Sum-Enchanted Evenings

The Fun and Joy of Mathematics

•

LECTURE 9

Peter Lynch
School of Mathematics & Statistics
University College Dublin

Evening Course, UCD, Autumn 2018

Outline

Outline

Meaning and Content of Mathematics

The word Mathematics comes from Greek $\mu\alpha\theta\eta\mu\alpha$ (máthéma), meaning "knowledge" or "study" or "learning".

It is the study of topics such as

- **Quantity (numbers)**
- **Structure (patterns)**
- **Space (geometry)**
- **Change (analysis).**

Outline

Ubiquity and Beauty of Symmetry

Symmetry is all around us.

- **Many buildings are symmetric.**
- **Our bodies have bilateral symmetry.**
- **Crystals have great symmetry.**
- **Viruses can display stunning symmetries.**
- **At the sub-atomic scale, symmetry reigns.**
- **Galaxies have many symmetries.**

The Taj Mahal

A Face with Symmetry: Halle Berry

Halle Berry   Berry Halle

An Asymmetric Face: You know Who!

Symmetry and Group Theory

Symmetry is an essentially *geometric* concept.

The mathematical theory of symmetry is *algebraic.*
The key concept is that of a group.

A group is a set of elements such that any two elements can be combined to produce another.

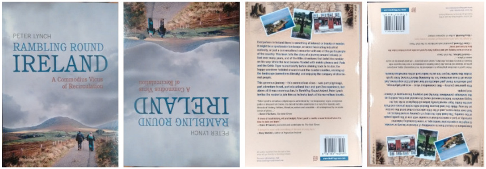
Instead of giving the mathematical definition,
I give an example to make things clear.

The Klein 4-Group

Take a book, place it on the table and draw a rectangle around it. In how many ways can the book fit into the rectangle?

Once a single corner of the book is put at the top left corner, there is no further lee-way.

There are four ways to fit the book in the rectangle.



The four orientations of the book can be described in terms of four simple rotations:

- **I: Place book upright with front cover upright**
- **X: Rotate 180° about horizontal through centre**
- **Y: Rotate 180° about vertical through centre**
- **Z: Rotate 180° about perp. through centre**

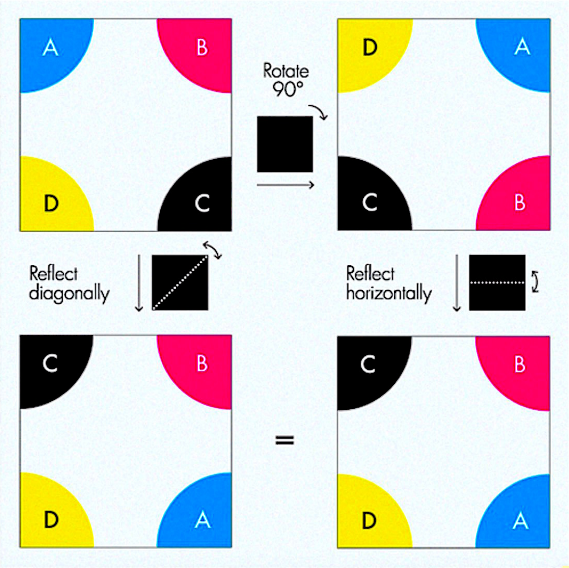
Multiplication Table

*	I	X	Y	Z
I	I	X	Y	Z
X	X	I	Z	Y
Y	Y	Z	I	X
Z	Z	Y	X	I

There are several sub-groups:

$$\{I, X, Y, Z\} \quad \{I, X\} \quad \{I, Y\} \quad \{I, Z\} \quad \{I\}$$

Non-Commutative Operations



Twelve-tone Music

Table : Klein 4-Group.

	P	R	I	RI
P	P	R	I	RI
R	R	P	RI	I
I	I	RI	P	R
RI	RI	I	R	P

The Klein 4-group is the basic group of transformations in twelve tone music.

The operations are retrogression (R), inversion (I) and the composion (RI), which is also a rotation operation.

Numbers of Low-Order Groups

Order *n*	# Groups[6]	Abelian	Non-Abelian
0	0	0	0
1	1	1	0
2	1	1	0
3	1	1	0
4	2	2	0
5	1	1	0
6	2	1	1
7	1	1	0
8	5	3	2
9	2	2	0
10	2	1	1
11	1	1	0
12	5	2	3
13	1	1	0
14	2	1	1
15	1	1	0
16	14	5	9

Table of number of groups of orders up to sixteen.

Commutative groups are called *Abelian* groups.

Groups that do not commute are *Non-Abelian.*

The smallest non-Abelian group is of order 6.

From 2 to 3 Dimensional Symmetry

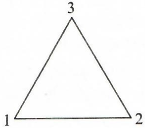


Fig. 4.6

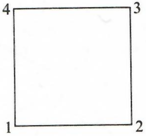
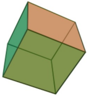
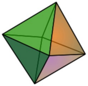
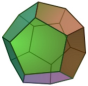
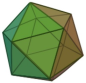
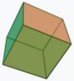
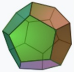
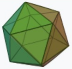


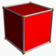
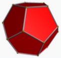
Fig. 4.7

Tetrahedron	Cube	Octahedron	Dodecahedron	Icosahedron
Four faces	Six faces	Eight faces	Twelve faces	Twenty faces
 (Animation)	 (Animation)	 (Animation)	 (Animation)	 (Animation)

The Five Platonic Solids

Polyhedron		Vertices	Edges	Faces
tetrahedron		4	6	4
cube		8	12	6
octahedron		6	12	8
dodecahedron		20	30	12
icosahedron		12	30	20

Platonic Solids: Euler's Gem

Name	Image	Vertices V	Edges E	Faces F	Euler characteristic: $V - E + F$
Tetrahedron		4	6	4	**2**
Hexahedron or cube		8	12	6	**2**
Octahedron		6	12	8	**2**
Dodecahedron		20	30	12	**2**
Icosahedron		12	30	20	**2**

Mnemonic: ***Very Easy Formula 2 remember!***

Dual Polyhedra

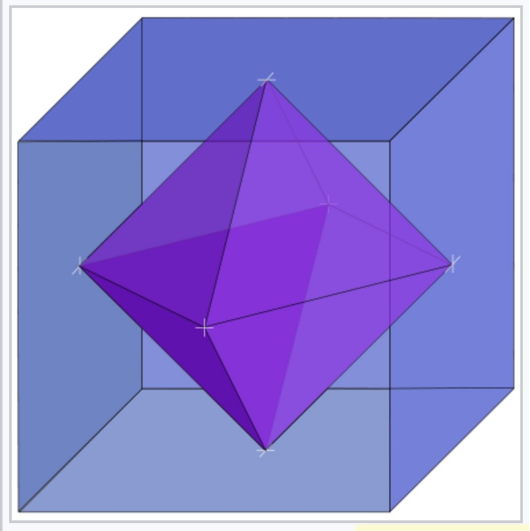
Every polyhedron is associated with a dual.

The vertices of the polyhedron correspond to the faces of its dual. The faces of the polyhedron correspond to the vertices of its dual.

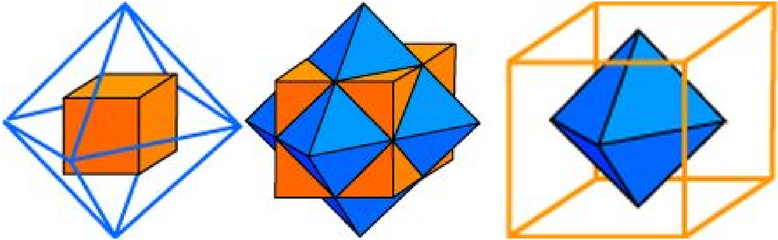
The dual of the dual is the original!

Duality preserves the symmetry of the polyhedron.

Cube and Octahedron are Dual



Cube and Octahedron are Dual



Dodecahedron and Icosahedron are Dual

Tetrhedron is its own Dual

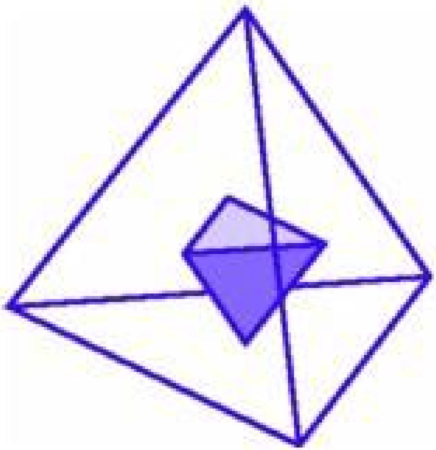


Figure : Tetrahedron and dual.

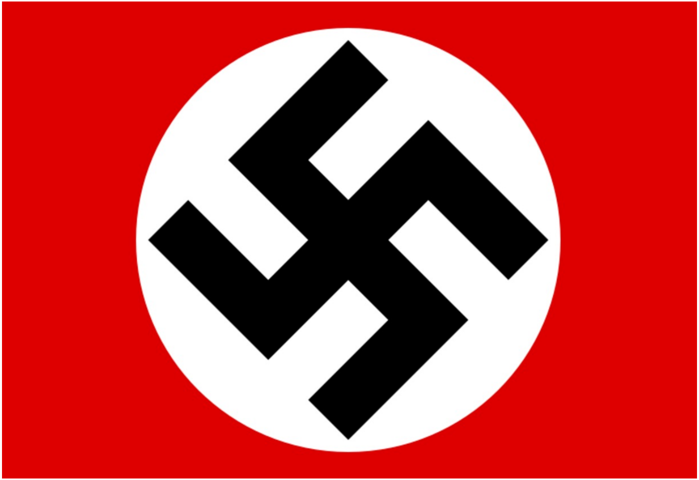
Threefold Symmetry: Z_3

Threefold Symmetry: Z_3

Threefold Symmetry: Z_3

Z_3 Symmetry

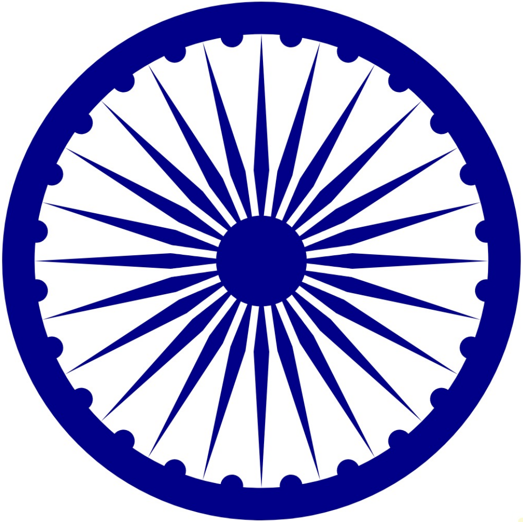
Z_4 Symmetry



Star of David (D_6 Symmetry)

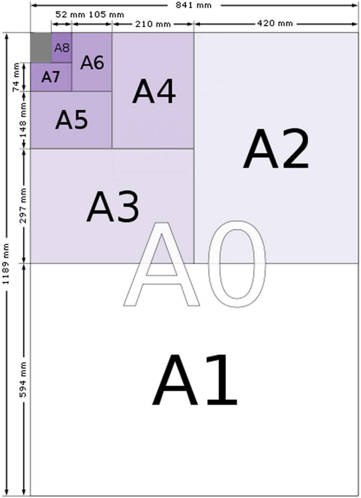
Flag of India (D_1)

Ashoka Chakra (D_{24})



Outline

Standard Paper Sizes



Standard sizes of A-series paper.

The ratio of heights to widths is always $\sqrt{2}$.

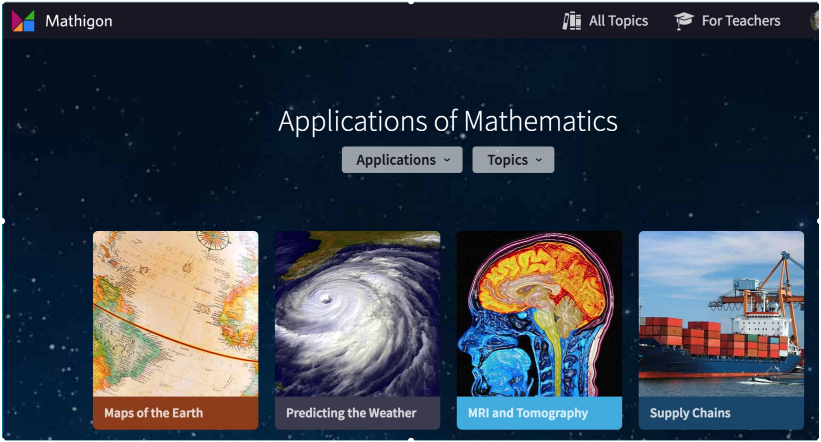
Making a Square

The standard sizes of paper are designed so that each has the same shape (or aspect ratio), and the largest, A0, has an area of one square metre.

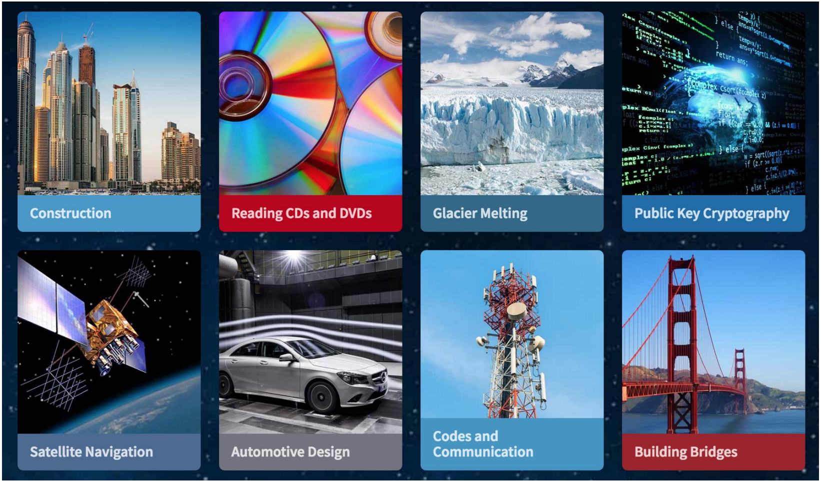
PUZZLE:
Is it possible to form a square out of sheets of A4 sized paper (without them overlapping)?

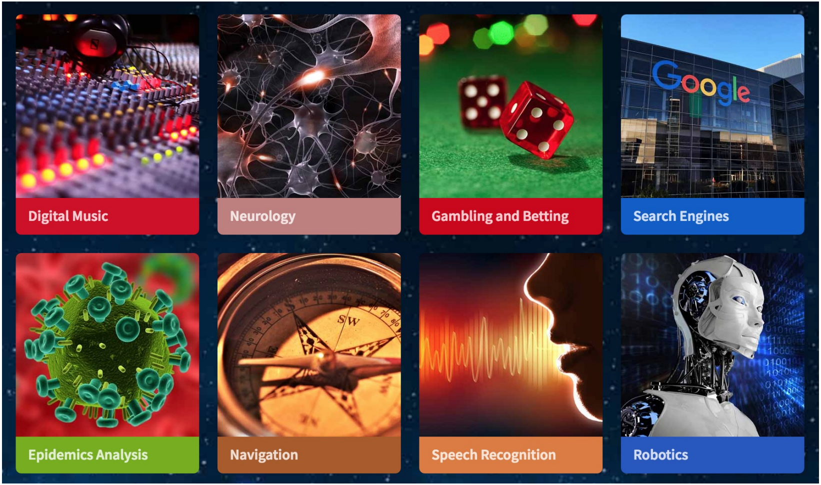
Outline

Applications on mathigon.org

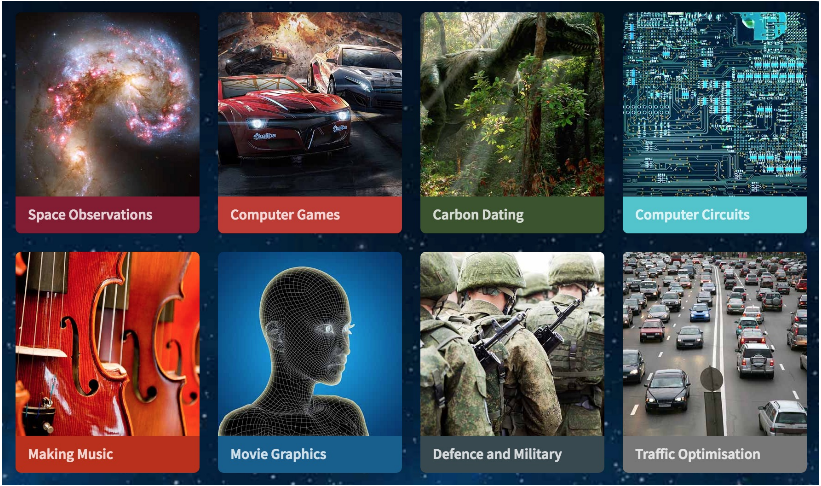


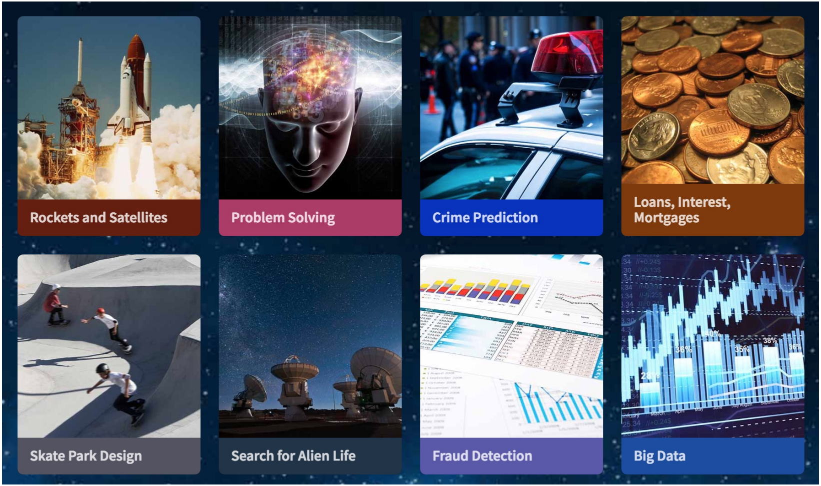

Maps of the Earth
Predicting the Weather
MRI and Tomography
Supply Chains
Finance and Banking
Internet and Phones
Cosmology
Computers

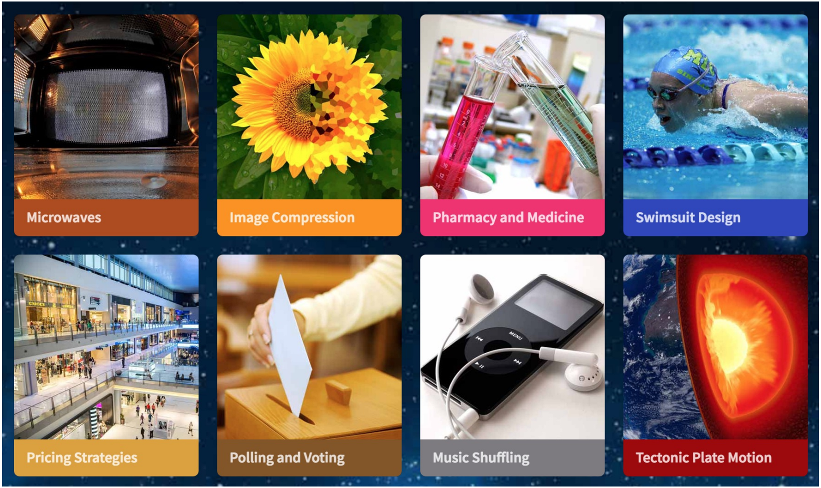

Construction
Reading CDs and DVDs
Glacier Melting
Public Key Cryptography
Satellite Navigation
Automotive Design
Codes and Communication
Building Bridges

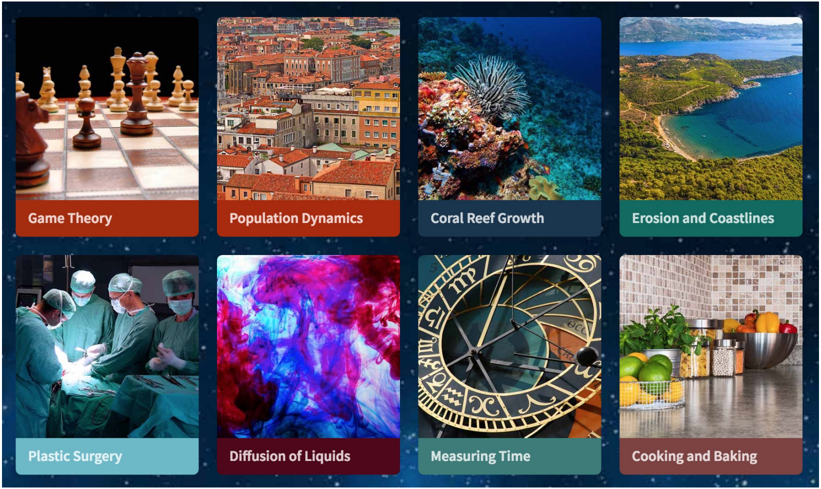

Digital Music
Neurology
Gambling and Betting
Google
Search Engines
Epidemics Analysis
SW
Navigation
Speech Recognition
Robotics

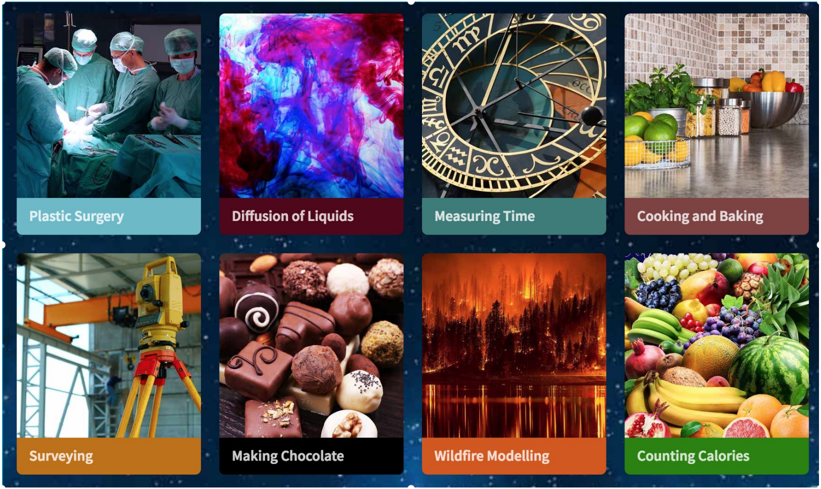

Football Scoring
Volcano Monitoring
Lottery
Roller Coaster Design
Breaking the Enigma
Public Transportation
Crowd Control
Insurance


Space Observations
Computer Games
Carbon Dating
Computer Circuits
Making Music
Movie Graphics
Defence and Military
Traffic Optimisation


Rockets and Satellites
Problem Solving
Crime Prediction
Loans, Interest, Mortgages
Skate Park Design
Search for Alien Life
Fraud Detection
Big Data


Microwaves
Image Compression
Pharmacy and Medicine
Swimsuit Design
Pricing Strategies
Polling and Voting
Music Shuffling
Tectonic Plate Motion


Game Theory
Population Dynamics
Coral Reef Growth
Erosion and Coastlines
Plastic Surgery
Diffusion of Liquids
Measuring Time
Cooking and Baking


Plastic Surgery
Diffusion of Liquids
Measuring Time
Cooking and Baking
Surveying
Making Chocolate
Wildfire Modelling
Counting Calories

Outline

Topology: a Major Branch of Mathematics

Topology is all about continuity and connectivity.

Here are some of the topics in Topology:

- **The Bridges of Königsberg**
- **Doughnuts and Coffee-cups**
- **Knots and Links**
- **Nodes and Edges: Graphs**
- **The Möbius Band**

In this lecture, we look at *Knots and Links.*

Pretzel Puzzle

Look at the two "pretzels" here:

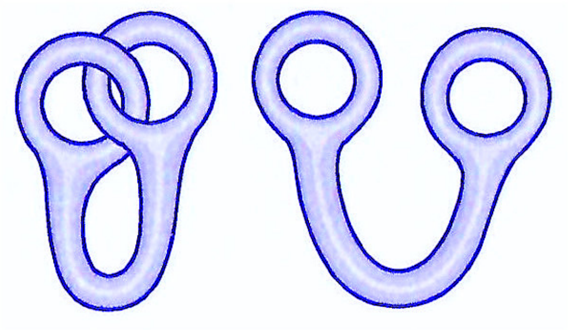


Figure : Two "Pretzels". Are they equivalent?

Knot Theory

A knot is an embedding of the unit circle S^1 into three-dimensional space R^3.

Two knots are equivalent if one can be distorted into the other without breaking it.

A knot is a mapping of the unit circle into three-space.

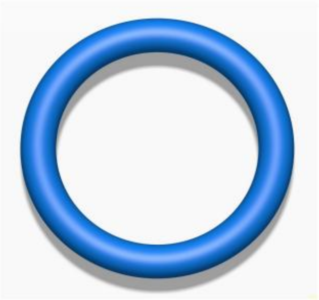
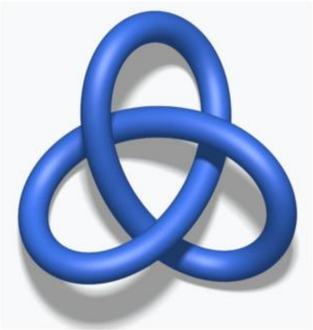



Figure : Left: Unknot. Right: Trefoil.

These two knots aren't equivalent: we can't distort the circle into the trefoil without breaking it.

Knots that are Mirror Images

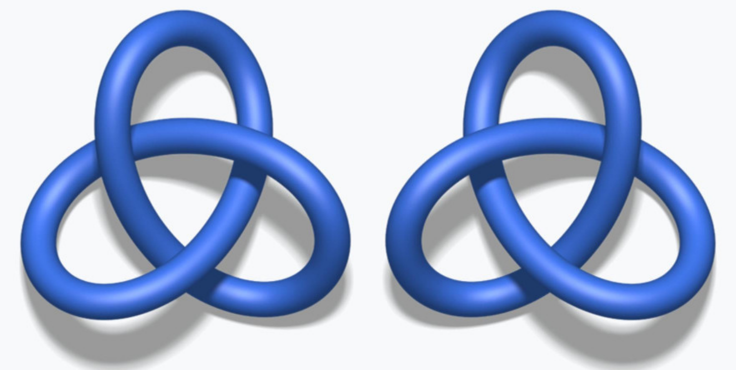


Figure : Levo and Dextro Trefoils.

These knots are mirror images but are not equivalent. We cannot change one into the other without breaking it.

The Simplest Knots and Links

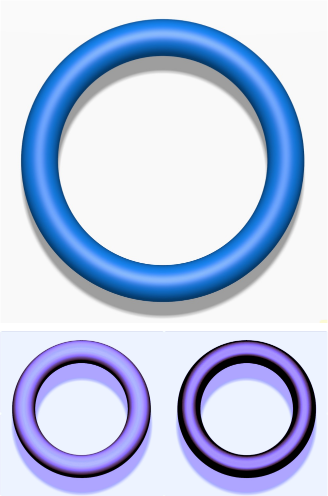


Figure : Top: The Unknot. Bottom: The Unlink.

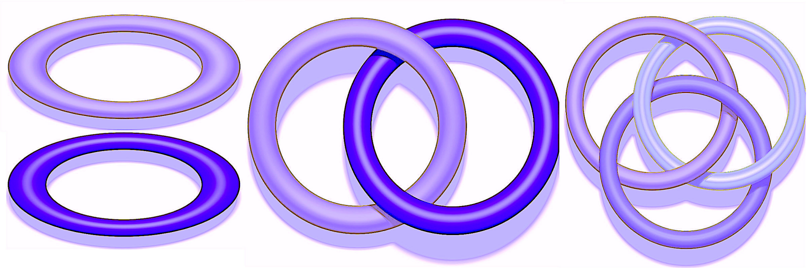


Figure : Unlink, Hopf Link and Borromean Rings.

The Hopf Link

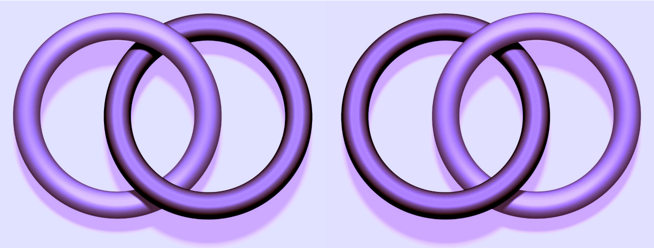


Figure : The Hopf Link and its mirror image. Equivalent?

Rings of Borromeo

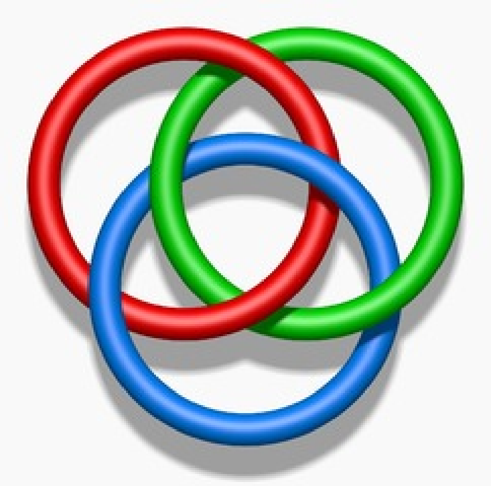


Figure : No two rings are linked! Are the three?

Genus of a Surface

**The genus of a topological surface is,
in simple terms, the number of holes in it.**

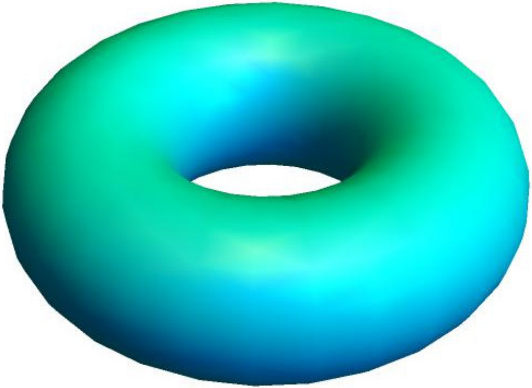
A sphere has no holes, so has genus 0.

A donut has one hole, so has genus 1.

Surfaces can have any number of holes; any genus.

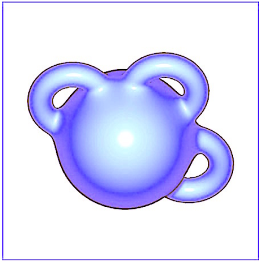
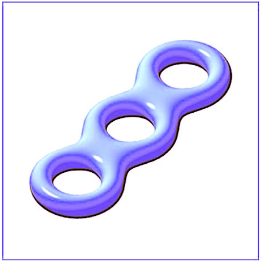
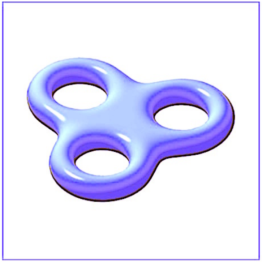
The Sphere, of Genus 0

The Torus, of Genus 1



The Double Torus, of Genus 2

Some Surfaces of Genus 3

Topologists have classified all surfaces in 3-space.

Link between Number Theory and Physics

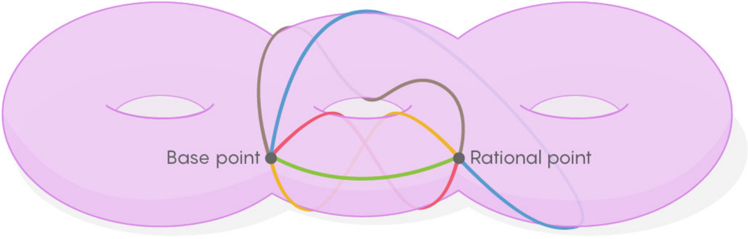
Forty years ago, physics and and topology had little or nothing to do with one another.

In the 1980s, mathematicians and physicists found ways to use physics to study the properties of shapes.

The field has never looked back.

`http://www.quantamagazine.org/secret-link-uncovered-between-pure-math-and-physics-20171201/`

Triple Torus



THREE-HOLED TORUS: Paths connect the base point with a rational point.

Figure : Rational solutions of $x^4 + y^4 = 1$ are on this surface

Pretzel Puzzle

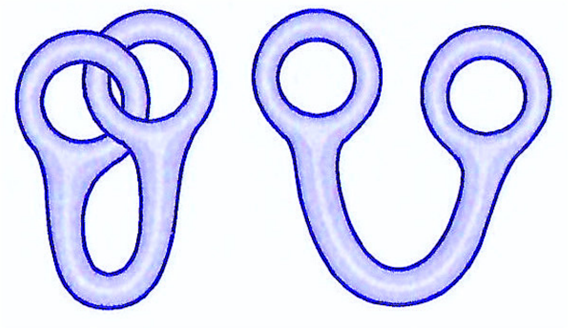


Figure : Two "Pretzels". Are they equivalent?

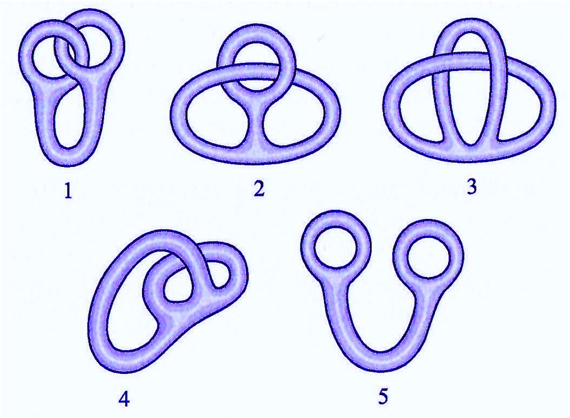


Figure : Equivalence!

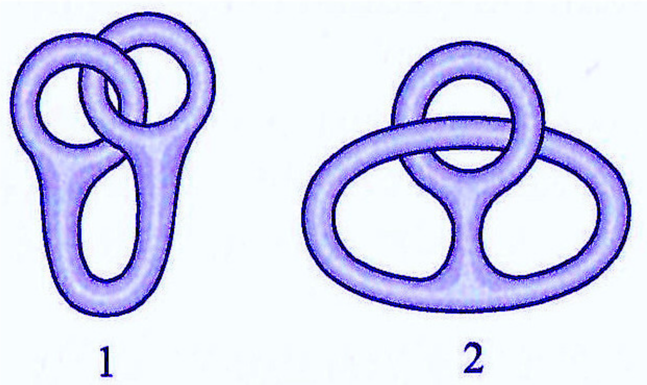


Figure : Make the left-hand loop bigger.

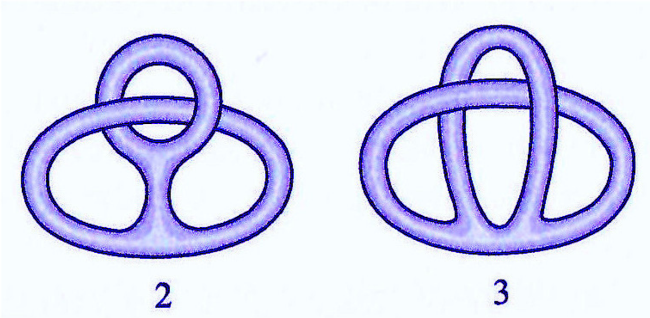


Figure : Make the other loop bigger.

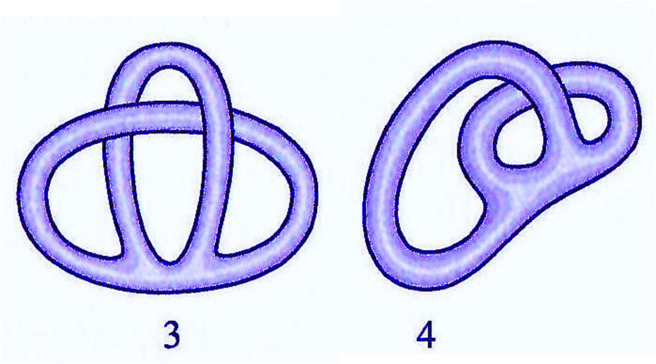


Figure : Pull the top loop away to the side.

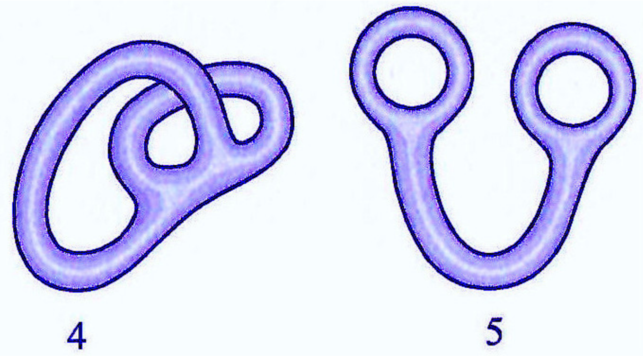


Figure : Smoothly distort to the final form.

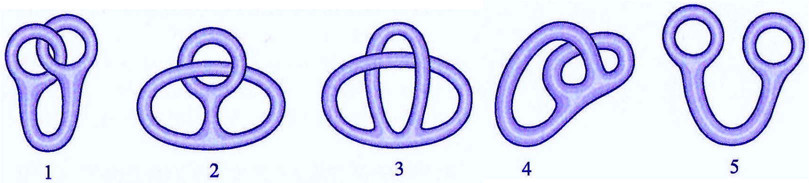


Figure : Combining all the distortions. Equivalence!

Another Surprising Result

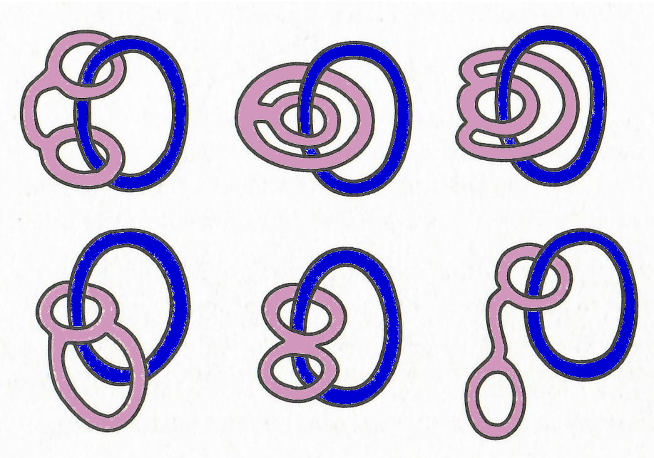


Figure : We can unlink one of the hand-cuffs.

Outline

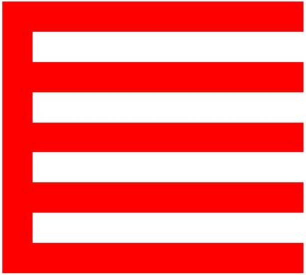
Source of Some Puzzles

Mathematical Lateral Thinking Puzzles
by
Paul Slone & Des MacHale

Slicing a Cake with One Cut

Bake a cake that you can slice into 6 equal pieces with one cut?

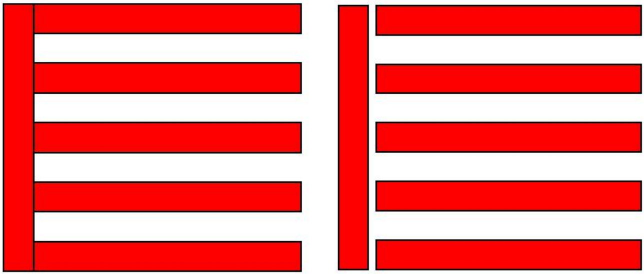
Hint: The cake can be any shape you like



Slicing a Cake with One Cut

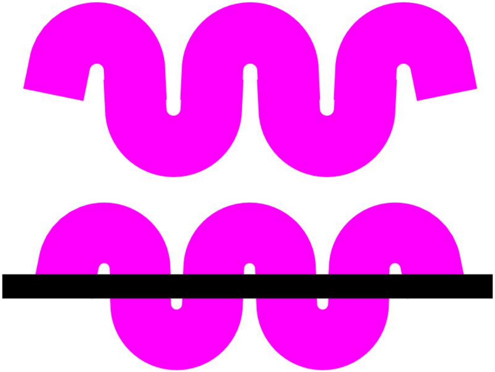
Bake a cake that you can slice into 6 equal pieces with one cut?

Hint: The cake can be any shape you like



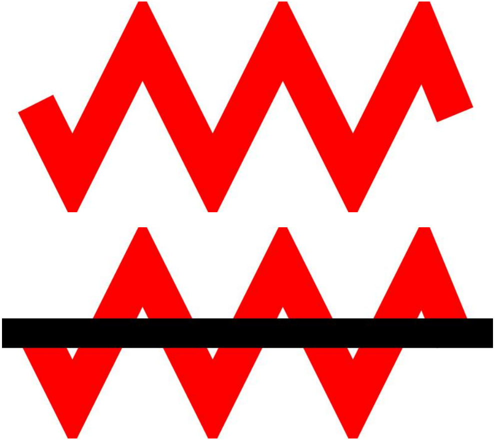
Student Solution: Snake Cake

Bake a cake that you can slice into 5 equal pieces with one cut?



Student Solution: Zigzag Cake

**Bake a cake that you can slice
into 6 equal pieces with one cut?**



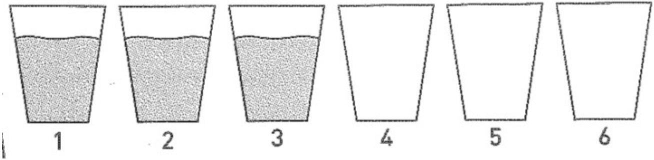
A Three-dimensional Cake

Cake in the form of a helix.

This is like twist . . .

. . . pastry twisted round
a stick and cooked over a
camp-fire.

Rearrange Six Glasses



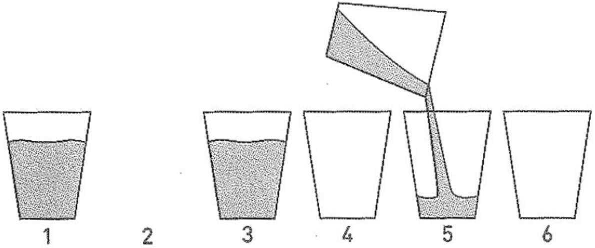
There are six glasses in a row.

Glasses 1, 2 and 3 are full.
Glasses 4, 5 and 6 are empty.

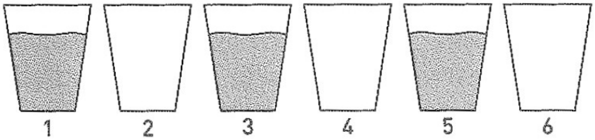
How can you arrange for the full and empty glasses to alternate, *moving only one glass?*

Rearrange Six Glasses

First, pour water from Glass 2 into glass 5:



Then, place Glass 2 in its original position:



Outline

David Hilbert (1862–1943)

David Hilbert, from a contemporary postcard.

Hilbert's Problems

In August 1900, David Hilbert addresed the *International Congress of Mathematicians* in the Sorbonne in Paris:

> ***"Who of us would not be glad to lift the veil behind which the future lies hidden; to cast a glance at the next advances of our science and at the secrets of its development during future centuries?"***

Hilbert presented 23 problems that challenged mathematicians through the twentieth century.

Hilbert's Problems

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 37, Number 4, Pages 407–436
S 0273-0979(00)00881-8
Article electronically published on June 26, 2000

MATHEMATICAL PROBLEMS

DAVID HILBERT

Lecture delivered before the International Congress of Mathematicians at Paris in 1900.

Hilbert's eighth problem concerned itself with what is called the Riemann Hypothesis (RH).

RH is generally regarded as the deepest and most important unproven mathematical problem.

Anyone who can prove it is assured of lasting fame.

Why is RH Important?

A large number of mathematical theorems (1000's) depend for their validity on the RH.

Were RH to turn out to be false, many of these mathematical arguments would simply collapse.

In 2000, industrialist Landon Clay donated $7M, with $1M for each of 7 problems in mathematics.

The Riemann hypothesis is one of these problems.

`http://www.claymath.org/millennium-problems`

Why is RH Important?

Whoever proves Riemann's hypothesis will have completed thousands of theorems that start like this:

"Assuming that the Riemann hypothesis is true . . . ".

He or she will be assured of lasting fame.

Those who establish fundamental mathematical results probably come closer to immortality than almost anyone else.

Thank you