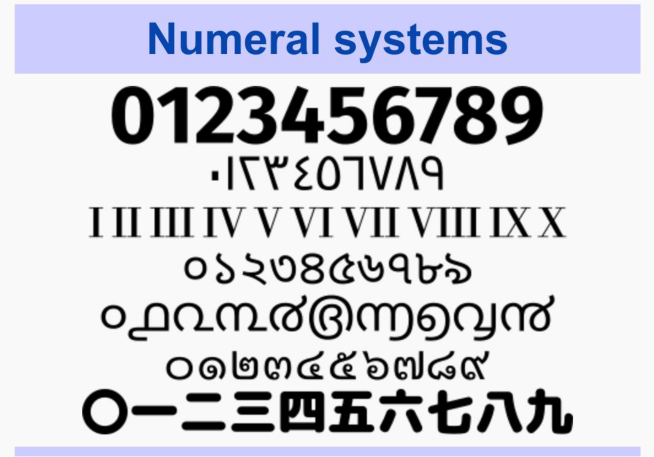
1	α	alpha	10	ι	iota	100	ρ	rho
2	β	beta	20	κ	kappa	200	σ	sigma
3	γ	gamma	30	λ	lambda	300	τ	tau
4	δ	delta	40	μ	mu	400	υ	upsilon
5	ϵ	epsilon	50	ν	nu	500	ϕ	phi
6	ϛ	vau*	60	ξ	xi	600	χ	chi
7	ζ	zeta	70	o	omicron	700	ψ	psi
8	η	eta	80	π	pi	800	ω	omega
9	θ	theta	90	ϙ	koppa*	900	ϡ	sampi

*vau, koppa, and sampi are obsolete characters

Mayan Numerals



Various Numeral Systems



Wikipedia: Hindu-Arabic Numeral System

Roman Numerals

I	1	XXI	21	XLI	41
II	2	XXII	22	XLII	42
III	3	XXIII	23	XLIII	43
IV	4	XXIV	24	XLIV	44
V	5	XXV	25	XLV	45
VI	6	XXVI	26	XLVI	46
VII	7	XXVII	27	XLVII	47
VIII	8	XXVIII	28	XLVIII	48
IX	9	XXIX	29	XLIX	49
X	10	XXX	30	L	50
XI	11	XXXI	31	LI	51
XII	12	XXXII	32	LII	52
XIII	13	XXXIII	33	LIII	53
XIV	14	XXXIV	34	LIV	54
XV	15	XXXV	35	LV	55
XVI	16	XXXVI	36	LVI	56
XVII	17	XXXVII	37	LVII	57
XVIII	18	XXXVIII	38	LVIII	58
XIX	19	XXXIX	39	LIX	59
XX	20	XL	40	LX	60

In order: $M\ D\ C\ L\ X\ V\ I\ = 1666$

How to Multiply Roman Numbers

Table : Multiplication Table for Roman Numbers.

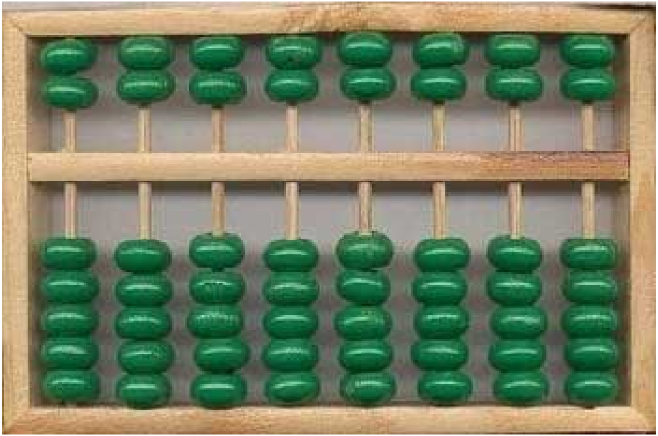
	I	**V**	**X**	**L**	**C**	**D**	**M**
I	*I*	*V*	*X*	*L*	*C*	*D*	*M*
V	*V*	*XXV*	*L*	*CCL*	*D*	*MMD*	$\overline{V}$
X	*X*	*L*	*C*	*D*	*M*	$\overline{V}$	$\overline{X}$
L	*L*	*CCL*	*D*	*MMD*	$\overline{V}$	$\overline{XXV}$	$\overline{L}$
C	*C*	*D*	*M*	$\overline{V}$	$\overline{X}$	$\overline{L}$	$\overline{C}$
D	*D*	*MMD*	$\overline{V}$	$\overline{XXV}$	$\overline{L}$	$\overline{CCL}$	$\overline{D}$
M	*M*	$\overline{V}$	$\overline{X}$	$\overline{L}$	$\overline{C}$	$\overline{D}$	$\overline{M}$

A Roman Abacus

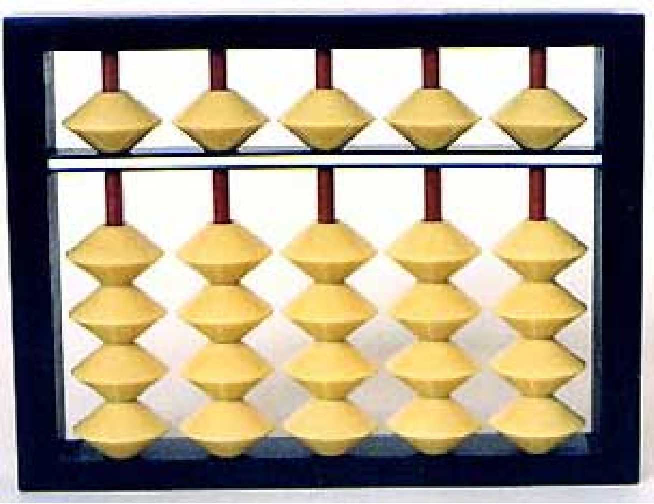
Replica of a Roman abacus from 1st century AD.

Abacus is a Latin word, which comes from the Greek $\alpha\beta\alpha\kappa\alpha\varsigma$ (board or table).

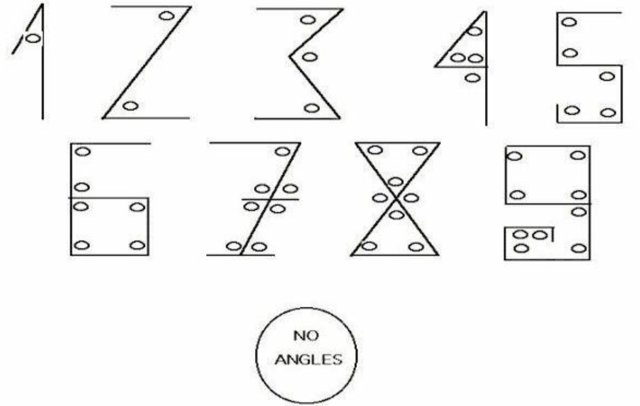
A Chinese Abacus: *Suan Pan*



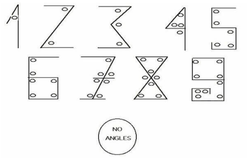
A Japanese Abacus: *Soroban*



A Different Angle on Numerals



Delightful theory. Almost certainly wrong.



Arguments "for"

1. It is a very simple idea
2. It links symbols to numerical values

Arguments "against"

1. Number forms modified to fit model
2. Complete lack of historical evidence.

The great tragedy of science —

the slaying of a beautiful hypothesis by an ugly fact (T H Huxley)

Outline

Estimating π with Series

There are many ways of estimating π.

For example, we can sum up the Basel Series:

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots .$$

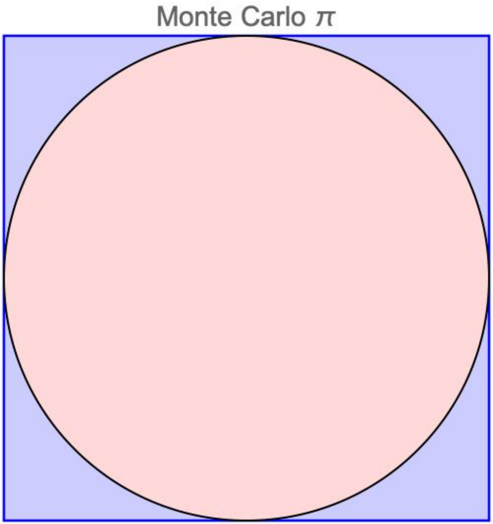
Another way is with the Gregory-Leibniz series, discovered much earlier by Madhava (c. 1340–1425).

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots .$$

We have already seen Archimedes' method.

We now give a completely different approach.

Estimating π with Probability



Estimating π with Probability

Area of Square: 4

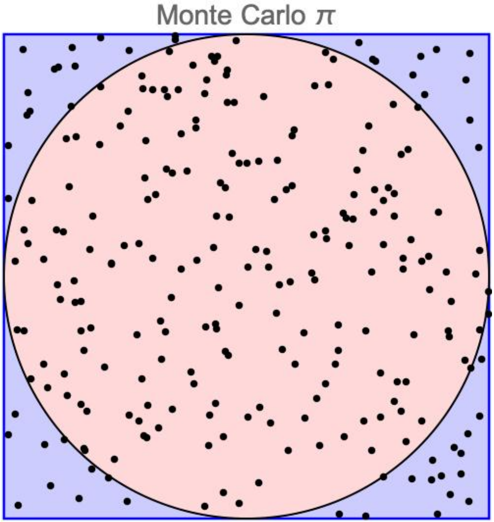
Area of Circle: π

Probability point is within circle: $\frac{\pi}{4}$

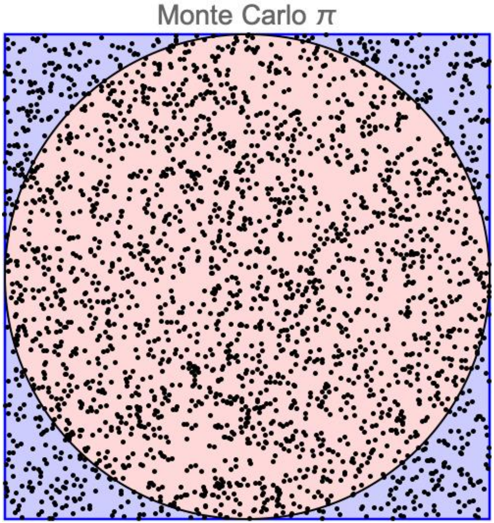
Thus, the following ratio should approach π**:**

$$4 \times \frac{\textbf{Number of points within Circle}}{\textbf{Number of points within Square}} \to \pi\,.$$

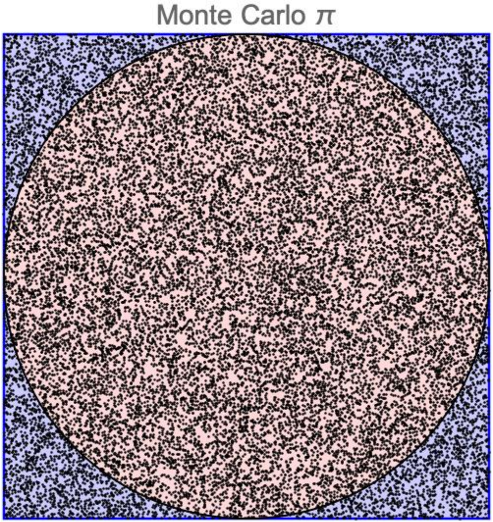
Estimating π with $n = 250$



Estimating π with $n = 2500$



Estimating π with $n = 25000$



Numerical Results

Table : Estimates of π

250	3.23506...
2500	3.15407...
25000	3.13177...
$\vdots$	$\vdots$
∞	3.14159...

Comment on uses of Monte Carlo method.

Outline

A Hierarchy of Numbers

We will introduce a *hierarchy of numbers.*

Each set is contained in the next one.

They are like a set of nested Russian Dolls:

Matryoshka

The Natural Numbers $\mathbb{N}$

The *counting numbers* were the first to emerge:

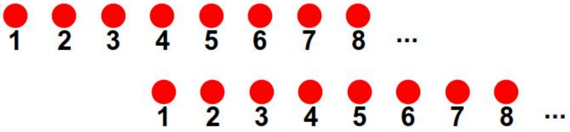
$$1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \ \ldots$$

They are also called the *Natural Numbers*.

We can arange the natural numbers in a list.

This list is like a toy computer.

A Primitive Sliderule



The Natural Numbers $\mathbb{N}$

The set of natural numbers is denoted $\mathbb{N}$.

If n is a natural number, we write $n \in \mathbb{N}$.

Natural numbers can be added: $4 + 2 = 6 \in \mathbb{N}$

But not always subtracted: $4 - 6 = -2 \notin \mathbb{N}$.

To allow for subtraction we have to extend $\mathbb{N}$.

The Integers $\mathbb{Z}$

We extend the set of counting numbers by including the negative whole numbers:

$$\ldots \quad -3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad \ldots$$

The whole numbers are also called the *Integers*.

The set of integers is denoted $\mathbb{Z}$.

If k is an integer, we write $k \in \mathbb{Z}$.

Clearly,

$$\mathbb{N} \subset \mathbb{Z}$$

Integers **can be added and subtracted.**

They can also multiplied:

$$6 \times 4 = 24 \in \mathbb{Z}\,.$$

However, they cannot usually be divided:

$$\frac{6}{4} = 1\tfrac{1}{2} \notin \mathbb{Z}\,.$$

To allow for division we have to extend $\mathbb{Z}$.

The Rational Numbers $\mathbb{Q}$

We extend the integers by including fractions:

$$r = \frac{p}{q} \qquad \text{where } p \text{ and } q \text{ are integers.}$$

These *rational numbers* are ratios of integers.

The set of rational numbers is denoted $\mathbb{Q}$.

If r is a rational number, we write $r \in \mathbb{Q}$.

Clearly,

$$\mathbb{Z} \subset \mathbb{Q}$$

With the Rational Numbers, we can:

Add, Subtract, Multiply and Divide

That is, for any $p \in \mathbb{Q}$ and $q \in \mathbb{Q}$, all of

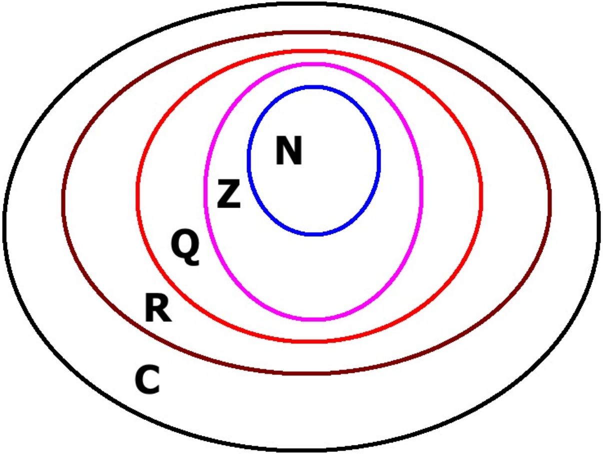
$$\{\, p+q \qquad p-q \qquad p\times q \qquad p\div q \,\}$$

are rational numbers.

We say that $\mathbb{Q}$ is *closed under addition, subtraction, multiplication and division.*

But we are not yet finished. $\mathbb{R}$ is yet to come.

The Hierarchy of Numbers



$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$$

The Hierarchy of Numbers

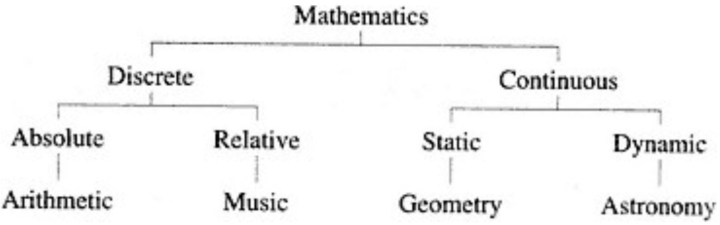
Each set is contained in the next one.

They are like a set of nested Russian Dolls:

Matryoshka

Outline

The Quadrivium



The Pythagorean model of mathematics

The Ancient Greeks

Mathematics and Astronomy are intimately linked.

Two of the strands of the Quadrivium were Geometry (static) and Cosmology (dynamic space).

Greek astronomer *Claudius Ptolemy* (c.90–168AD) placed the Earth at the centre of the universe.

The Sun and planets move around the Earth in orbits that are of the most perfect of all shapes: *circles.*

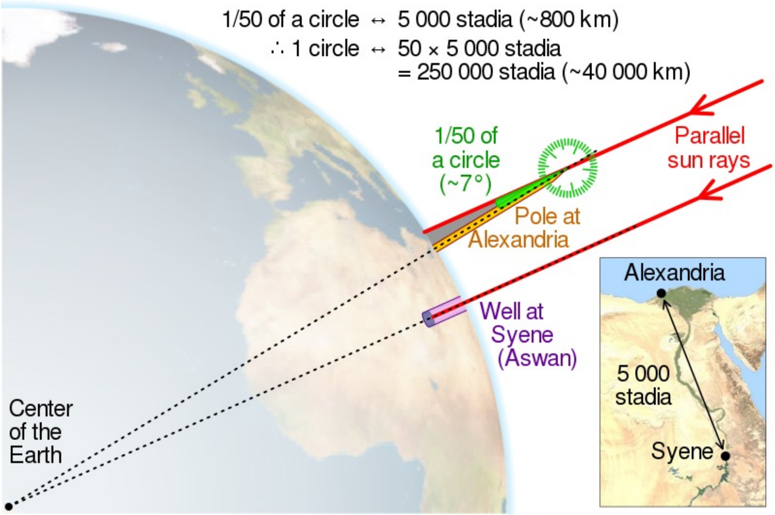
Aristarchus of Samos (c.310–230 BC)

Aristarchus of Samos ('A$\rho\iota\sigma\tau\alpha\rho\chi o\varsigma$), astronomer and mathematician, presented the first model that placed the Sun at the center of the universe.

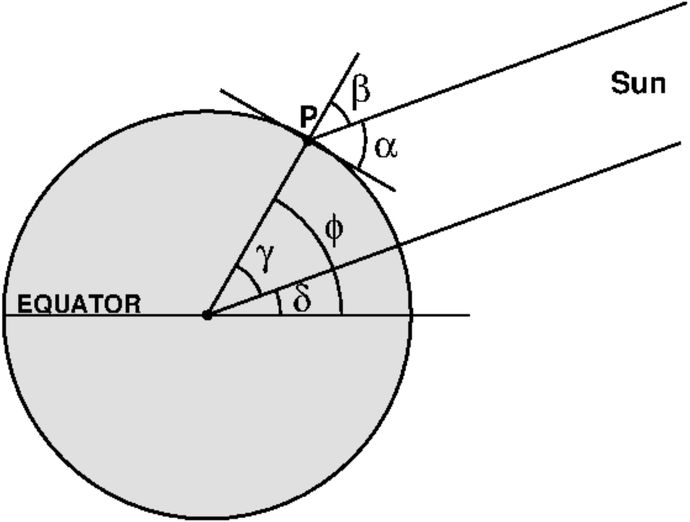
The original writing of Aristarchus is lost, but Archimedes wrote in his *Sand Reckoner:*

"His hypotheses are that the fixed stars and the Sun remain unmoved, that the Earth revolves about the Sun on the circumference of a circle, ... "

Eratosthenes (c.276–194 BC)



Eratosthenes (c.276–194 BC)



Hipparchus (c.190–120 BC)

Hipparchus of Nicaea ($'I\pi\pi\alpha\rho\chi o\varsigma$) was a Greek astronomer, geographer, and mathematician.

Regarded as the greatest astronomer of antiquity.

Often considered to be the *founder of trigonometry.*

Famous for

- Precession of the equinoxes
- First comprehensive *star catalog*
- Invention of the *astrolabe*
- Invention (perhaps) of the *armillary sphere.*

Claudius Ptolemy (c.AD 100–170)

Claudius Ptolemy was a Greco-Roman astronomer, mathematician, geographer and astrologer.

He lived in the city of Alexandria.

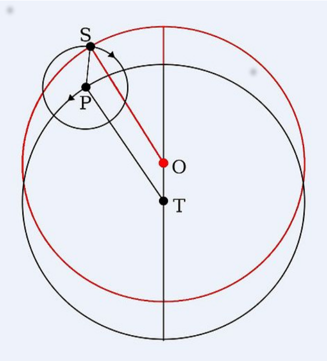
Ptolemy wrote several scientific treatises:

- **An astronomical treatise (*the Almagest*) originally called Mathematical Treatise (Mathematike Syntaxis).**
- **A book on geography.**
- **An astrological treatise.**

Ptolemy's Almagest is the only surviving comprehensive ancient treatise on astronomy.

Ptolemy's Model

Ptolemy's model was universally accepted until the appearance of simpler heliocentric models during the scientific revolution.



O is the earth and S the planet.

“Adding Epicycles”

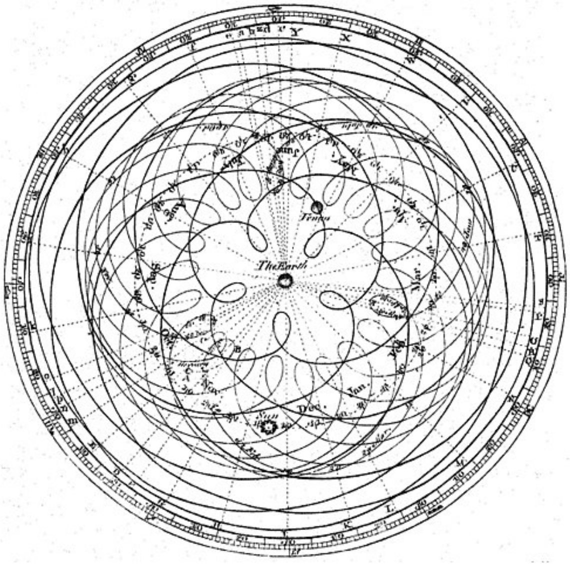
According to *Norwood Russell Hanson* (science historian):

There is no bilaterally symmetrical, nor eccentrically periodic curve used in any branch of astrophysics or observational astronomy which could not be smoothly plotted as the resultant motion of a point turning within a constellation of epicycles, finite in number, revolving around a fixed deferent.

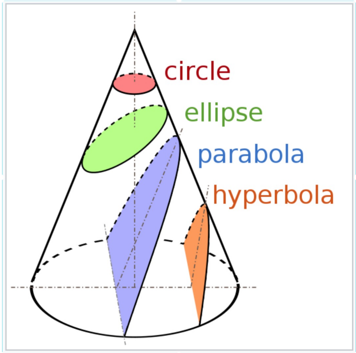
“The Mathematical Power of Epicyclical Astronomy", 1960

Any path — periodic or not, closed or open — can be represented by an infinite number of epicycles.

Ptolemaic Epicycles



Conic Sections



Circles are special cases of conic sections.

They are formed by a plane cutting a cone at an angle.

Conics were studied by *Apollonius of Perga* (late 3rd – early 2nd centuries BC).

`https://en.wikipedia.org/wiki/Conic_section`

The Scientific Revolution

TRAILER

Next week, we will look at developments in the sixteenth and seventeenth centuries.

Figure from mathigon.org

Thank you