# AweSums

## Marvels and Mysteries of Mathematics

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## LECTURE 5

**Peter Lynch**
**School of Mathematics & Statistics**
**University College Dublin**

**Evening Course, UCD, Autumn 2019**

# Outline

# Outline

## Introduction

## Quadrivium

## Greek 4

## Theorem of Pythagoras

## Lateral Thinking 2

## The Unary System

## Topology II

# Meaning and Content of Mathematics

**The word Mathematics comes from Greek** $\mu\alpha\theta\eta\mu\alpha$ **(máthéma), meaning "knowledge" or "study" or "learning".**

**It is the study of topics such as**

- ► **Quantity (numbers)**
- ► **Structure (patterns)**
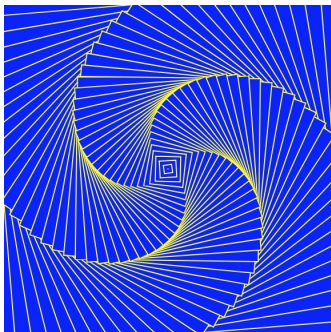- ► **Space (geometry)**
- ► **Change (analysis).**

# Demo: Φ-TOP in a Magnetic Field

# Demo: Spinning and Shrinking Polygons

**Run Mathematica Notebook
$HOME/Dropbox/AweSums/Miscellaneous/
SpinAndShrinkSquares.nb**



**An Emergent Geometric Pattern**

# Outline

# The Quadrivium

# The Quadrivium

The Quadrivium originated with the Pythagoreans around 500 BC.

The Pythagoreans' quest was to find *the eternal laws of the Universe,* and they organized their studies into the scheme later known as the Quadrivium.

It comprised four disciplines:

- ▶ **Arithmetic**
- ▶ **Geometry**
- ▶ **Music**
- ▶ **Astronomy**

# The Quadrivium

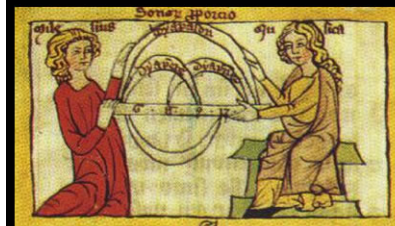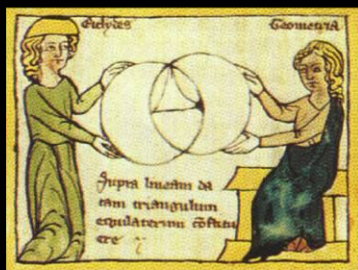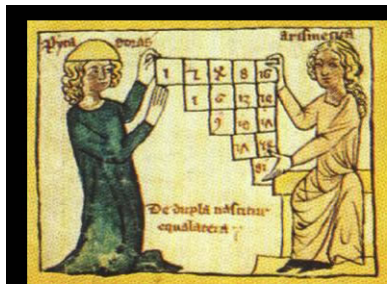**First comes Arithmetic, concerned with the infinite linear array of numbers.**

**Moving beyond the line to the plane and 3D space, we have Geometry.**

**The third discipline is Music, which is an application of the science of numbers.**

**Fourth comes Astronomy, the application of Geometry to the world of space.**

# The Quadrivium

# Static/Dynamic. Pure/Applied

- **Arithmetic (static number)**
- **Music (moving number)**
- **Geometry (measurement of static Earth)**
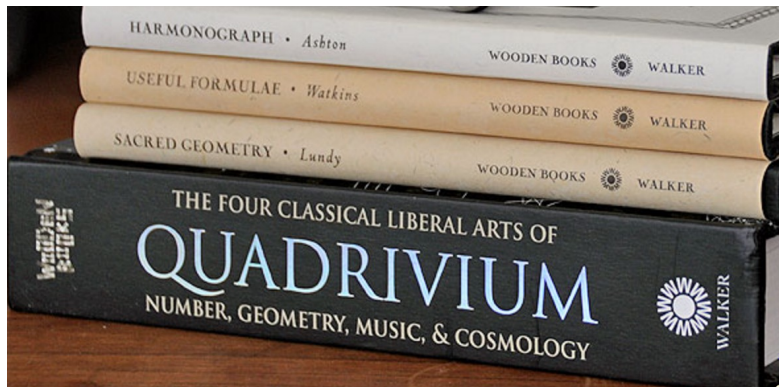- **Astronomy (measurement of moving Heavens)**

**Arithmetic represents numbers at rest,**
**Geometry is magnitudes at rest,**

**Music is numbers in motion and**
**Astronomy is geometry in motion.**

**The first two are *pure* in nature,**
**while the last two are *applied.***

# The Quadrivium



**For the Greeks, *Mathematics* embraced all four areas.**

# The Pythagoreans

**Pythagoras distinguished between quantity and magnitude.**

**Objects that can be counted yield quantities or numbers.**

**Substances that are measured provide magnitudes.**

**Thus, *cattle are counted* whereas *milk is measured.***

**Arithmetic studies quantities or numbers and Music involves the relationship between numbers and their evolution in time.**

**Geometry deals with magnitudes, and Astronomy with their distribution in space.**

# Archytas (428–350 BC): *APXΥ TAΣ*
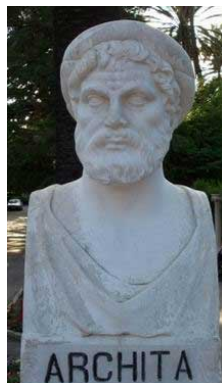


*Aρχυτας.*

**Born in Tarentum, son of Hestiaeus.**

**Mathematician and philosopher.**

**Pythagorean, student of Philolaus.**

**Provided a solution for the Delian problem of doubling the cube.**

***Said to have tutored Plato in mathematics(?)***

# Archytas (428–350 BC)

**Archytas lived in Tarentum (now in Southern Italy).**

**One of the last scholars of the Pythagorean School and was a good friend of Plato.**

*The designation of the four disciplines of the Quadrivium was ascribed to Archytas.*

**His views were to dominate pedagogical thought for over two millennia.**

**Partly due to Archytas, mathematics has played a prominent role in education ever since.**

# Plato's Academy

**According to Plato, mathematical knowledge was essential for an understanding of the Universe. The curriculum was outlined in Plato's *Republic.***

**Inscription over the entrance to Plato's Academy:**



*"Let None But Geometers Enter Here".*

**This indicated that the Quadrivium was a prerequisite for the study of philosophy in ancient Greece.**

# Boethius (AD 480–524)

**The Western Roman Empire was in many ways static for centuries.**

**No new mathematics between the conquest of Greece and the fall of the Roman Empire in AD 476.**

*Boethius,* **the 6th century Roman philosopher, was one of the last great scholars of antiquity.**

**The organization of the Quadrivium was formalized by Boethius.**

**It was the mainstay of the medieval monastic system of education.**

# The Quadrivium

# Typus Arithmeticae

**A woodcut from the book *Margarita Philosophica,* by Gregor Reisch, Freiburg, 1503.**

**The central figure is *Dame Arithmetic,* watching a competition between Boethius, using pen and Hindu-Arabic numerals, and Pythagoras, using a counting board or *tabula.***

**She looks favourably toward Boethius.**

**But how did Boetius know about Hindu-Arabic numerals?**

# The Liberal Arts

**The seven liberal arts comprised
the *Trivium* and the *Quadrivium.***

**The Trivium was centred on three arts of language:**

- **Grammar: proper structure of language.**
- **Logic: for arriving at the truth.**
- **Rhetoric: the beautiful use of language.**

**Aim of the Trivium: *Goodness, Truth and Beauty.***

**Aristotle traced the origin of the Trivium back to Zeno.**

# The Ultimate Goal

The goal of studying the Quadrivium was
*an insight into the nature of reality,
an understanding of the Universe.*

The Quadrivium offered the seeker of wisdom
*an understanding of the integral nature of
the Universe and the role of humankind within it.*

That is our aim in AweSums!

# Outline

# The Greek Alphabet, Part 4

| | | | | | |
|---|---|---|---|---|---|
| $\alpha$ | $\beta$ | $\gamma$ | $\delta$ | $\varepsilon$ | $\zeta$ |
| Alpha | Beta | Gamma | Delta | Epsilon | Zeta |
| $\eta$ | $\theta$ | $\iota$ | $\varkappa$ | $\lambda$ | $\mu$ |
| Eta | Theta | I ota | Kappa | Lambda | Mu |
| $\nu$ | $\xi$ | $o$ | $\pi$ | $\rho$ | $\sigma$ |
| Nu | Xi | Omicron | Pi | Rho | Sigma |
| $\tau$ | $\upsilon$ | $\varphi$ | $\chi$ | $\psi$ | $\omega$ |
| Tau | Upsilon | Phi | Chi | Psi | Omega |
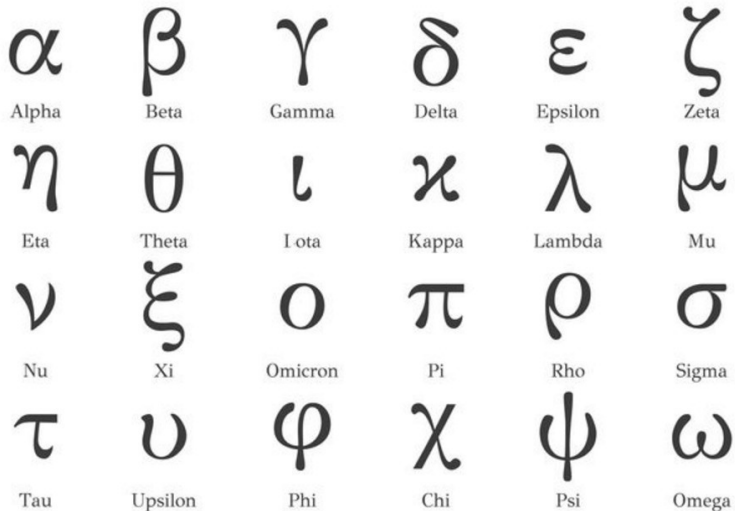
**Figure :** 24 beautiful letters

# The Last Six Letters

**We will consider the final group of six letters.**

$$\tau \qquad \upsilon \qquad \phi \qquad \chi \qquad \psi \qquad \omega$$

$$\mathrm{T} \qquad \Upsilon \qquad \Phi \qquad \mathrm{X} \qquad \Psi \qquad \Omega$$

**Let us focus first on the *small letters*
and come back to the big ones later.**

$$\tau \qquad \upsilon \qquad \phi \qquad \chi \qquad \psi \qquad \omega$$

**Tau: You have certainly heard of a Tau-cross: $\tau$.**

**Upsilon ($\upsilon$) or u-psilon means 'bare u'.**
**It is often transliterated as 'y'.**

**Phi ($\phi$) is ' f ', often used for latitude**
**(as $\lambda$ is often used for longitude).**

**Chi ($\chi$) has a 'ch' or 'k' sound.**

**Psi ($\psi$) is very common: psychology, etc.**

**Omega ($\omega$) is the end: Alpha and Omega $\left( \dfrac{\mathrm{A}}{\Omega} \right)$.**

**Now you know 24 letters. You should get a diploma.**

# A Few Greek Words (for practice)

$\kappa\omega\mu\alpha$                 **Coma:** $\kappa\omega\mu\alpha$
$\psi\upsilon\kappa\eta$                 **Psyche:** $\psi\upsilon\kappa\eta$
$\kappa\rho\iota\sigma\iota\varsigma$                 **Crisis:** $\kappa\rho\iota\sigma\iota\varsigma$

$\alpha\nu\alpha\theta\epsilon\mu\alpha$              **Anathema:** $\alpha\nu\alpha\theta\epsilon\mu\alpha$
$\alpha\mu\beta\rho o\sigma\iota\alpha$             **Ambrosia:** $\alpha\mu\beta\rho o\sigma\iota\alpha$
$\kappa\alpha\tau\alpha\sigma\tau\rho o\phi\eta$       **Catastrophe:** $\kappa\alpha\tau\alpha\sigma\tau\rho o\phi\eta$

**End of Greek 104**

# Outline

# Theorem of Pythagoras

**The Theorem of Pythagoras is of fundamental importance in Euclidean geometry**

*It encapsulates the structure of space.*

**In the BBC series, The Ascent of Man, Jacob Bronowski said**

**"The theorem of Pythagoras remains the most important single theorem in mathematics."**

# Theorem of Pythagoras

**YouTube video with Danny Kaye**

**Google search for
"Danny Kaye Hypotenuse"**

```
https:
//www.youtube.com/watch?v=oeRCsAGQVy8
```

# Hypotenuse

**The side of a right triangle opposite to the right angle.**

**1570s, from Late Latin hypotenusa, from Greek hypoteinousa "stretching under" (the right angle).**

**Fem. present participle of hypoteinein, from hypo- "under" + teinein "to stretch"**

From Online Etymology Dictionary: **http://www.etymonline.com/**

# Mathigon.org

**Mathigon.org video on *Proofs without Formulas:***

- ▶ **What is the sum of the angles in a triangle?**
- ▶ **What is the sum of the angles in a polygon?**
- ▶ **What is the area of a triangle?**
- ▶ **How does Pythagoras' Theorem work?**

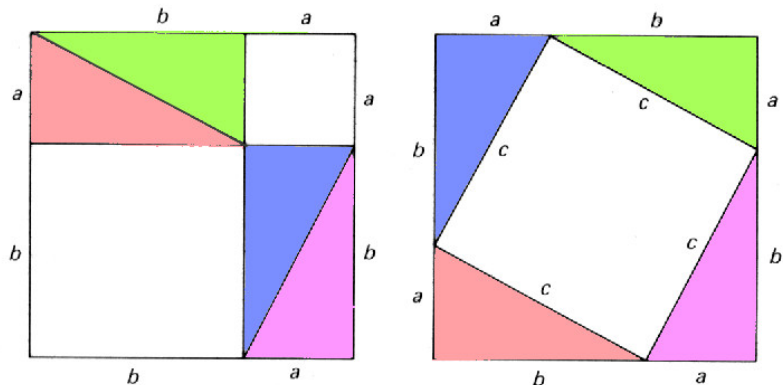**In the video below, these and other important concepts are explained in only two minutes using nothing but graphics.**

```
https://youtu.be/IUCK8bk0xPo
```

# Proof without Formulae
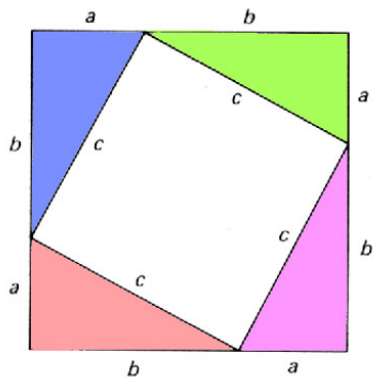
# Proof without Formulae
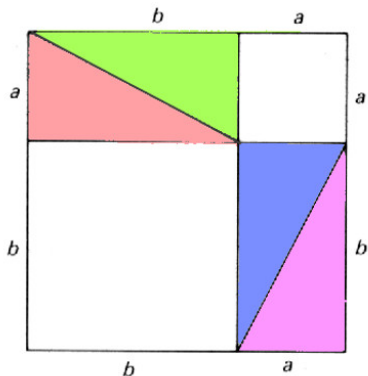
# Proof without Formulae



$$a^2 + b^2 = c^2$$

# Why is this Important / Interesting?

**Squares on the sides of triangles don't seem much.**

*But the theorem gives us distances.*

**If one point is at $(0,0)$ and another at $(x, y)$, the theorem gives the distance:**

$$r^2 = x^2 + y^2 \qquad \textbf{or} \qquad r = \sqrt{x^2 + y^2}$$

**This tells us about the structure of space.**

**I should expand on this topic (e.g., SAR)**

# Outline

# Set Theory Puzzle

**In a small Canadian village, everyone
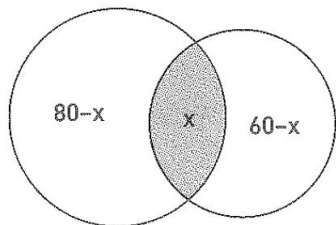speaks either English or French, or both.**

    **80% of the people speak French**

    **60% of the people speak English**

**What percentage speak both English and French?**
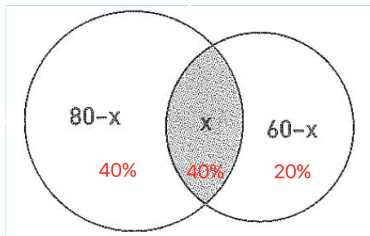
**Answer next week!**

$$(80 - x) + x + (60 - x) = 100.$$

**Therefore**

$$140 - x = 100 \quad \textbf{or} \quad x = 40.$$

# Outline

# The Unary System

**The simplest numeral system is the *unary system.***

**Each natural number is represented by a corresponding number of symbols.**

**If the symbol is " | " , the number seven would be represented by | | | | | | | .**

*Tally marks represent one such system, which is still in common use.*

**The unary system is only useful for small numbers.**

**The unary notation can be abbreviated, with new symbols for certain values.**

# Sign-Value Notation

**The *five-bar gate* system groups 5 strokes together.**

**Normally, distinct symbols are used for powers of 10.**

**If " | " stands for one, " ∧ " for ten and " ⸎ " for 100, then the number *123* becomes  ⸎ ∧∧ | | |**
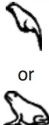
**There is no need for a symbol for zero.**

**This is called sign-value notation.**

**Ancient Egyptian numerals were of this type.**

**Roman numerals were a modification of this idea.**

# Egypyian Numerals

| Value | 1 | 10 | 100 | 1,000 | 10,000 | 100,000 | 1 million, or many |
|---|---|---|---|---|---|---|---|
| Hieroglyph | I | ∩ | ℮ | (Water lily glyph) | (Bent finger glyph) | or (Tadpole/Frog glyphs) | (Man with raised hands glyph) |
| Description | Single stroke | Heel bone | Coil of rope | Water lily (also called Lotus) | Bent Finger | Tadpole or Frog | Man with both hands raised, perhaps Heh.[3] |

**Figure :** From Wikipedia page `https://en.wikipedia.org/wiki/Egyptian_numerals`
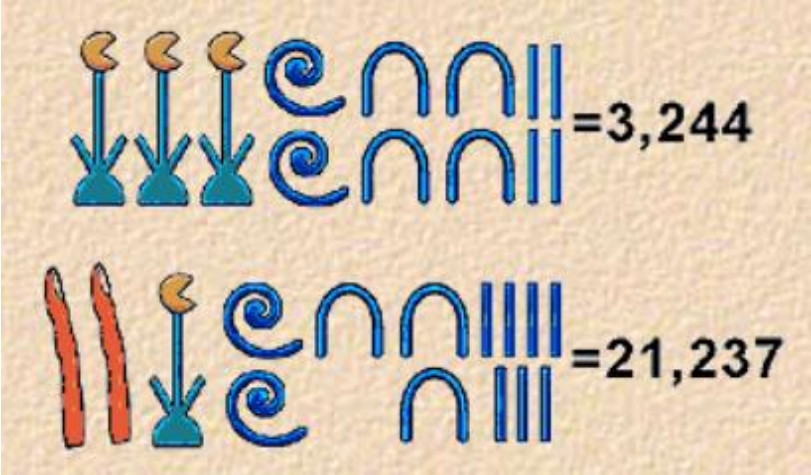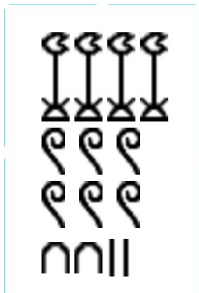
# Egypyian Numerals
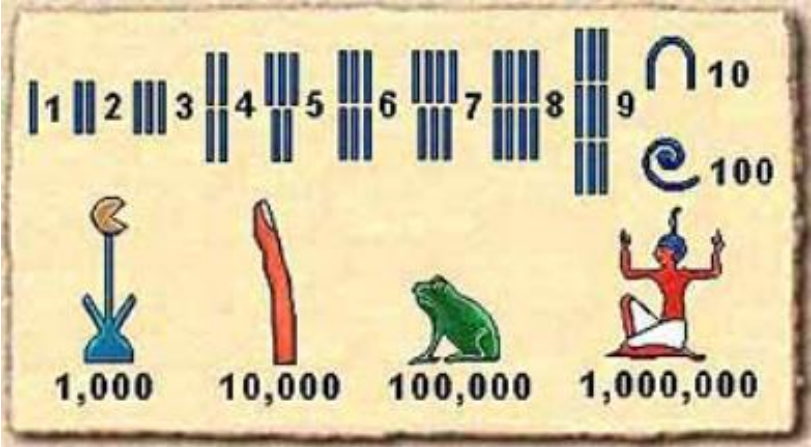
# Egypyian Numerals

**The arrangement of symbols is not important.**

**What number is this?**

**This pattern represents**

**4622.**

# Hebrew Numerals

Hebrew Number Values

| | | |
|---|---|---|
| א | Aleph - | 1 |
| ב | Beth - | 2 |
| ג | Gimel - | 3 |
| ד | Daleth - | 4 |
| ה | Heh - | 5 |
| ו | Vav - | 6 |
| ז | Zain - | 7 |
| ח | Cheth - | 8 |
| ט | Teth - | 9 |
| י | Yod - | 10 |
| כ | Kaph - | 20 |

| | | |
|---|---|---|
| ל | Lamed - | 30 |
| מ | Mem - | 40 |
| נ | Nun - | 50 |
| ס | Samekh - | 60 |
| ע | Ayin - | 70 |
| פ | Peh - | 80 |
| צ | Tzaddi - | 90 |
| ק | Qoph - | 100 |
| ר | Resh - | 200 |
| ש | Shin - | 300 |
| ת | Tau - | 400 |

**The 22 letters of the Hebrew alphabet were used also as numerals.**

*Each letter corresponded to a numerical value.*

# Greek Numerals

|   | Units | Tens | Hundreds |
|---|---|---|---|
| 1 | α alpha | ι iota | ρ rho |
| 2 | β beta | κ kappa | σ sigma |
| 3 | γ gamma | λ lambda | τ tau |
| 4 | δ delta | μ mu | υ upsilon |
| 5 | ε epsilon | ν nu | φ phi |
| 6 | ϝ digamma | ξ xi | χ chi |
| 7 | ζ zeta | ο omicron | ψ psi |
| 8 | η eta | π pi | ω omega |
| 9 | θ theta | ϙ koppa | ϡ sampi |

*The 24 letters of the Greek alphabet had corresponding numerical values.*

**They were supplemented by three additional letters, which are now archaic.**

$$\sigma\mu\gamma = ?$$

$$243 = \sigma\mu\gamma$$

# Greek Numerals

| Arabic number | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| Greek number | $\alpha$ | $\beta$ | $\gamma$ | $\delta$ | $\varepsilon$ | F | $\zeta$ | $\eta$ | $\theta$ |
| Greek name | alpha | beta | gamma | delta | epsilon | digamma | zeta | eta | theta |
| Sound | a | b | g | d | short e | | z | long e | th |

| Arabic number | 10 | 20 | 30 | 40 | 50 | 60 | 70 | 80 | 90 |
|---|---|---|---|---|---|---|---|---|---|
| Greek number | $\iota$ | $\kappa$ | $\lambda$ | $\mu$ | $\nu$ | $\xi$ | o | $\pi$ | $\varsigma$ |
| Greek name | iota | kappa | lambda | mu | nu | xi | omicron | pi | koppa |
| Sound | i | k/c | l | m | n | x | short o | p | |

| Arabic number | 100 | 200 | 300 | 400 | 500 | 600 | 700 | 800 | 900 |
|---|---|---|---|---|---|---|---|---|---|
| Greek number | $\varrho$ | $\sigma$ | $\tau$ | $\upsilon$ | $\phi$ | $\chi$ | $\psi$ | $\omega$ | $\pi$ |
| Greek name | rho | sigma | tau | upsilon | phi | chi | psi | omega | sampi |
| Sound | r | s | t | u | f/ph | ch | ps | long o | |

# Outline

# Topology: a Major Branch of Mathematics

**Topology is all about continuity and connectivity, but the meaning of that will appear later.**

**We will look at a few aspects of Topology.**

- ► **The Bridges of Königsberg**
- ► **Doughnuts and Coffee-cups**
- ► **Knots and Links**
- ► **Nodes and Edges: Graphs**
- ► **The Möbius Band**
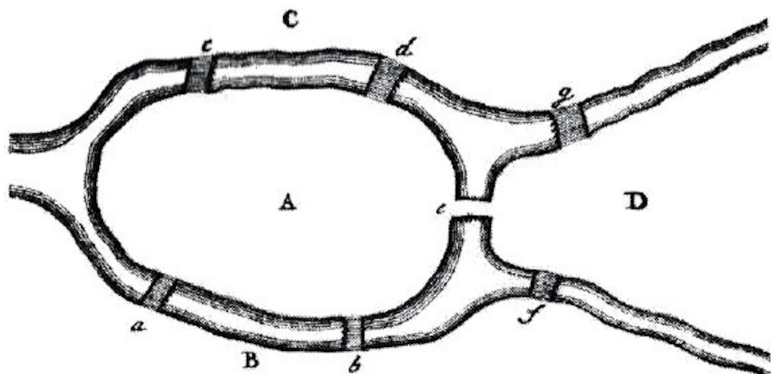
**In this lecture, we study *The Bridges of Königsberg.***

# The Bridges of Königsberg

One of the earliest topological puzzles was studied by the renowned Swiss mathematician *Leonard Euler.*

It is called 'The Seven Bridges of Königsberg'.

The goal is to find a route through that city, crossing each of seven bridges exactly once.

# The Bridges of Königsberg

# The Bridges of Königsberg
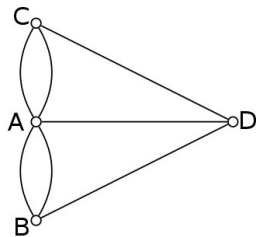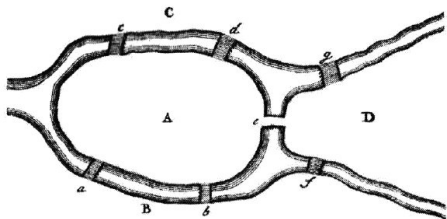
**Euler reduced the problem to its essentials, removing all extraneous details.**
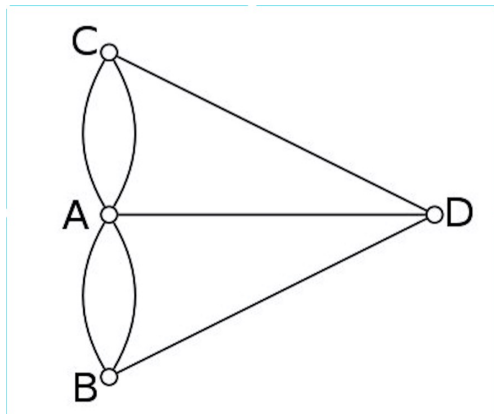
**He replaced the map above by the graph on the right.**

**A simple argument showed that no journey that crosses each bridge exactly once is possible.**

**Except at the termini of the route, each path arriving at a vertex must have a corresponding path leaving it.**

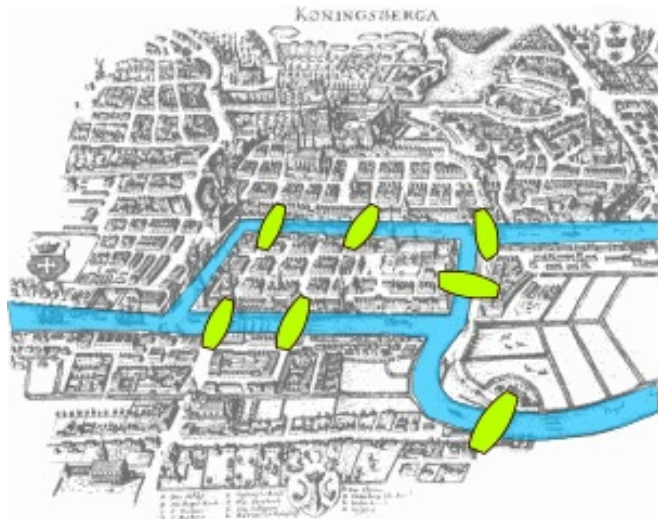**Only two vertices with an odd number of edges are possible for a solution to exist.**
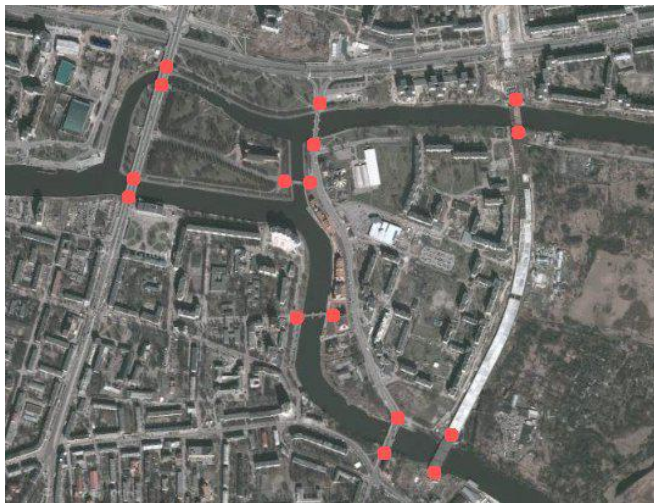
# The Bridges of Königsberg



**Exercise: Draw the diagram with *A*, *B*, *C* and *D* arranged clockwise at the corners of a square.**
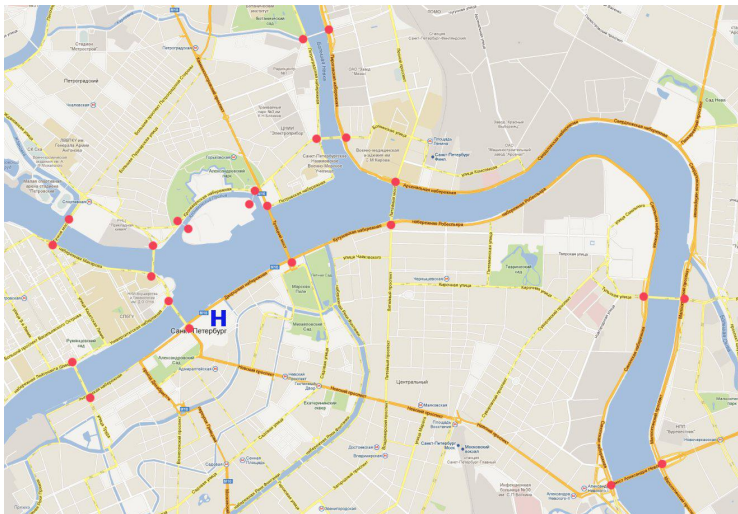
# The Bridges of Königsberg



KONINGSBERGA

# Königsberg Today

# The Bridges of St Petersburg

# The Bridges of St Petersburg

**Euler spend much of his life in St Petersburg, a city with many rivers, canals and bridges.**

**Did he think about another problem like the Königsberg Bridges problem while there?**

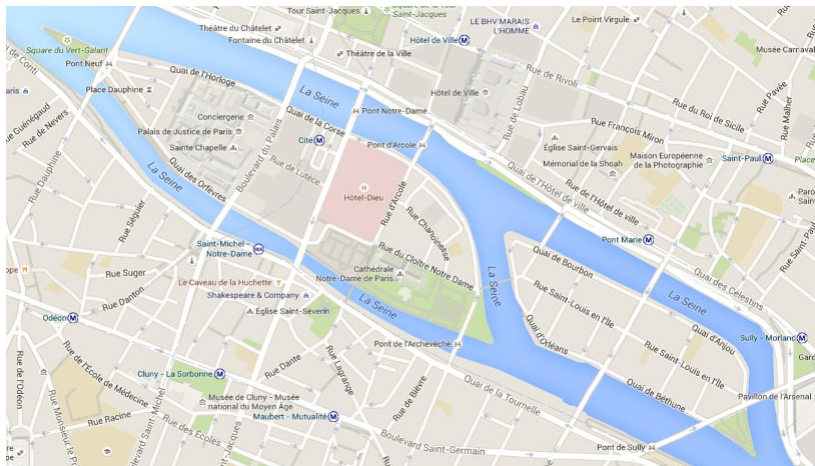**The map of central St Petersburg has twelve bridges.**

**An Euler cycle is a route that crosses all bridges exactly once and returns to the starting point?**

**Is there an Euler cycle starting at the Hermitage (marked "H" on the map)?**

# The Bridges of Paris

**Cue romantic music**

# The Bridges of Paris

**In central Paris, two small islands, Île de la Cité and Île Saint-Louis, are linked to the Left and Right Banks of the Seine and to each other.**

**The number of bridges for each land-mass are:**
- **Left Bank: 7 bridges**
- **Right Bank: 7 bridges**
- **Île de la Cité: 10 bridges**
- **Île Saint-Louis: 6 bridges**

**The total is 30. How many bridges are there?**

# The Bridges of Paris



RIGHT BANK

LEFT BANK

# The Bridges of Paris

1. **Starting from Saint-Michel on the Left Bank, walk continuously so as to cross each bridge once.**
2. **Start at Saint-Michel but find a closed route that ends back at the starting point.**
3. **Start at Notre-Dame Cathedral, on Île de la Cité, and cross each bridge exactly once.**
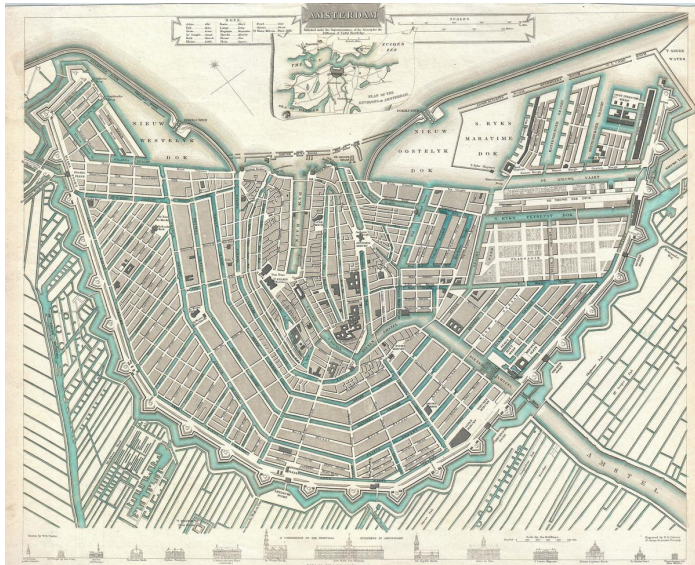4. **Find a closed route that crosses each bridge once and arrives back at Notre-Dame.**

**Try these puzzles yourself. *Use logic, not brute force!***

# The Bridges of Paris
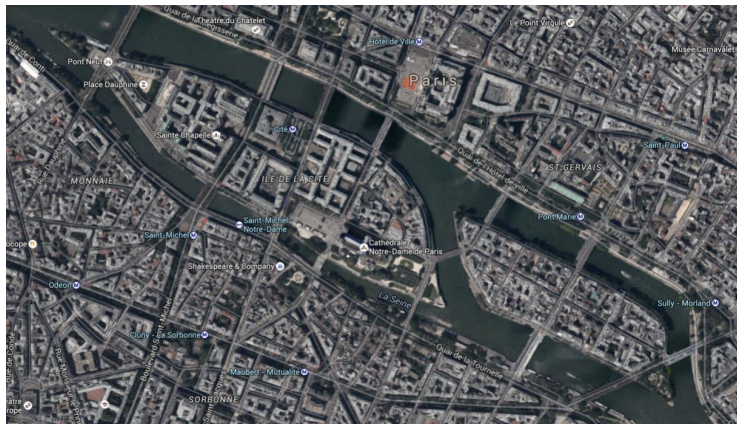
# The Bridges of Amsterdam

# Wikipedia Article

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## Seven Bridges of Königsberg

From Wikipedia, the free encyclopedia

Coordinates: 54°42'12"N 20°30'56"E

*This article is about an abstract problem. For the historical group of bridges in the city once known as Königsberg, and those of them that still exist, see § Present state of the bridges.*

This article **needs additional citations for verification**. Please help improve this article by adding citations to reliable sources. Unsourced material may be challenged and removed. *(July 2015) (Learn how and when to remove this template message)*

The **Seven Bridges of Königsberg** is a historically notable problem in mathematics. Its negative resolution by Leonhard Euler in 1736 laid the foundations of graph theory and prefigured the idea of topology.[1]

The city of Königsberg in Prussia (now Kaliningrad, Russia) was set on both sides of the Pregel River, and included two large islands which were connected to each other, or to the two mainland portions of the city, by seven bridges. The problem was to devise a walk through the city that would cross each of those bridges once and only once.

By way of specifying the logical task unambiguously, solutions involving either

1. reaching an island or mainland bank other than via one of the bridges, or
2. accessing any bridge without crossing to its other end

are explicitly unacceptable.

Euler proved that the problem has no solution. The difficulty he faced was the development of a suitable technique of analysis, and of subsequent tests that established this assertion with mathematical rigor.
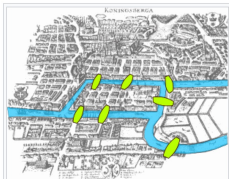
Map of Königsberg in Euler's time showing the actual layout of the seven bridges, highlighting the river Pregel and the bridges

**Thank you**