Sum-Enchanted Evenings

The Fun and Joy of Mathematics

•

LECTURE 10

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School of Mathematics & Statistics
University College Dublin

Evening Course, UCD, Autumn 2018

Outline

Outline

Meaning and Content of Mathematics

The word Mathematics comes from Greek $\mu\alpha\theta\eta\mu\alpha$ (máthéma), meaning "knowledge" or "study" or "learning".

It is the study of topics such as

- Quantity (numbers)
- Structure (patterns)
- Space (geometry)
- Change (analysis).

Reminder: A square from A4 paper sheets.

PUZZLE:
Is it possible to form a square out of sheets of A4 sized paper (without them overlapping)?

Remember: Ratio of width to height is $1 : \sqrt{2}$.

A Square from A4 Paper Sheets

Let dimensions be: Width = 1 unit. Height = $\sqrt{2}$ units.

Suppose there are a short sides and b long sides along the lower horizontal edge of the big square.

Then the length of the horizontal edge is

$$H = a.1 + b.\sqrt{2}$$

Suppose there are c short sides and d long sides along the left vertical edge of the big square.

So the length of the vertical edge is

$$V = c.1 + d\sqrt{2}$$

Since the region is square, $V = H$ and we must have

$$a.1 + b.\sqrt{2} = c.1 + d\sqrt{2}$$

Therefore

$$\begin{aligned} a + b\sqrt{2} &= c + d\sqrt{2} \\ a - c &= (d - b)\sqrt{2} \\ \left(\frac{a-c}{d-b}\right) &= \sqrt{2} \end{aligned}$$

But the left side is a ratio of two whole numbers, whereas the right side is irrational.

This is impossible. There is no solution!

Outline

Symmetries of the Triangle and Square: The Dihedral Groups D_3 and D_4

Let's look at symmetries of the triangle and square.

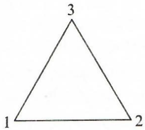


Fig. 4.6

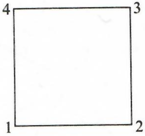
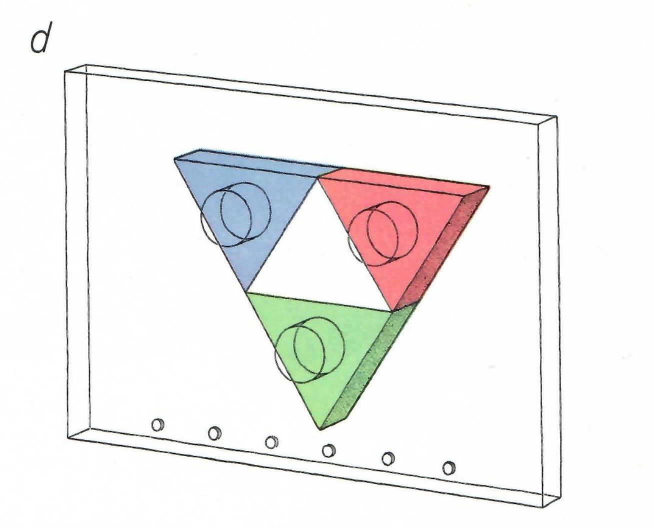
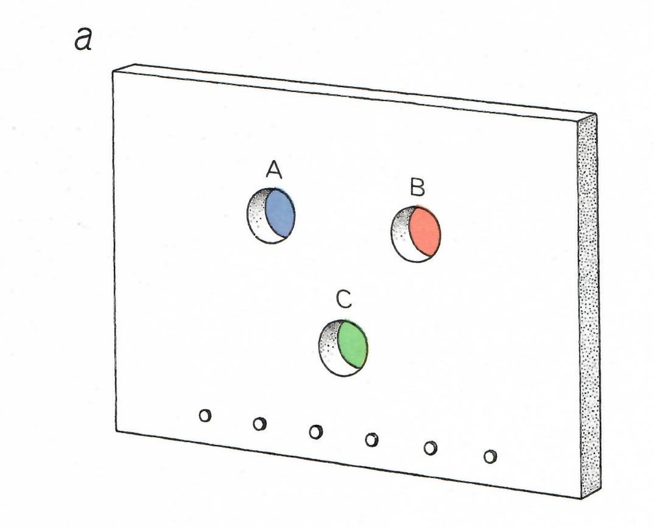


Fig. 4.7

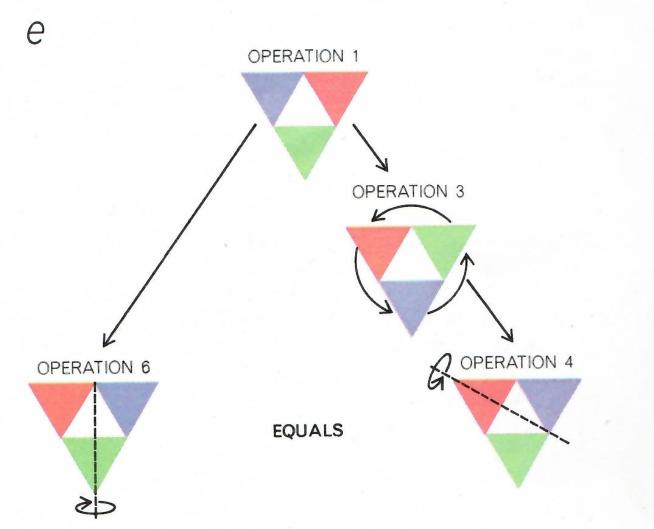
They correspond to the dihedral groups D_3 and D_4.

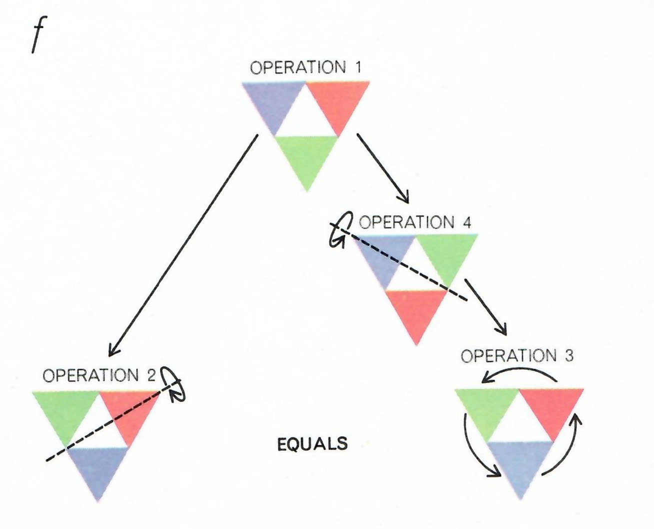

d

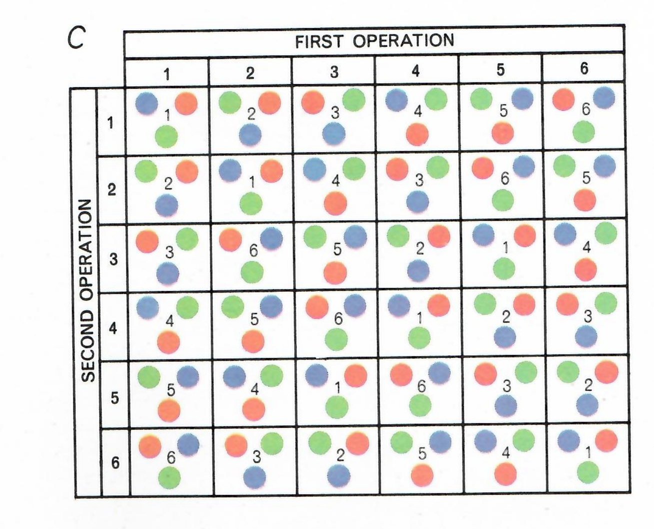

a
A
B
C

b

OPERATION	RESULT
1. NO CHANGE:	
2. SWITCH A AND C:	
3. REPLACE A BY B, B BY C, C BY A:	
4. SWITCH C AND B:	
5. REPLACE A BY C, B BY A, C BY B:	
6. SWITCH A AND B:	


e
OPERATION 1
OPERATION 3
OPERATION 6
EQUALS
OPERATION 4


f
OPERATION 1
OPERATION 4
OPERATION 2
OPERATION 3
EQUALS


C
FIRST OPERATION
1 2 3 4 5 6
SECOND OPERATION
1: 1 2 3 4 5 6
2: 2 1 4 3 6 5
3: 3 6 5 2 1 4
4: 4 5 6 1 2 3
5: 5 4 1 6 3 2
6: 6 3 2 5 4 1

Skip to end of Section: Counting Symmetries

Symbols for Transformations of Triangle

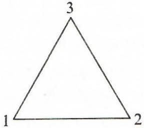


Fig. 4.6

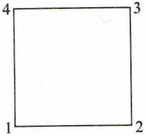


Fig. 4.7

$$\rho_0 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \quad \mu_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix},$$

$$\rho_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \quad \mu_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix},$$

$$\rho_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, \quad \mu_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}.$$

The Third Dihedral Group D_3

	ρ_0	ρ_1	ρ_2	μ_1	μ_2	μ_3
ρ_0	ρ_0	ρ_1	ρ_2	μ_1	μ_2	μ_3
ρ_1	ρ_1	ρ_2	ρ_0	μ_2	μ_3	μ_1
ρ_2	ρ_2	ρ_0	ρ_1	μ_3	μ_1	μ_2
μ_1	μ_1	μ_3	μ_2	ρ_0	ρ_2	ρ_1
μ_2	μ_2	μ_1	μ_3	ρ_1	ρ_0	ρ_2
μ_3	μ_3	μ_2	μ_1	ρ_2	ρ_1	ρ_0

Fig. 4.5

Subgroup Z_3 of Third Dihedral Group D_3

	ρ_0	ρ_1	ρ_2
ρ_0	ρ_0	ρ_1	ρ_2
ρ_1	ρ_1	ρ_2	ρ_0
ρ_2	ρ_2	ρ_0	ρ_1

Fig. 4.5

The Third Dihedral Group D_3

	ρ_0	ρ_1	ρ_2	μ_1	μ_2	μ_3
ρ_0	ρ_0	ρ_1	ρ_2	μ_1	μ_2	μ_3
ρ_1	ρ_1	ρ_2	ρ_0	μ_2	μ_3	μ_1
ρ_2	ρ_2	ρ_0	ρ_1	μ_3	μ_1	μ_2
μ_1	μ_1	μ_3	μ_2	ρ_0	ρ_2	ρ_1
μ_2	μ_2	μ_1	μ_3	ρ_1	ρ_0	ρ_2
μ_3	μ_3	μ_2	μ_1	ρ_2	ρ_1	ρ_0

Fig. 4.5

Subgroup Z_2 of Third Dihedral Group D_3

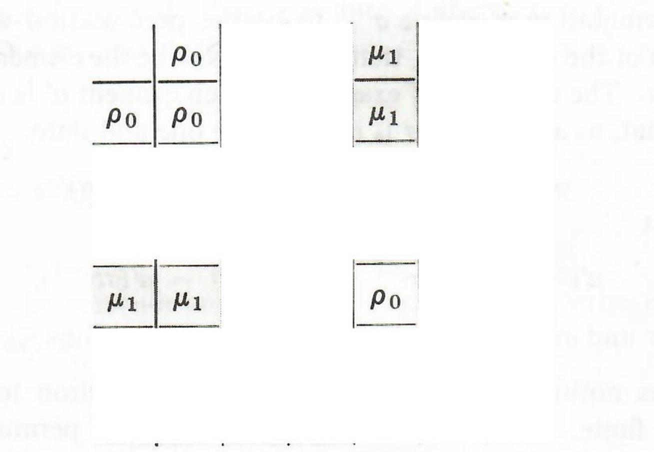


Fig. 4.5

Symbols for Transformations of Square

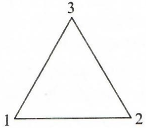


Fig. 4.6

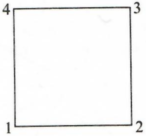


Fig. 4.7

$$\rho_0 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}, \quad \mu_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix},$$

$$\rho_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}, \quad \mu_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix},$$

$$\rho_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}, \quad \delta_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix},$$

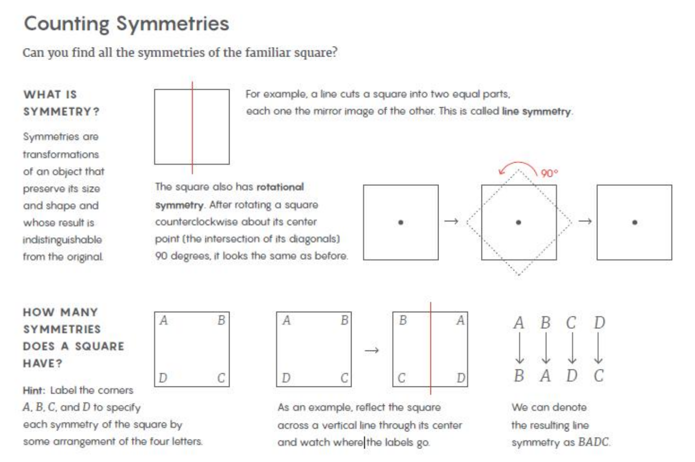
$$\rho_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}, \quad \delta_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix}.$$

The Fourth Dihedral Group D_4

	ρ_0	ρ_1	ρ_2	ρ_3	μ_1	μ_2	δ_1	δ_2
ρ_0	ρ_0	ρ_1	ρ_2	ρ_3	μ_1	μ_2	δ_1	δ_2
ρ_1	ρ_1	ρ_2	ρ_3	ρ_0	δ_2	δ_1	μ_1	μ_2
ρ_2	ρ_2	ρ_3	ρ_0	ρ_1	μ_2	μ_1	δ_2	δ_1
ρ_3	ρ_3	ρ_0	ρ_1	ρ_2	δ_1	δ_2	μ_2	μ_1
μ_1	μ_1	δ_1	μ_2	δ_2	ρ_0	ρ_2	ρ_1	ρ_3
μ_2	μ_2	δ_2	μ_1	δ_1	ρ_2	ρ_0	ρ_3	ρ_1
δ_1	δ_1	μ_2	δ_2	μ_1	ρ_3	ρ_1	ρ_0	ρ_2
δ_2	δ_2	μ_1	δ_1	μ_2	ρ_1	ρ_3	ρ_2	ρ_0

Fig. 4.8

Counting Symmetries of the Square



Counting Symmetries

Can you find all the symmetries of the familiar square?

WHAT IS SYMMETRY?

Symmetries are transformations of an object that preserve its size and shape and whose result is indistinguishable from the original.

For example, a line cuts a square into two equal parts, each one the mirror image of the other. This is called **line symmetry**.

The square also has **rotational symmetry**. After rotating a square counterclockwise about its center point (the intersection of its diagonals) 90 degrees, it looks the same as before.

HOW MANY SYMMETRIES DOES A SQUARE HAVE?

Hint: Label the corners *A*, *B*, *C*, and *D* to specify each symmetry of the square by some arrangement of the four letters.

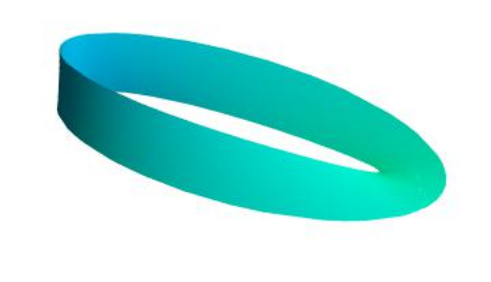
As an example, reflect the square across a vertical line through its center and watch where the labels go.

We can denote the resulting line symmetry as *BADC*.

See worksheet: CountingSymmetriesWorksheet.pdf

Outline

The Möbius Band



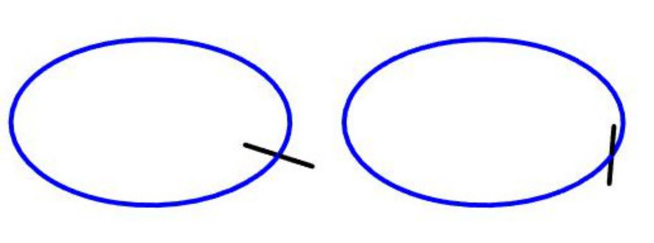
You may be familiar with the Möbius strip or Möbius band. It has one side and one edge.

It was discovered independently by August Möbius and Johann Listing in the same year, 1858.

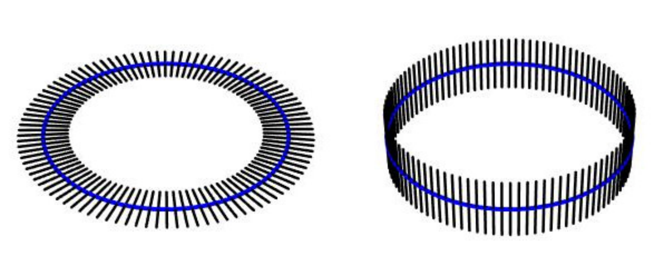
Building the Band

It is easy to make a Möbius band from a paper strip.

For a geometrical construction, we start with a circle and a small line segment with centre on this circle.



Now move the line segment around the circle:



To show the boundary of the surface, we color one end of the line segment red and the other magenta.

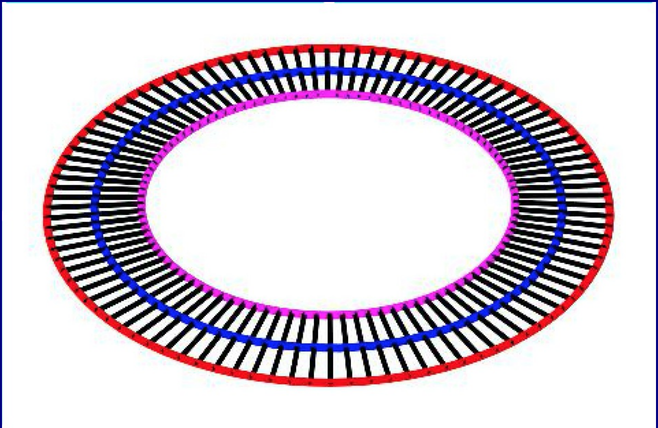


Figure: The boundary comprises two unlinked circles

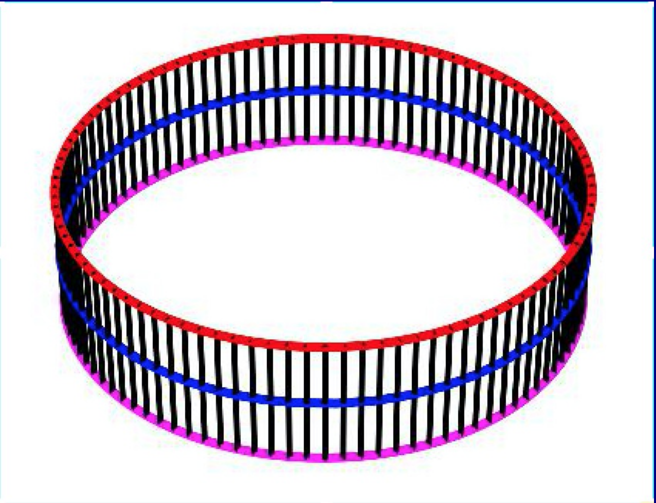
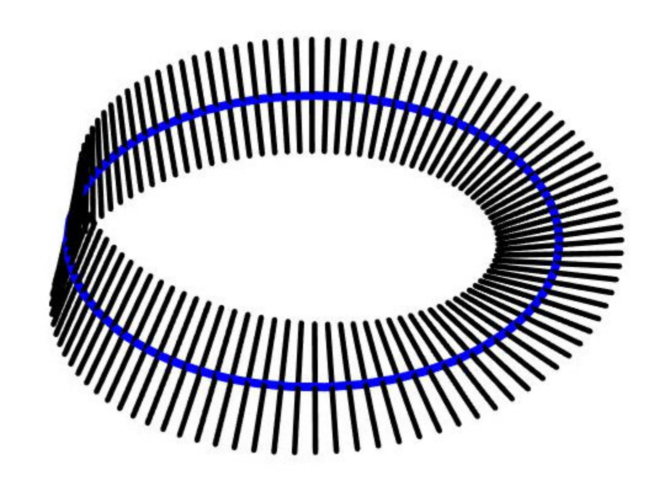


Figure: The boundary comprises two unlinked circles

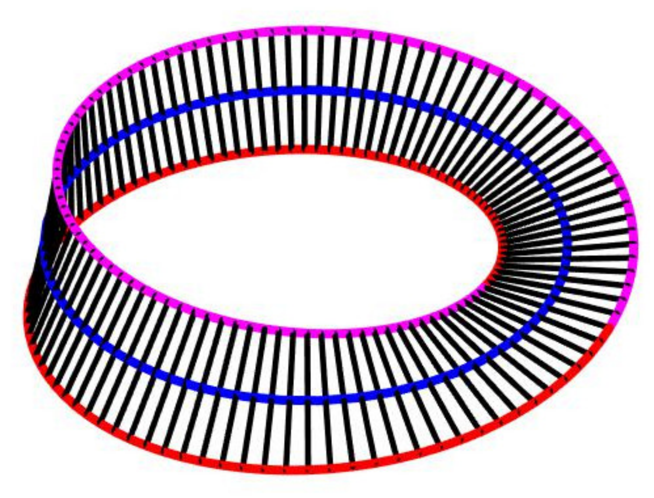
The Möbius Band

Now, as the line moves, we give it a half-twist:



The Möbius Band

The two boundary curves now join up to become one:



The Möbius Band

The Möbius Band has only one side.

It is possible to get from any point on the surface to any other point without crossing the edge.

The surface also has just one edge.

Band with a Full Twist

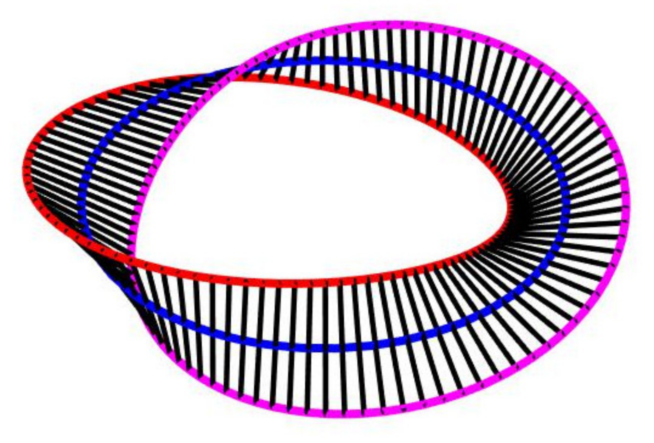


Figure: The boundary comprises two linked circles

Band with Three Half-twists

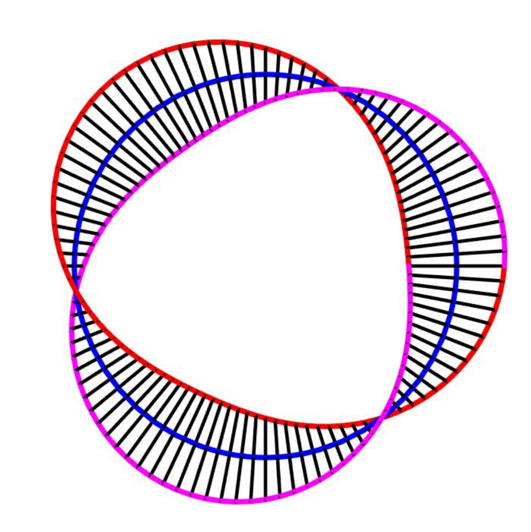
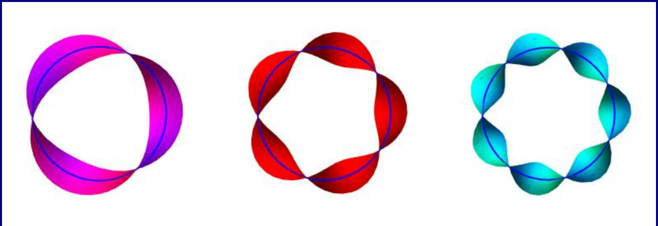
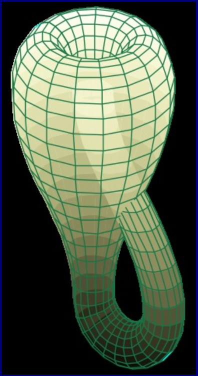


Figure: The boundary is a knot, a trefoil curve



Two Möbius Bands make a Klein Bottle



A mathematician named Klein
Thought the Möbius band was divine.
    Said he: “If you glue
    The edges of two,
You’ll get a weird bottle like mine.”

Equations for the Möbius Band

The process of moving the line segment around the circle leads us to the equations for the Möbius band.

In cylindrical polar coordinates the circle is $(r, \theta, z) = (a, \theta, 0)$.

The tip of the segment, relative to its centre, is

$$(r, \theta, z) = (b \cos \phi, 0, b \sin \phi)$$

where $b = \frac{1}{2}\ell$ is half the segment length and $\phi = \alpha\theta$, with α determining the amount of twist.

The tip of the line has $(r, z) = (a + b \cos \alpha\theta, b \sin \alpha\theta)$.

Equations for the Möbius Band

In cartesian coordinates, the equations become

$$\begin{aligned} x &= (a + b\cos\alpha\theta)\cos\theta \\ y &= (a + b\cos\alpha\theta)\sin\theta \\ z &= (b\sin\alpha\theta) \end{aligned}$$

These are the parametric equations for the twisted bands, with $\theta \in [0, 2\pi]$ and $b \in [-\ell, \ell]$.

For the Möbius band, $\alpha = \frac{1}{2}$.

Outline

Alfred Moessner's Conjecture

Aus den Sitzungsberichten der Bayerischen Akademie der Wissenschaften
Mathematisch-naturwissenschaftliche Klasse 1951 Nr. 3

Eine Bemerkung über die Potenzen der natürlichen Zahlen

Von Alfred Moessner in Gunzenhausen

Vorgelegt von Herrn O. Perron am 2. März 1951

A Remark on the Powers of the Natural Numbers

Moessner’s Construction: n=2

We start with the sequence of natural numbers:

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 ...

Now we delete every second number and calculate the sequence of partial sums:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1		4		9		16		25		36		49		64	

Moessner's Construction: n=2

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1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 ...

Now we delete every second number and calculate the sequence of partial sums:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1		4		9		16		25		36		49		64	

The result is the sequence of perfect squares:

$$1^2 \quad 2^2 \quad 3^2 \quad 4^2 \quad 5^2 \quad 6^2 \quad 7^2 \quad 8^2 \quad \ldots$$

Moessner's Construction: n=3

Now we delete **every third number** and calculate the sequence of partial sums.

Then we delete **every second number** and calculate the sequence of partial sums:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	3		7	12		19	27		37	48		61	75		91
1			8			27			64			125			216

The result is the sequence of perfect cubes:

$$1^3 \quad 2^3 \quad 3^3 \quad 4^3 \quad 5^3 \quad 6^3 \quad \ldots$$

Moessner's Construction: n=4

The Moessner Construction also works for larger n:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	3	6		11	17	24		33	43	54		67	81	96	
1	4			15	32			65	108			175	256		
1				16				81				256			

Moessner's Construction: n=4

The Moessner Construction also works for larger n:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	3	6		11	17	24		33	43	54		67	81	96	
1	4			15	32			65	108			175	256		
1				16				81				256			

The result is the sequence of fourth powers:

$$1^4 \quad 2^4 \quad 3^4 \quad 4^4 \quad \ldots$$

Moessner's Constructions

Remark:

Using Moessner's construction, we can generate a table of squares, cubes or higher powers.

The only arithmetical operations used are counting and addition!

Moessner's Constructions

Remark:

Using Moessner's construction, we can generate a table of squares, cubes or higher powers.

The only arithmetical operations used are counting and addition!

Are there any other sequences generated in this way?

Moessner's Construction for n!

We begin by striking out the triangular numbers, $\{1, 3, 6, 10, 15, 21, \ldots\}$ **and form partial sums.**

Moessner's Construction for n!

We begin by striking out the triangular numbers, $\{1, 3, 6, 10, 15, 21, \ldots\}$ and form partial sums.

Next, we delete the final entry in each group and form partial sums. This process is repeated indefinitely:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
	2		6	11		18	26	35		46	58	71	85		101
			6			24	50			96	154	225			326
						24				120	274				600
										120					720

Moessner's Construction for n!

We begin by striking out the triangular numbers, $\{1, 3, 6, 10, 15, 21, \ldots\}$ **and form partial sums.**

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1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
	2		6	11		18	26	35		46	58	71	85		101
			6			24	50			96	154	225			326
						24				120	274				600
										120					720

This yields the factorial numbers:

1! 2! 3! 4! 5! 6! . . .

Beautiful Math

The beauty of maths?
What do mathematicians think?

VIDEO: Beautiful Maths, available at

`http://momath.org/home/beautifulmath/`

Video by James Tanton

Try to disregard the antipodean exuberance!

Wikipedia Mathematics Portal

Topics in mathematics edit

General

- Mathematicians
- History of mathematics
- Philosophy of mathematics
- Mathematical notation
- Mathematical beauty
- Mathematics education
- Areas of mathematics
- Outline of mathematics
- List of mathematical symbols
- Wikipedia Books: Mathematics

Foundations

- Foundations of mathematics
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  - Proof theory
    - Gödel's incompleteness theorems
  - Model theory
  - Recursion theory
- Set theory (portal)
  - Naive set theory
  - Axiomatic set theory
- Category theory (portal)
  - Topos theory

Number theory

- Number theory (portal)
- Algebraic number theory
- Analytic number theory
- Arithmetic
  - Fundamental theorem of arithmetic
- Numbers
  - Natural numbers
  - Prime numbers
  - Rational numbers
  - Algebraic numbers

Discrete mathematics

- Discrete mathematics (portal)
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  - Coding theory
  - Combinatorial design
  - Enumerative combinatorics
  - Combinatorial optimization
- Graph theory
- Order theory
  - Lattice theory
- Digital Signal Processing

Algebra

- Algebra (portal)
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- Abstract algebra
  - Group theory
  - Ring theory
  - Field theory
  - Commutative algebra
- Geometric algebra
- Linear algebra
  - Matrix theory
  - Multilinear algebra
- Universal algebra
- Fundamental theorem of algebra

Analysis

- Analysis (portal)
- Calculus
  - Fundamental theorem of calculus
  - Vector calculus
  - Geometric calculus
- Measure theory
- Real analysis
- Complex analysis
- Differential equations
  - Ordinary differential equations
  - Partial differential equations
- Integral equations
- Approximation theory
- Special functions
- Potential theory
- Harmonic analysis
  - Fourier analysis
- Functional analysis
- Operator theory

Geometry and topology

- Geometry (portal)
- Euclidean geometry
  - Trigonometry
- Analytic geometry
- Non-Euclidean geometry
- Affine geometry
- Projective geometry
- Convex geometry
- Discrete geometry
- Algebraic geometry
- Differential geometry
  - Riemannian geometry
  - Lie groups
- Topology (portal)
  - General topology
  - Algebraic topology
  - Geometric topology
  - Differential topology

Applied mathematics

- Applied mathematics
- Mathematical modeling
- Mathematical physics
- Dynamical systems
  - Control theory
    - Calculus of variations
- Optimization
- Mathematical economics
  - Game theory
  - Mathematical finance
- Statistics (portal)
- Probability theory
  - Stochastic processes
- Numerical analysis
- Theoretical computer science
  - Computability theory
  - Complexity theory
- Cryptography (portal)
- Information theory

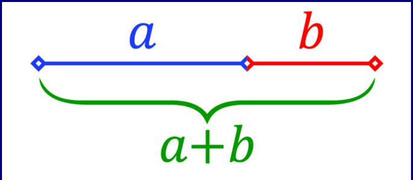
Index of mathematics articles edit

Outline

Golden Rectangle in Your Pocket

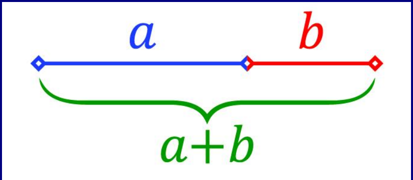
Aspect ratio is about $\phi = \frac{1+\sqrt{5}}{2} \approx 1.618$.

Geometric Ratio: $a + b$ is to a as a is to b.



$$\left[\frac{\textbf{Short Bit}}{\textbf{Long Bit}}\right] = \left[\frac{\textbf{Long Bit}}{\textbf{Full Line}}\right] \quad \textbf{or} \quad \frac{b}{a} = \frac{a}{a+b}$$

Geometric Ratio: $a + b$ is to a as a is to b.



$$\left[\frac{\text{Short Bit}}{\text{Long Bit}}\right] = \left[\frac{\text{Long Bit}}{\text{Full Line}}\right] \quad \text{or} \quad \frac{b}{a} = \frac{a}{a+b}$$

Let the blue segment be $a = 1$ and the whole line ϕ.

Then $b = \phi - 1$ and we have

$$\frac{\phi - 1}{1} = \frac{1}{\phi}$$

$$\phi - 1 = \frac{1}{\phi}$$

This means ϕ solves a quadratic equation:

$$\phi^2 - \phi - 1 = 0$$

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Recall the two solutions of a quadratic equation

$$ax^2 + bx + c = 0 \qquad \textbf{are} \qquad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

In the present case, this means that the roots are

$$\phi = \frac{1 \pm \sqrt{1 + 4}}{2}$$

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We take the positive root, giving

$$\phi = \frac{1+\sqrt{5}}{2} \approx 1.618$$

This is the golden ratio.

Golden Rectangle

Ratio of breath to height is $\phi = \frac{1+\sqrt{5}}{2}$.

Golden Rectangle in Your Pocket

Aspect ratio is about $\phi = \frac{1+\sqrt{5}}{2} \approx 1.618$.

Terminology

- Golden Ratio. Golden Number. Golden Mean.
- Golden Proportion. Golden Cut.
- Golden Section. Medial Section.
- Divine Proportion. Divine Section.
- Extreme and Mean Ratio.
- Various Other Terms.

Fibonacci Numbers

The Fibonacci sequence is the sequence

$$\{0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots\}$$

where each number is the sum of the previous two.

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Can we solve this recurrence relation for all F_n?

Fibonacci Numbers

The recurrence relation is

$$F_{n+1} = F_n + F_{n-1}$$

We assume that the solution is of the form $F_n = k\phi^n$, where we have to find ϕ (this is called an Ansatz).

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Substitute this solution into the recurrence relation:

$$k\phi^{n+1} = k\phi^n + k\phi^{n-1}$$

Divide by $k\phi^{n-1}$ to get the quadratic equation

$$\phi^2 = \phi + 1 \qquad \text{or} \qquad \phi^2 - \phi - 1 = 0$$

This is the quadratic we got for the golden number.

Fibonacci Numbers

We found that $F_n = k\phi^n$ where ϕ is a root of

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The two roots are

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Then the full solution for the Fibonacci numbers is

$$F_n = \frac{1}{\sqrt{5}}\left[\frac{1+\sqrt{5}}{2}\right]^n - \frac{1}{\sqrt{5}}\left[\frac{1-\sqrt{5}}{2}\right]^n$$

Check that the conditions $F_0 = 0$ and $F_1 = 1$ are true.

Fibonacci Numbers

$$F_n = \frac{1}{\sqrt{5}}\left[\frac{1+\sqrt{5}}{2}\right]^n - \frac{1}{\sqrt{5}}\left[\frac{1-\sqrt{5}}{2}\right]^n$$

The first term in square brackets is greater than 1, so the powers grow rapidly with n.

The second term in square brackets is less than 1, so the powers become small rapidly with n.

Fibonacci Numbers

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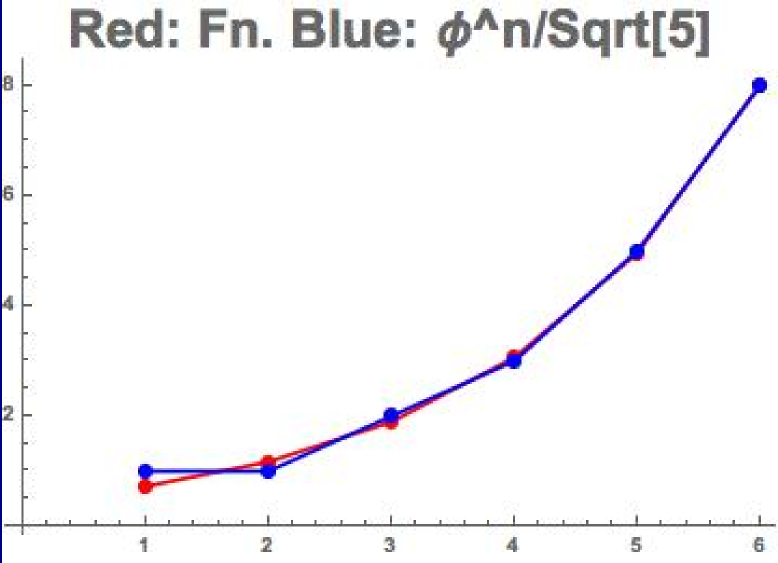
The first term in square brackets is greater than 1, so the powers grow rapidly with n.

The second term in square brackets is less than 1, so the powers become small rapidly with n.

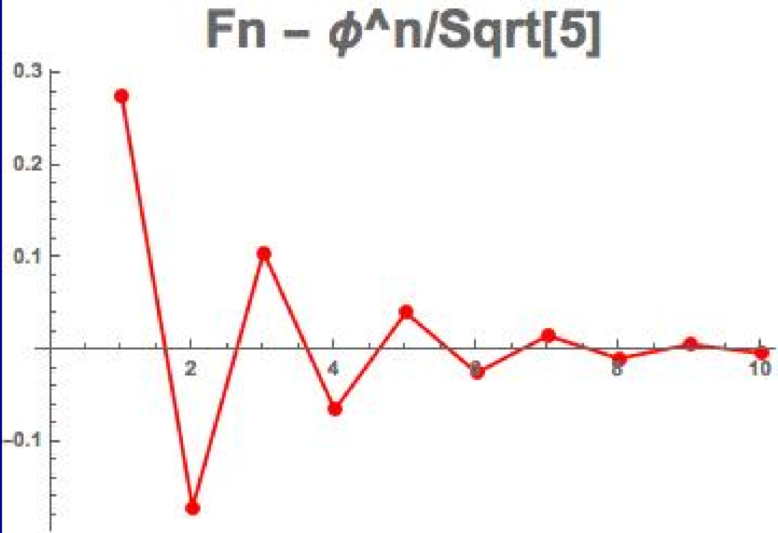
So, we ignore the second term and write

$$F_n \approx \frac{1}{\sqrt{5}}\left[\frac{1+\sqrt{5}}{2}\right]^n \qquad \textbf{or} \qquad F_n \approx \frac{\phi^n}{\sqrt{5}}$$

Approximation to F_n



Oscillating Error of Approximation



Ratio F_n/F_{n-1}

$$F_n \approx \frac{\phi^n}{\sqrt{5}} \implies \frac{F_n}{F_{n-1}} \approx \phi$$

Let's consider the sequence of ratios of terms

$$\frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \frac{21}{13}, \frac{34}{21}, \dots$$

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$$\frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \frac{21}{13}, \frac{34}{21}, \dots$$

The ratios get closer and closer to ϕ:

$$\frac{F_{n+1}}{F_n} \to \phi \quad \textbf{as} \quad n \to \infty$$

Continued Fraction for ϕ

$$\phi^2 - \phi - 1 = 0 \implies \phi = 1 + \frac{1}{\phi}$$

Now use the equation to replace ϕ on the right:

$$\phi = 1 + \frac{1}{\phi} = 1 + \frac{1}{1 + \frac{1}{\phi}} = 1 + \frac{1}{1 + \frac{1}{1+\frac{1}{\phi}}}$$

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Eventually

$$\phi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}}$$

Continued Root for ϕ

$$\phi^2 - \phi - 1 = 0 \implies \phi = \sqrt{1+\phi}$$

Now use the equation to replace ϕ on the right:

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Eventually

$$\phi = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\cdots}}}}}$$

Fibonacci Numbers in Nature

Look at post

Sunflowers and Fibonacci: Models of Efficiency

on the *ThatsMaths* blog.

Vi Hart's Videos

Vi Hart has many mathematical videos on YouTube.

- On Fibonacci Numbers: https://www.youtube.com/watch?v=ahXIMUkSXX0
- On the Three Utilities Problem: https://www.youtube.com/watch?v=CruQylWSfoU&feature=youtu.be
- On Continued Fractions: https://www.youtube.com/watch?v=a5z-OEIfw3s

Outline

What is Randomness?

Randomness is a slippery concept,
defying precise definition.

Toss a coin and get a sequence like 1001110100.

Some uses of Random Numbers:

- Computer simulations of fluid flow.
- Interactions of subatomic particles.
- Evolution of galaxies.

Tossing coins is impractical.
We need more effective methods.

Defining Randomness?

Richard von Mises (1919):

A binary sequence is random if the proportion of zeros and ones approaches 50% and if this is also true for any sub-sequence.

Consider (0101010101).

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Andrey Kolmogorov defined the complexity of a binary sequence as the length of a computer program or algorithm that generates it.

The phrase a sequence of one million 1s completely defines a sequence.

Non-random sequences are compressible.
Randomness and incompressibility are equivalent.

Pseudo-random versus Truly Random

Pseudo-random number generators are algorithms that use mathematical formulae to produce sequences of numbers.

The sequences appear completely random and satisfy various statistical conditions for randomness.

Pseudo-random numbers are valuable for many applications but they have serious difficiencies.

Truly Random Number Generators

True random number generators extract randomness from physical phenomena that are completely unpredictable.

Atmospheric noise is the static generated by lightning [globally there are 40 flashes/sec]. It can be detected by an ordinary radio.

Truly Random Number Generators

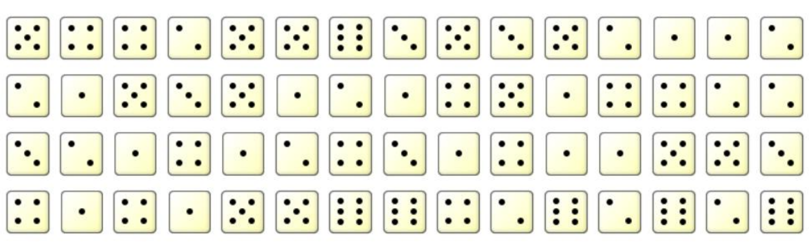
Atmospheric noise passes all the statistical checks for randomness.

Dr Mads Haahr of Trinity College, Dublin uses atmospheric noise to produce random numbers.

Results available on on the website: `random.org`.

20 Random Coin Tosses

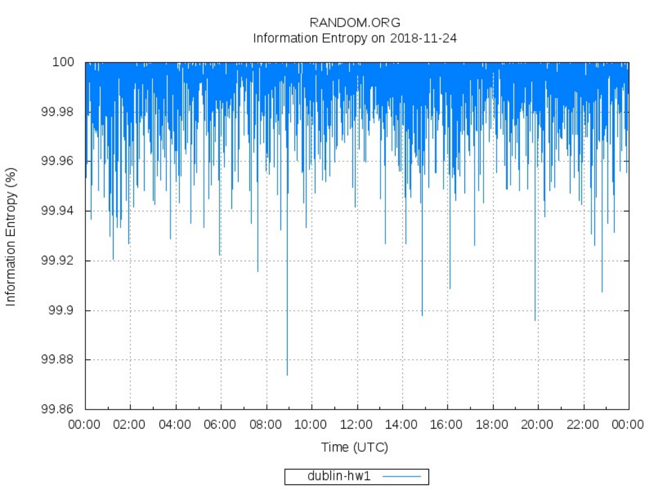
60 Dice Rolls



100 Random Numbers in [0,99]

17	60	57	66	4	71	59	36	8	49
87	64	94	82	6	38	14	87	76	72
97	38	44	59	56	24	20	6	24	97
0	40	14	77	18	98	41	39	6	79
21	59	49	86	91	81	65	64	3	11
92	17	65	6	37	98	84	17	70	93
60	52	1	98	20	2	65	9	57	3
48	86	27	3	71	51	57	56	2	2
13	14	73	65	11	32	17	7	91	37
3	8	10	67	0	72	0	42	15	24

Quality of Random Numbers



PRNG versus TRNG

Characteristic	Pseudo-Random Number Generators	True Random Number Generators
Efficiency	Excellent	Poor
Determinism	Determinstic	Nondeterministic
Periodicity	Periodic	Aperiodic

Outline

Prime & Composite Numbers

A prime number is a number that cannot be broken into a product of smaller numbers.

The first few primes are 2, 3, 5, 7, 11, 13, 17 and 19.

There are 25 primes less than 100.

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There are 25 primes less than 100.

Numbers that are not prime are called **composite.** They can be expressed as **products of primes.**

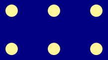
Thus, $6 = 2 \times 3$ is a composite number.

The number 1 is neither prime nor composite.

The Atoms of the Number System

A line of six spots

can be arranged in a rectangular array:



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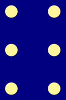

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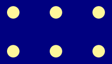

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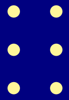

Note that

$$2 \times 3 = 3 \times 2$$

This is the **commutative law of multiplication.**

The Atoms of the Number System

The primes play a role in mathematics analogous to the elements of Mendeleev's Periodic Table.

Just as a chemical molecule can be constructed from the 100 or so fundamental elements, any whole number be constructed by combining prime numbers.

The primes 2, 3, 5 are the hydrogen, helium and lithium of the number system.

Some History

In 1792 Carl Friedrich Gauss, then only 15 years old, found that the proportion of primes less that n decreased approximately as $1/\log n$.

Around 1795 Adrien-Marie Legendre noticed a similar logarithmic pattern of the primes, but it was to take another century before a proof emerged.

In a letter written in 1823 the Norwegian mathematician Niels Henrik Abel described the distribution of primes as *the most remarkable result in all of mathematics.*

Percentage of Primes Less than N

Table: Percentage of Primes less than N

100	25	25.0%
1,000	168	16.8%
1,000,000	78,498	7.8%
1,000,000,000	50,847,534	5.1%
1,000,000,000,000	37,607,912,018	3.8%

We can see that the percentage of primes is falling off with increasing size.

But the rate of decrease is very slow.

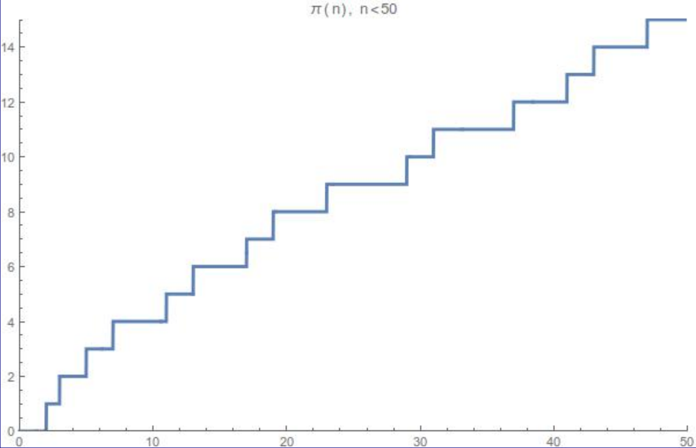

Figure: The prime counting function $\pi(n)$ for $0 \leq n \leq 50$.

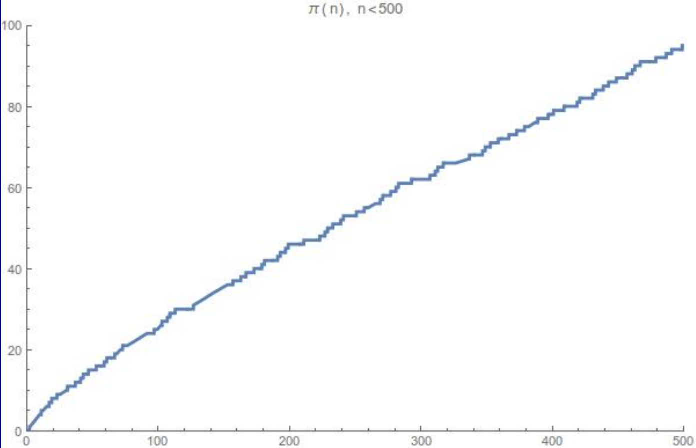

Figure: The prime counting function $\pi(n)$ for $0 \leq n \leq 500$.

Is There a Pattern in the Primes?

It is a simple matter to make a list of all the primes less that 100 or 1000.

It becomes clear very soon that there is no clear pattern emerging.

The primes appear to be scattered at random.

Figure: Prime numbers up to 100

Is There a Pattern in the Primes?

Do the primes settle down as n becomes larger?

Between 9,999,900 and 10,000,000 (100 numbers) there are 9 primes.

Between 10,000,000 and 10,000,100 (100 numbers) there are just 2 primes.

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(100 numbers) there are just 2 primes.

What kind of function could generate this behaviour?

We just do not know.

Is There a Pattern in the Primes?

The gaps between primes are very irregular.

- Can we find a pattern in the primes?
- Can we find a formula that generates primes?
- How can we determine the hundreth prime?
- What is the thousanth? The millionth?

WolframAlpha©

WolframAlpha is a Computational Knowledge Engine.

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It is freely available through a web browser.

Euler’s Formula for Primes

No mathematician has ever found a *useful* formula that generates all the prime numbers.

Euler found a beautiful little formula:

$$n^2 - n + 41$$

This gives prime numbers for n between 1 and 40.

Euler's Formula for Primes

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Euler found a beautiful little formula:

$$n^2 - n + 41$$

This gives prime numbers for n between 1 and 40.

But for $n = 41$ we get

$$41^2 - 41 + 41 = 41 \times 41$$

a composite number.

The Infinitude of Primes

Euclid proved that there is no finite limit to the number of primes.

His proof is a masterpiece of symplicity.

(See Dunham book).

Some Unsolved Problems

There appear to be an infinite number of prime pairs

$$(2n - 1, 2n + 1)$$

There are also gaps of arbitrary length:

for example, there are 13 consecutive composite numbers between 114 and 126.

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$$(2n - 1, 2n + 1)$$

There are also gaps of arbitrary length:

for example, there are 13 consecutive composite numbers between 114 and 126.

We can find gaps as large as we like:

Show that *N! + 1* is followed by a sequence of *N - 1* composite numbers.

Primes have been used as markers of civilization.

In the novel Cosmos, by Carl Sagan,
the heroine detects a signal:

- First 2 pulses
- Then 3 pulses
- Then 5 pulses
- ...
- Then 907 pulses.

In each case, a prime number of pulses.
This could hardly be due to any natural phenomenon.

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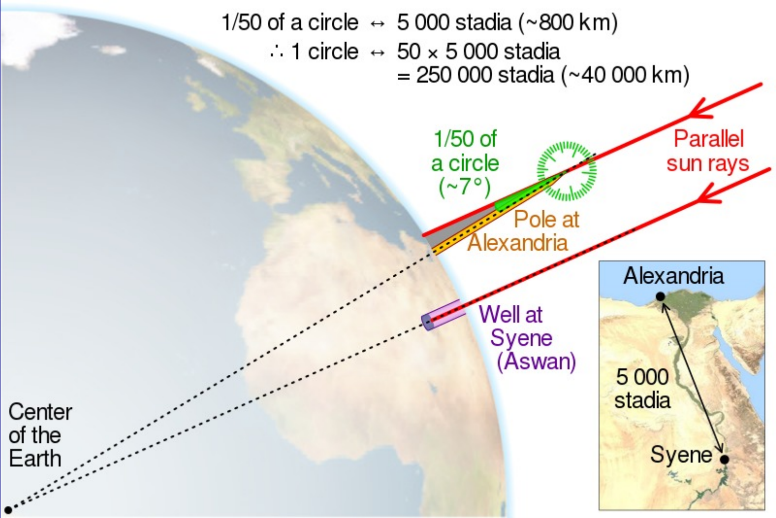
Is this evidence of extra-terrestrial intelligence?

Outline

Eratosthenes Measured the Earth



The Sieve of Eratosthenes

Eratosthenes was the Librarian in Alexandria when Archimedes flourished in Syracuse.

They were "pen-pals".

Eratosthenes estimated size of the Earth.

He devised a systematic procedure for generating the prime numbers: the Sieve of Eratosthenes.

The Sieve of Eratosthenes

The idea:

- List all natural numbers up to n.
- Circle 2 and strike out all multiples of two.
- Move to the next number, 3.
- Circle it and strike out all multiples of 3.
- Continue till no more numbers can be struck out.

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- Continue till no more numbers can be struck out.

The numbers that have been circled are the prime numbers. Nothing else survives.

It is sufficient to go as far as $\sqrt{n}$.

The Sieve of Eratosthenes

	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

The Sieve of Eratosthenes

	2	3		5		7		9	
11		13		15		17		19	
21		23		25		27		29	
31		33		35		37		39	
41		43		45		47		49	
51		53		55		57		59	
61		63		65		67		69	
71		73		75		77		79	
81		83		85		87		89	
91		93		95		97		99	

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61				65		67			
71		73				77		79	
		83		85				89	
91				95		97			

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		53						59	
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51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
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81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

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Figure: Prime numbers up to 100

The grand challenge is to find patterns in the sequence of prime numbers.

This is an enormously difficult problem that has taxed the imagination of the greatest mathematicians for centuries.

Thank you