AweSums

Marvels and Mysteries of Mathematics

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LECTURE 2

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Evening Course, UCD, Autumn 2019

Outline

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Meaning and Content of Mathematics

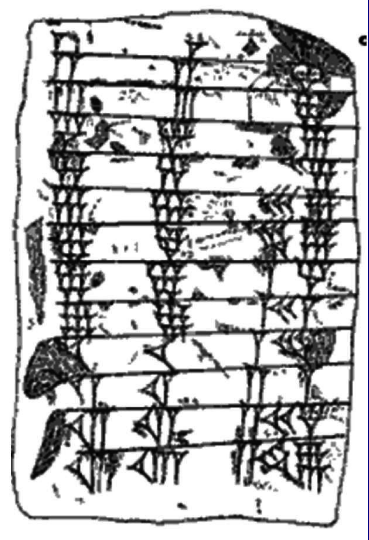
The word Mathematics comes from Greek $\mu\alpha\theta\eta\mu\alpha$ (máthéma), meaning "knowledge" or "study" or "learning".

It is the study of topics such as

- Quantity (numbers)
- Structure (patterns)
- Space (geometry)
- Change (analysis).

Outline

The Nippur Tablet



What is the last line?

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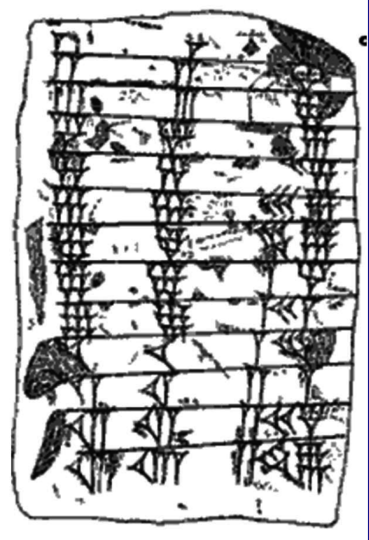

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The last line states that

$$13 \times 13 = 2 \times 60 + 49 = 169$$

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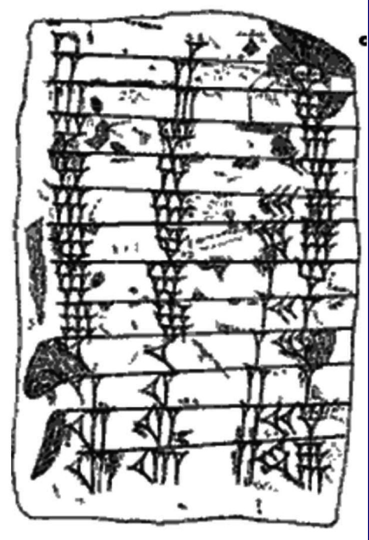

What is the last line?
The last line states that

$$13 \times 13 = 2 \times 60 + 49 = 169$$

But it could be

$$13 \times 13 = 2 \times 60^2 + 40 \times 60 + 9$$

which comes to 9609. Babylonian numeration is ambiguous.

There is no zero!

The Nippur Tablet

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What use could there be for a list of squares?

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What use could there be for a list of squares?

Perhaps it was used for multiplication!

After a brief refresher on school maths,
we show how this can be done.

Refresher: Some School Maths

How do we do multiplication of binomials

$$(a + b) \times (c + d)\ ?$$

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This can be evaluated by expanding twice:

$$a\cdot(c+d)+b\cdot(c+d)=a\cdot c+a\cdot d+b\cdot c+b\cdot d$$

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A special case where the two factors are equal:

$$(a + b) \cdot (a + b) \quad = \quad a \cdot a + a \cdot b + b \cdot a + b \cdot b$$

so that

$$(a + b)^2 \quad = \quad a^2 + 2ab + b^2$$

Multiplication by Squaring

Let a and b be any two numbers:

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Thus, we can find the product using squares:

$$ab = \frac{1}{4}\left[(a+b)^2 - (a-b)^2\right].$$

Every product is the difference of two squares ($\div 4$).

Multiplication by Squaring

$$\frac{1}{4}\left[(a+b)^2-(a-b)^2\right]=ab$$

Let us take a particular example: $37 \times 13 = ?$

$$a = 37 \quad b = 13 \quad a+b = 50 \quad a-b = 24\,.$$

Multiplication by Squaring

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Let us take a particular example: $37 \times 13 = ?$

$$a = 37 \quad b = 13 \quad a+b = 50 \quad a-b = 24\,.$$

$$\begin{aligned} \tfrac{1}{4}[50^2 - 24^2] &= \tfrac{1}{4}[2500 - 576] \\ &= \tfrac{1}{4}[1924] \\ &= 481 \\ &= 37 \times 13\,. \end{aligned}$$

Perhaps this was the function of the Nippur tablet.

Practicalities in Babylon

$$ab = \frac{1}{4}\left[(a+b)^2 - (a-b)^2\right] .$$

Suppose it was important to be able to multiply numbers up to, say, 100.

A full multiplication table would have 10,000 entries. With 20 products on each tablet, this would mean 500 clay tablets!

A table of squares up to 200 would require only 10 clay tablets.

Outline

Introduction

The Nippur Tablet

Distraction 2: Simpsons

Georg Cantor

Greek 1

Set Theory I

Distraction: The Simpsons

Several writers of the Simpsons scripts have advanced mathematical training.

They "sneak" jokes into the programmes.

Books on a Shelf

Ten books are arranged on a shelf.
They include an Almanac (A) and a Bible (B).

Suppose A must be to the left of B
(not necssarily beside it).

How many possible arrangements are there?

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BIG IDEA: SYMMETRY.

Every SOLUTION correponds to a NON-SOLUTION:
Just switch the positions of A and B!

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BIG IDEA: SYMMETRY.

Every SOLUTION correponds to a NON-SOLUTION:
Just switch the positions of A and B!

The total number of arrangements is 10!.
For half of these, A is to the left of B.

So, answer is $\frac{1}{2}(10 \times 9 \times \cdots \times 1) = \frac{1}{2} \times 10!$ Q.E.D.

Stirling's Formula

$$n! = 1 \times 2 \times 3 \times \cdots \times n$$

An illustration of the ubiquity of π and e.
Stirling's approximation for factorials is

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

where the numbers π and e are

$$\pi = 3.14159\ldots \qquad \textbf{and} \qquad e = 2.71828\ldots$$

Outline

Georg Cantor

Inventor of Set Theory

Born in St. Petersburg, Russia in 1845.

Moved to Germany in 1856 at the age of 11.

His main career was at the University of Halle.

Georg Cantor (1845–1918)

- Invented Set Theory.
- One-to-one Correspondence.
- Infinite and Well-ordered Sets.
- Cardinal and Ordinal Numbers.
- Proved: $\#(\mathbb{Q}) = \#(\mathbb{N})$.
- Proved: $\#(\mathbb{R}) > \#(\mathbb{N})$.
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Outline Galileo's arguments on infinity.

Set Theory: Controversy

Cantor was strongly criticized by

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Set Theory is a "grave disease" (HP).
Set Theory is "nonsense; laughable; wrong!" (LW).

Set Theory: Controversy

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- Henri Poincaré.
- Ludwig Wittgenstein.

Cantor is a "corrupter of youth" (LK).
Set Theory is a "grave disease" (HP).
Set Theory is "nonsense; laughable; wrong!" (LW).

Adverse criticism like this may well have contributed to Cantor's mental breakdown.

Set Theory: A Difficult Birth

Set Theory brought into prominence several paradoxical results.

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To illustrate the difficulty of accepting new ideas, let's consider the problem of a river flowing uphill.

Describe the blog post "Paddling Uphill".

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Gösta Mittag-Leffler was reluctant to publish it in his *Acta Mathematica*. He said the work was "100 years ahead of its time".

David Hilbert said:
"We shall not be expelled from the paradise that Cantor has created for us."

A Passionate Mathematician

In 1874, Cantor married Vally Guttmann.

They had six children. The last one, a son named Rudolph, was born in 1886.

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According to Wikipedia:

“During his honeymoon in the Harz mountains, Cantor spent much time in mathematical discussions with Richard Dedekind.”

[Cantor had met the renowned mathematician Dedekind two years earlier while he was on holiday in Switzerland.]

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The Greek Alphabet, Part 1

Ελληνικό αλφάβητο

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Some Motivation

- Greek letters are used extensively in maths.
- Greek alphabet is the basis of the Roman one.
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Ελληνικό αλφάβητο

Some Motivation

- Greek letters are used extensively in maths.
- Greek alphabet is the basis of the Roman one.
- Also the basis of the Cyrillic and others.
- A great advantage for touring in Greece.
- You already know several of the letters.
- It is simple to learn in small sections.

Ursa Major

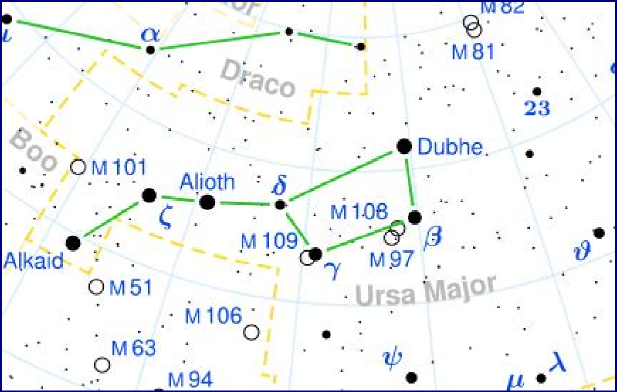

Figure: The Great Bear: Dubhe is α-Ursae Majoris.

Letter	Name	Sound	
		Ancient[5]	Modern[6]
Α α	alpha, άλφα	[a] [aː]	[a]
Β β	beta, βήτα	[b]	[v]
Γ γ	gamma, γάμμα	[g], [ŋ][7]	[ɣ] ~ [ʝ], [ŋ][8] ~ [ɲ][9]
Δ δ	delta, δέλτα	[d]	[ð]
Ε ε	epsilon, έψιλον	[e]	[e]
Ζ ζ	zeta, ζήτα	[zd][A]	[z]
Η η	eta, ήτα	[ɛː]	[i]
Θ θ	theta, θήτα	[tʰ]	[θ]
Ι ι	iota, ιώτα	[i] [iː]	[i], [ʝ],[10] [ɲ][11]
Κ κ	kappa, κάππα	[k]	[k] ~ [c]
Λ λ	lambda, λάμδα	[l]	[l]
Μ μ	mu, μυ	[m]	[m]

Letter	Name	Sound	
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Ν ν	nu, νυ	[n]	[n]
Ξ ξ	xi, ξι	[ks]	[ks]
Ο ο	omicron, όμικρον	[o]	[o]
Π π	pi, πι	[p]	[p]
Ρ ρ	rho, ρώ	[r]	[r]
Σ σ/ς[13]	sigma, σίγμα	[s]	[s]
Τ τ	tau, ταυ	[t]	[t]
Υ υ	upsilon, ύψιλον	[y] [yː]	[i]
Φ φ	phi, φι	[pʰ]	[f]
Χ χ	chi, χι	[kʰ]	[x] ~ [ç]
Ψ ψ	psi, ψι	[ps]	[ps]
Ω ω	omega, ωμέγα	[ɔː]	[o]

Figure: The Greek Alphabet (from Wikipedia)

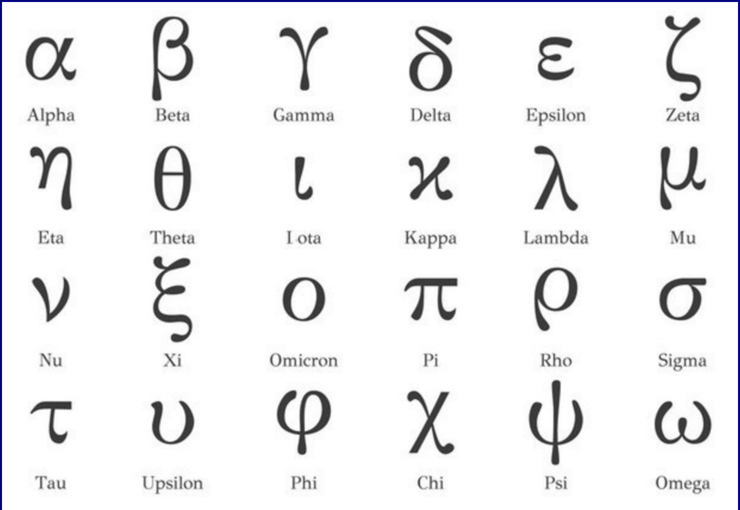

Figure: 24 beautiful letters

The First Six Letters

We will take the alphabet in groups of six letters.

α	β	γ	δ	ϵ	ζ
A	B	Γ	Δ	E	Z

Let us focus first on the small letters
and come back to the big ones later.

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Both δ and ϵ are widely used in maths. For example, the definition of **continuity** of function $f(x)$ at $x = a$ is

$$\forall\epsilon > 0 \; \exists\delta > 0 : |x - a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon$$

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A famous unsolved maths problem, Riemann's Hypothesis, is concerned with zeros of the **Riemann zeta-function:**

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Now we already know the first six letters!

End of Greek 101

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Set Theory I

The concept of set is very general.

Sets are the basic building-blocks of mathematics.

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Definition: A set is a collection of objects.

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Examples:

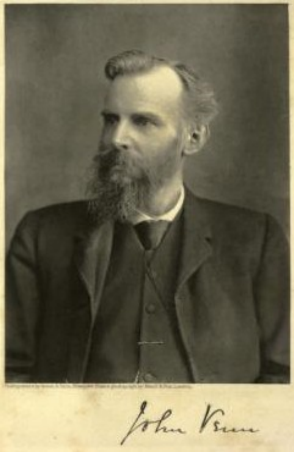
- All the prime numbers, $\mathbb{P}$
- All even numbers: $\mathbb{E} = \{2, 4, 6, 8 \dots\}$
- All the people in Ireland: See Census returns.
- The colours of the rainbow: {Red, . . . , Violet}.
- Light waves with wavelength $\lambda \in [390 - 700\text{nm}]$

Do You Remember Venn?

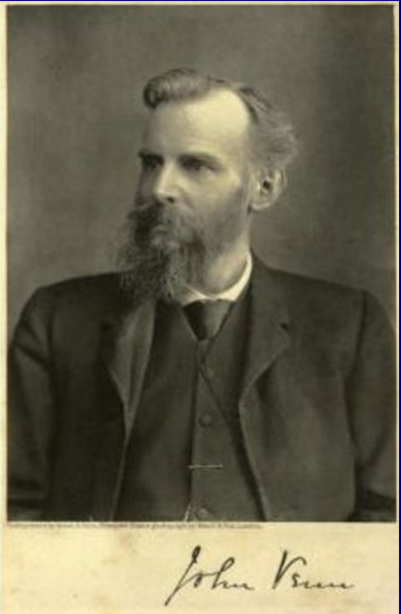
John Venn was a logician and philosopher, born in Hull, Yorkshire in 1834.

He studied at Cambridge University, graduating in 1857 as sixth Wrangler.

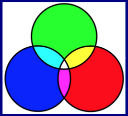
Venn introduced his diagrams in *Symbolic Logic*, a book published in 1881.




JOHN VENN
FELLOW
PRESIDENT


John Venn

Venn Diagrams



Venn diagrams are very valuable for showing elementary properties of sets.

They comprise a number of overlapping circles.

The interior of a circle represents a collection of numbers or objects or perhaps a more abstract set.

The Universe of Discourse

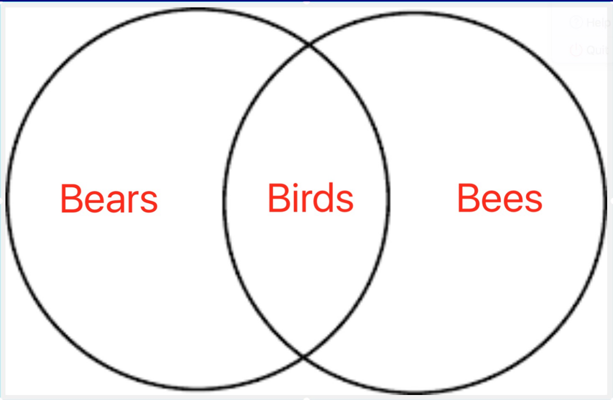
We often draw a rectangle to represent the universe, the set of all objects under current consideration.

For example, suppose we consider all species of animals as the universe.

A rectangle represents this universe.

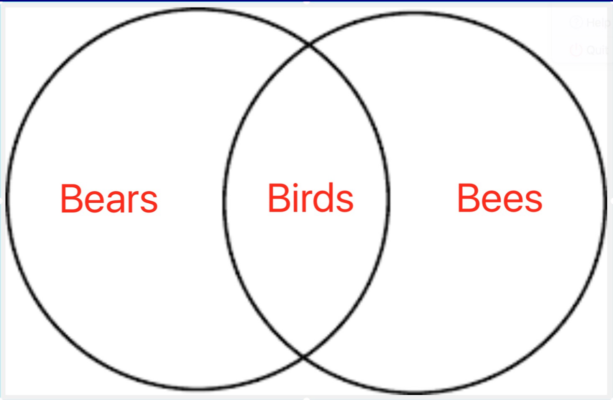
Two circles indicate subsets of animals of two different types.

The Birds and the Bees



Two-legged Animals **Flying Animals**

The Birds and the Bees



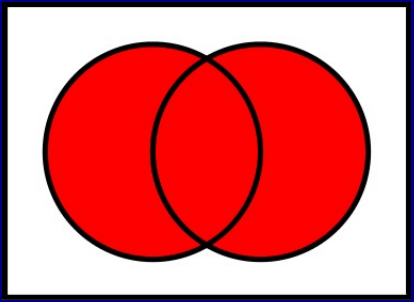
Two-legged Animals Flying Animals

Where do we fit in this diagram?

The Union of Two Sets

The aggregate of two sets is called their union.

Let one set contain all two-legged animals and the other contain all flying animals.



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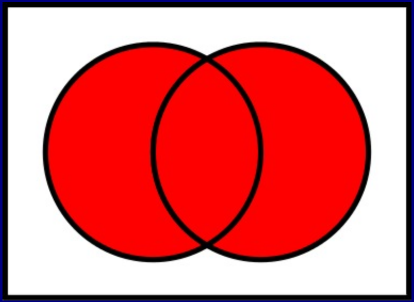
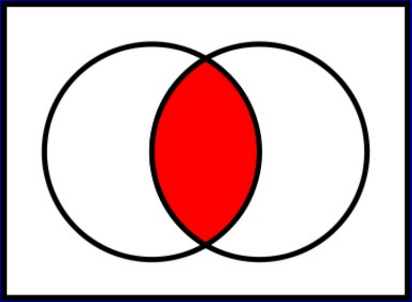


Bears, birds and bees (and we) are in the union.

The Intersection of Two Sets

The elements in both sets make up the intersection.

Let one set contain all two-legged animals
and the other contain all flying animals.



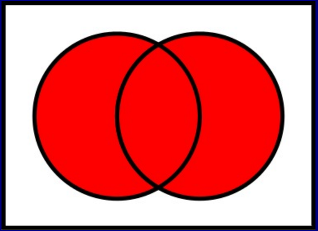
Birds are in the intersection. Bears and bees are not.

The Notation for Union and Intersection

Let A and B be two sets

The union of the sets is

$$A \cup B$$



The intersection is

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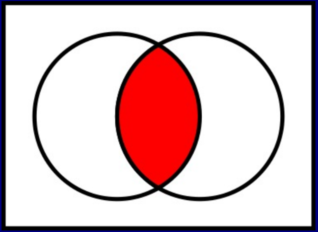

The Technical (Logical) Definitions

Let A and B be two sets.

The **union** of the sets $A \cup B$ is defined by

$$[x \in A \cup B] \iff [(x \in A) \vee (x \in B)]$$

The **intersection** of the sets $A \cap B$ is defined by

$$[x \in A \cap B] \iff [(x \in A) \wedge (x \in B)]$$

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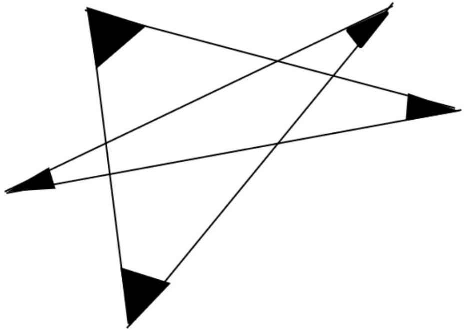
There is an intimate connection between
Set Theory and Symbolic Logic.

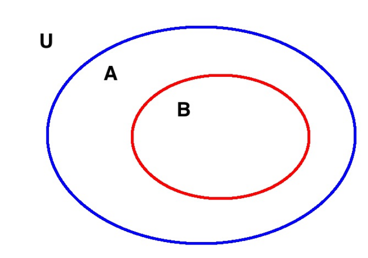
Digression: A Simple Puzzle

Puzzle - Seeing Stars

What is the sum of all the marked angles in the five-pointed star?



Subset of a Set

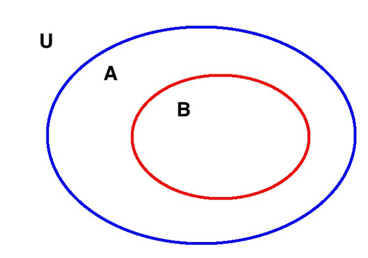


For two sets A and B we write

$$B \subset A \qquad \text{or} \qquad B \subseteq A$$

to denote that B is a subset of A.

Subset of a Set



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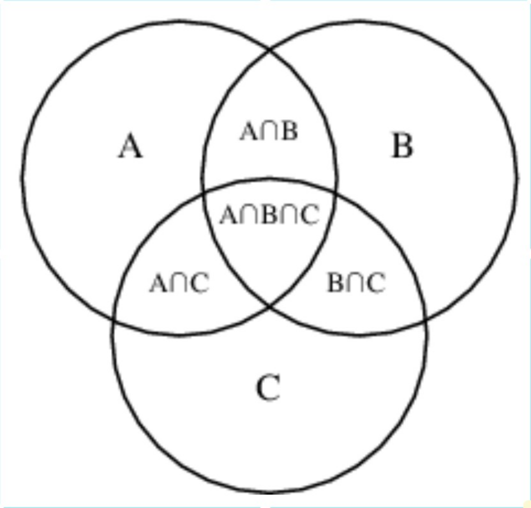
For more on set theory, see website of Claire Wladis
http://www.cwladis.com/math100/Lecture4Sets.htm

Intersections between 3 Sets



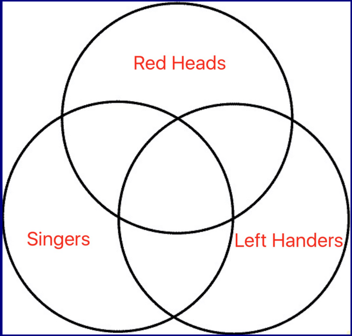
Example: Intersection of 3 Sets

In the diagram the elements of the universe are all the people from Connacht.

Three subsets are shown:

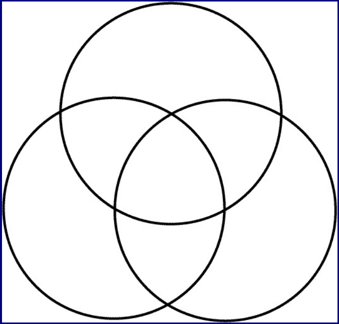
- Red-heads
- Singers
- Left-handers.

All are from Connacht.

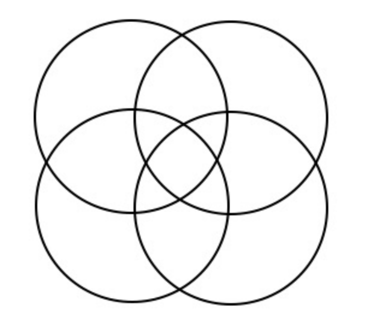


These sets overlap and, indeed, there are some copper-topped, crooning cithogues in Connacht.

Three and Four Sets



8 Domains



14 Domains

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With two sets, A and B, there are 4 possibilities:

$$(x \in A) \wedge (x \in B) \qquad \textbf{or} \qquad (x \in A) \wedge (x \notin B)$$
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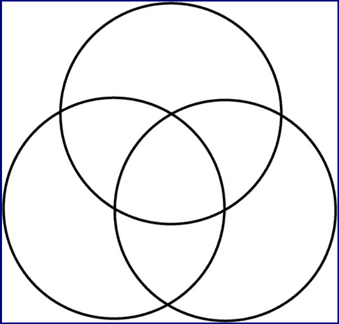
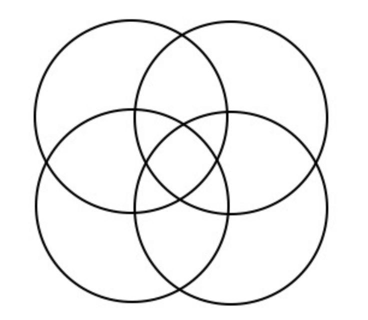
With three sets there are 8 possible conditions.

With four sets there are 16 possible conditions.

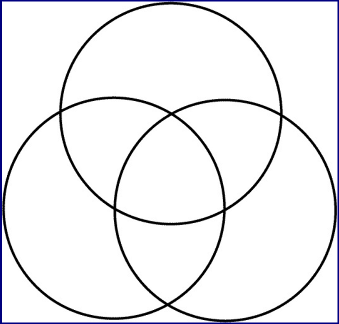
Three and Four Sets



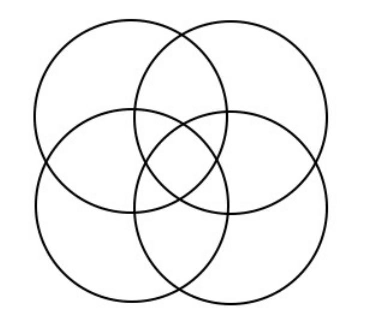


8 Domains **14 Domains**

Three and Four Sets



8 Domains



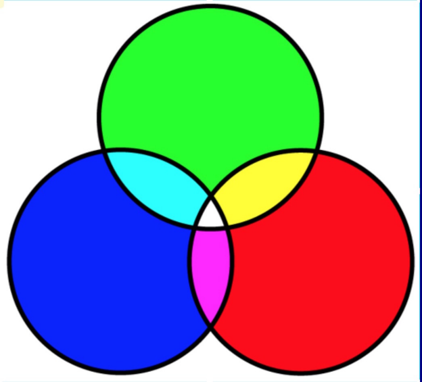
14 Domains

With three sets there are 8 possible conditions.
With four sets there are 16 possible conditions.

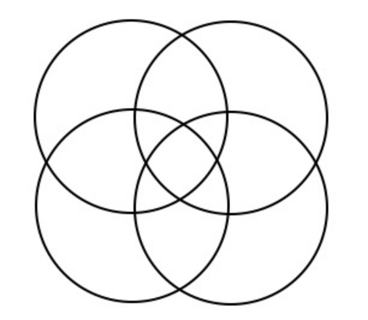
The Intersection of 3 Sets

The three overlapping circles have attained an iconic status, seen in a huge range of contexts.

It is possible to devise Venn diagrams with four sets, but the simplicity of the diagram is lost.



Exercise: Four Set Venn Diagram



Can you modify the 4-set diagram to cover all cases. (You will not be able to do it with circles only)

Hint: Venn Diagrams for 5 and 7 Sets

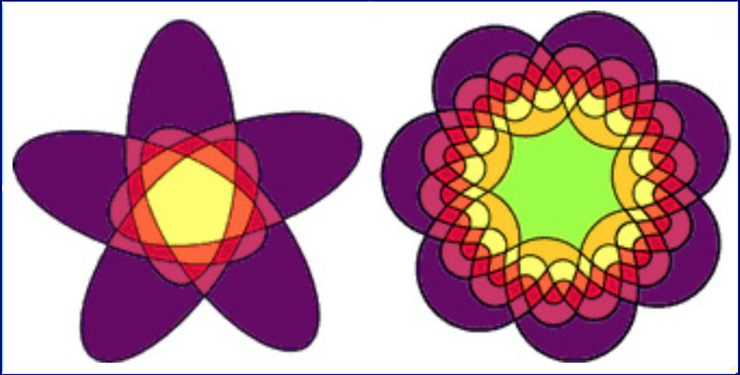
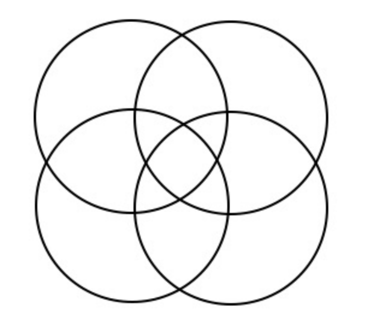


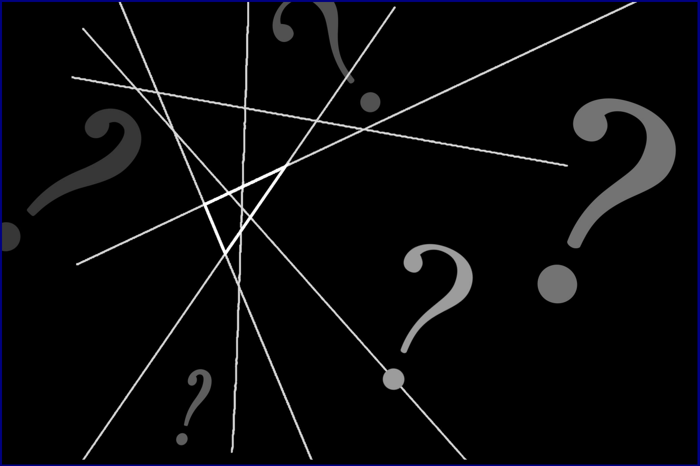
Image from Wolfram MathWorld: Venn Diagram

Solution: Next Week (if you are lucky)



We will find a surprising connection with a Cube

Digression: A Simple Puzzle



Thank you