

# Sum-Enchanted Evenings

The Fun and Joy of Mathematics



## LECTURE 8

**Peter Lynch**

**School of Mathematics & Statistics  
University College Dublin**

**Evening Course, UCD, Autumn 2018**



# Outline

**Introduction**

**Euler's Gem**

**Distraction 6A: Slicing a Pizza (Again)**

**Distraction 7: Plus Magazine**

**Astronomy II**

**Distraction 8: Sum by Inspection**

**Carl Friedrich Gauss**



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# Meaning and Content of Mathematics

The word **Mathematics** comes from Greek *μαθημα* (*máthéma*), meaning “knowledge” or “study” or “learning”.

It is the study of topics such as

- ▶ Quantity (numbers)
- ▶ Structure (patterns)
- ▶ Space (geometry)
- ▶ Change (analysis).



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Carl Friedrich Gauss

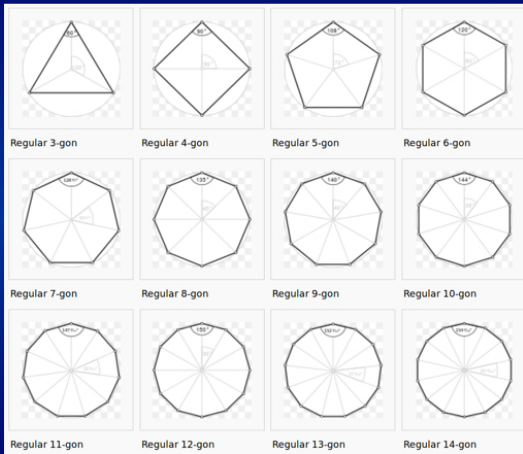


# Euler's polyhedron formula.






Carving up the globe.



# Regular Polygons



# The Platonic Solids (polyhedra)

Tetrahedron (four faces)	Cube or hexahedron (six faces)	Octahedron (eight faces)	Dodecahedron (twelve faces)	Icosahedron (twenty faces)
				

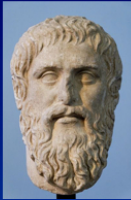
These five regular polyhedra were discovered in ancient Greece, perhaps by **Pythagoras**.

**Plato** used them as models of the universe.

They are analysed in Book XIII of **Euclid's Elements**.





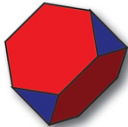


There are only five **Platonic** solids.

But **Archimedes** found, using different types of polygons, that he could construct 13 new solids.



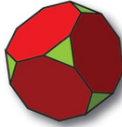
# The Thirteen Archimedean Solids



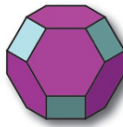
TRUNCATED TETRAHEDRON



CUBOCTAHEDRON



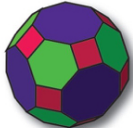
TRUNCATED CUBE



TRUNCATED OCTAHEDRON



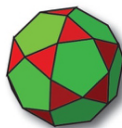
RHOMBICUBOCTAHEDRON



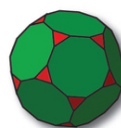
TRUNCATED CUBOCTAHEDRON



SNUB CUBE



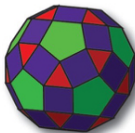
ICOSIDODECAHEDRON



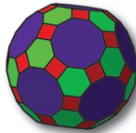
TRUNCATED DODECAHEDRON



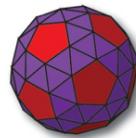
TRUNCATED ICOSAHEDRON



RHOMBICOSIDODECAHEDRON



TRUNCATED ICOSIDODECAHEDRON



SNUB DODECAHEDRON

Check  $V - E + F$  for the Truncated Cube



# Euler's Polyhedron Formula

The great Swiss mathematician, **Leonard Euler**, noticed that, for all (convex) polyhedra,

$$V - E + F = 2$$

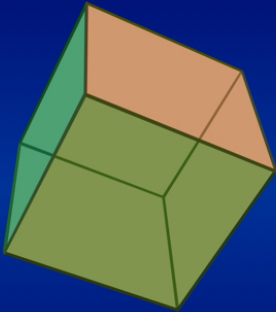
where

- **V** = Number of vertices
- **E** = Number of edges
- **F** = Number of faces

Mnemonic: Very Easy Formula



## For example, a Cube



Number of vertices:  $V = 8$

Number of edges:  $E = 12$

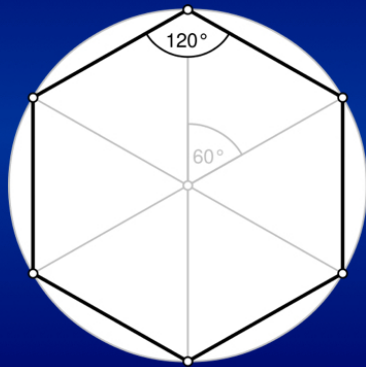
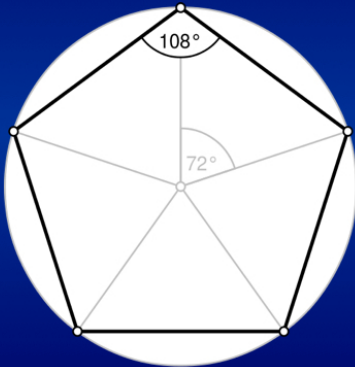
Number of faces:  $F = 6$

$$(V - E + F) = (8 - 12 + 6) = 2$$

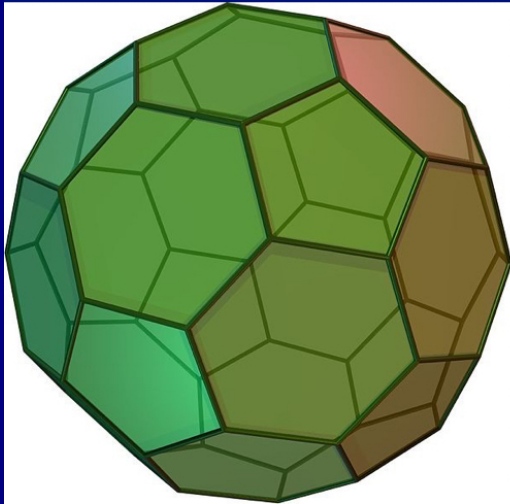
Mnemonic: Very Easy Formula



# Pentagons and Hexagons



# The Truncated Icosahedron



**An Archimedean solid  
with  
pentagonal and  
hexagonal faces.**



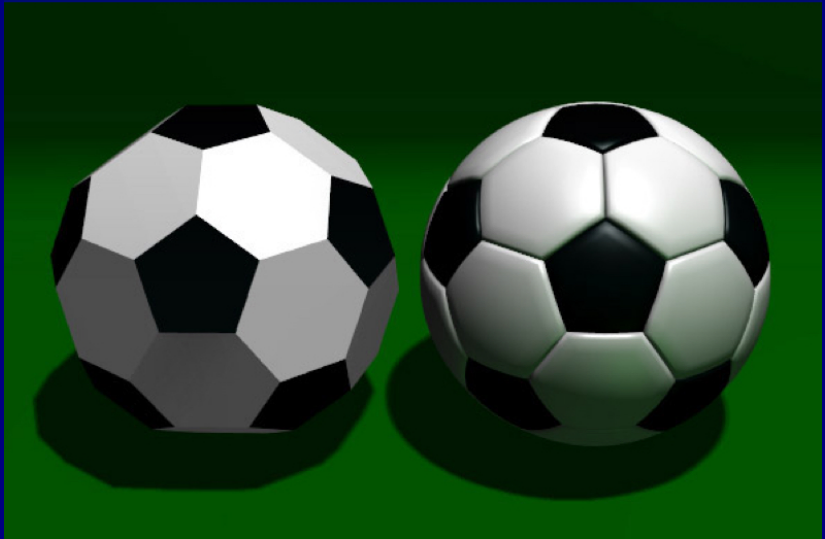
# The Truncated Icosahedron



Where have  
you seen this  
before?



# The Truncated Icosahedron







The "**Buckyball**", introduced at the 1970 World Cup Finals in Mexico.

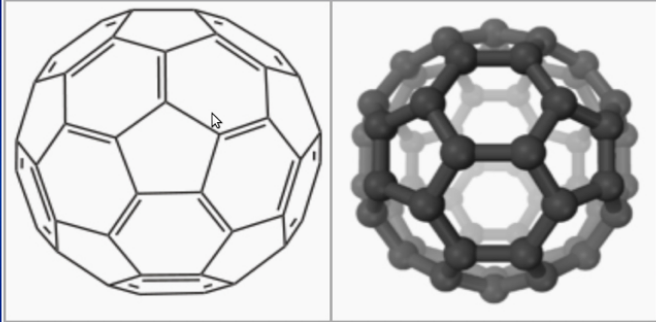
It has 32 panels: 20 hexagons and 12 pentagons.



**A Geodesic Dome designed by the American architect  
Richard Buckminster "Bucky" Fuller.**



# Buckminsterfullerene

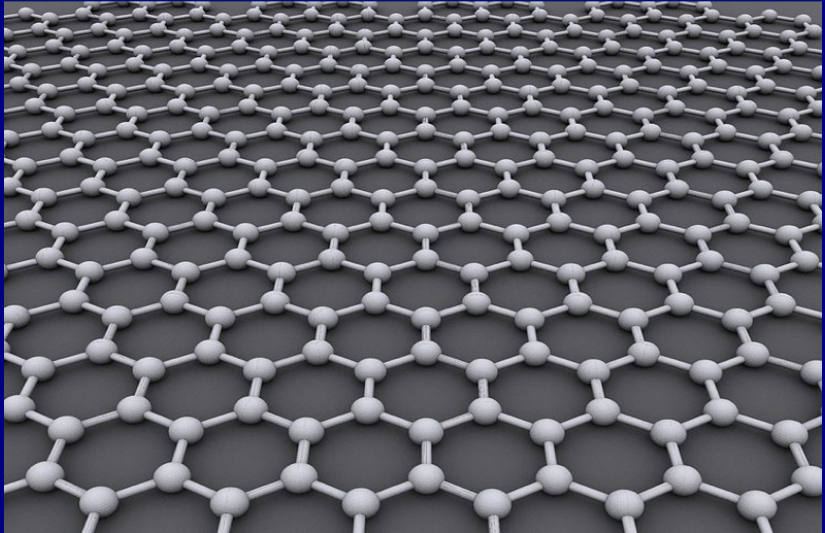


**Buckminsterfullerene is a molecule with formula  $C_{60}$**

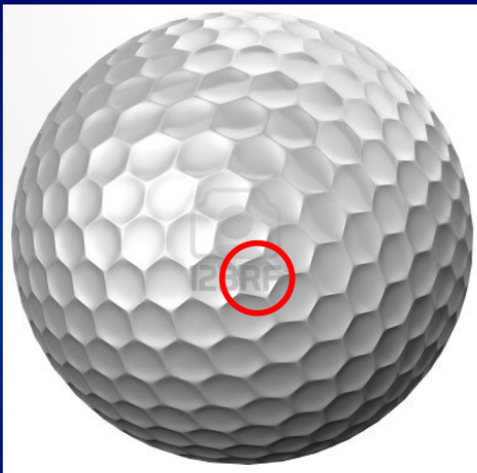
**It was first synthesized in 1985.**

# Graphene

A hexagonal pattern of carbon one atom thick



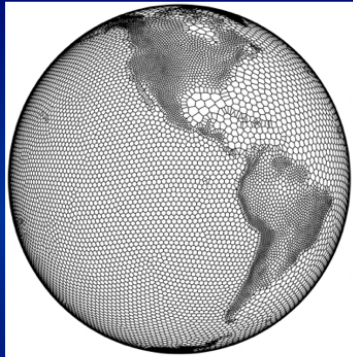




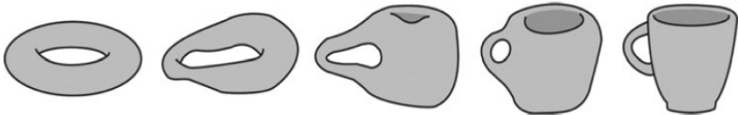
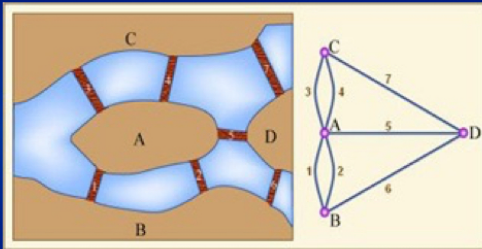
# Euler's Polyhedron Formula

$$V - E + F = 2$$

still holds.

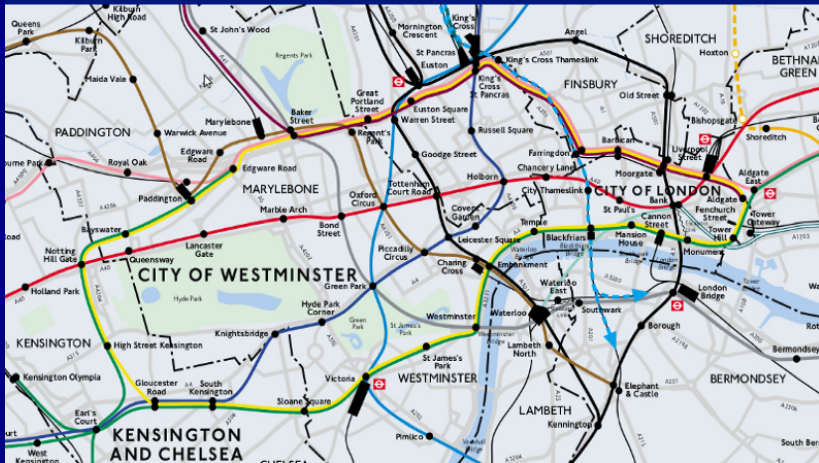


# Topology is often called Rubber Sheet Geometry





# Topology and the London Underground Topographical Map



# Topology and the London Underground

## Topological Map



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Carl Friedrich Gauss



# Distraction 6A: Slicing a Pizza (Again)



Cut the pizza using only straight cuts.

There should be exactly one piece of pepperoni on each slice of pizza.

**Minimum number of cuts?**



# Abstract Formulation

**Last Week's Problem:**

**Plane cut by  $n$  lines. How many regions are formed?**



# Abstract Formulation

Last Week's Problem:

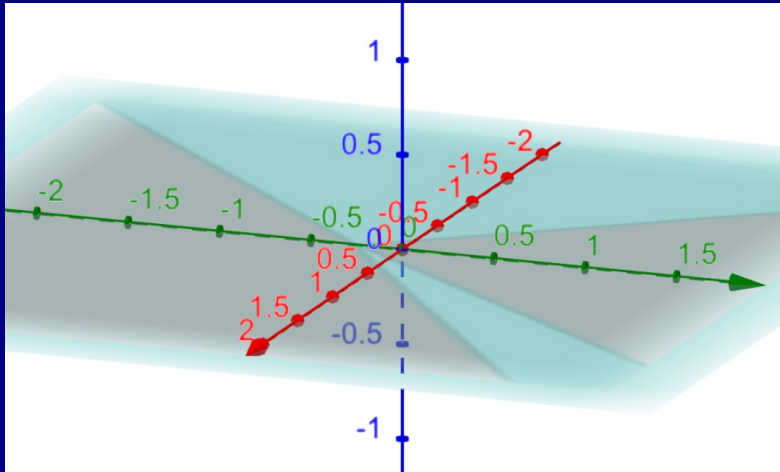
Plane cut by  $n$  lines. How many regions are formed?

Cuts	Segments (1D)	Regions (2D)	Solids (3D)
0	1	1	1
1	2	2	2
2	3	4	4
3	4	7	8
4	5	11	15
5	6	16	26
6	7	22	42

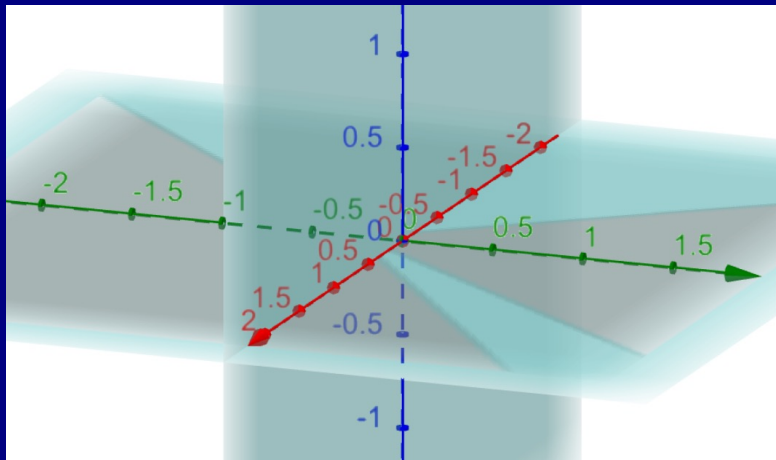
What is the pattern here?



# Distraction 6A: Slicing a Pizza

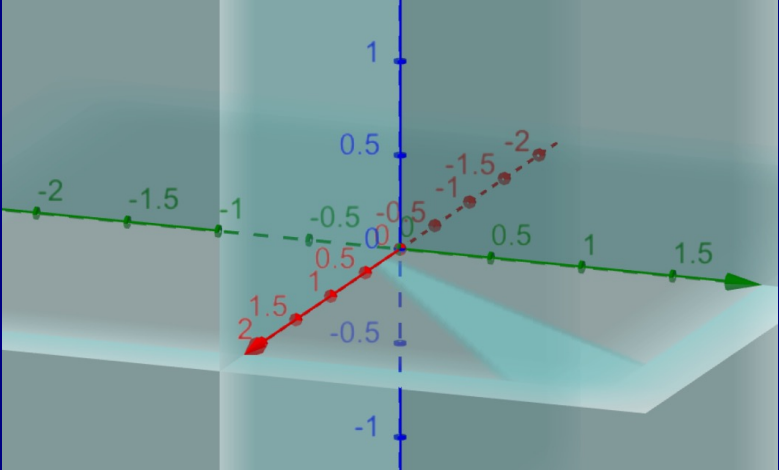


# Distraction 6A: Slicing a Pizza

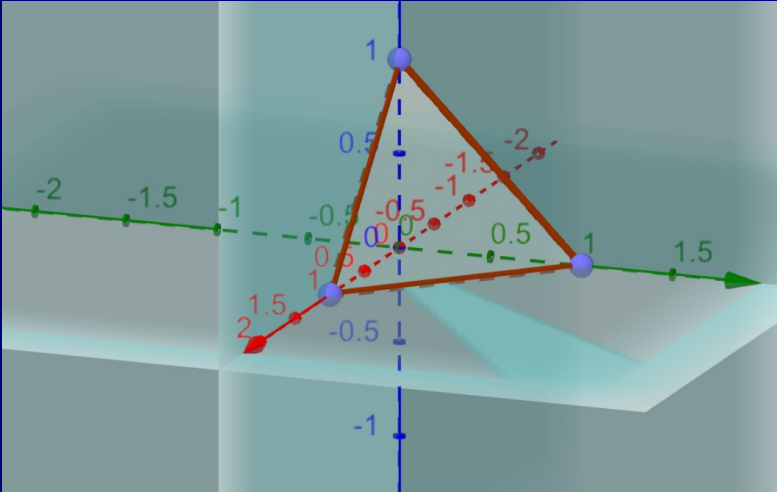




# Distraction 6A: Slicing a Pizza



# Distraction 6A: Slicing a Pizza



# Cutting Lines, Planes and Spaces

Cuts	Segments (1D)	Regions (2D)	Solids (3D)
0	1	1	1
1	2	2	2
2	3	4	4
3	4	7	8
4	5	11	15
5	6	16	26
6	7	22	42

There is a pattern here.  
It is reminiscent of Pascal's Triangle.



# Distraction 6A: Doughnut-Slicing Problem



BRILLIANT

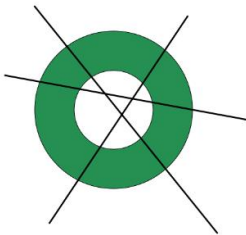
Courses

Practice

Community

## Cutting an Annulus

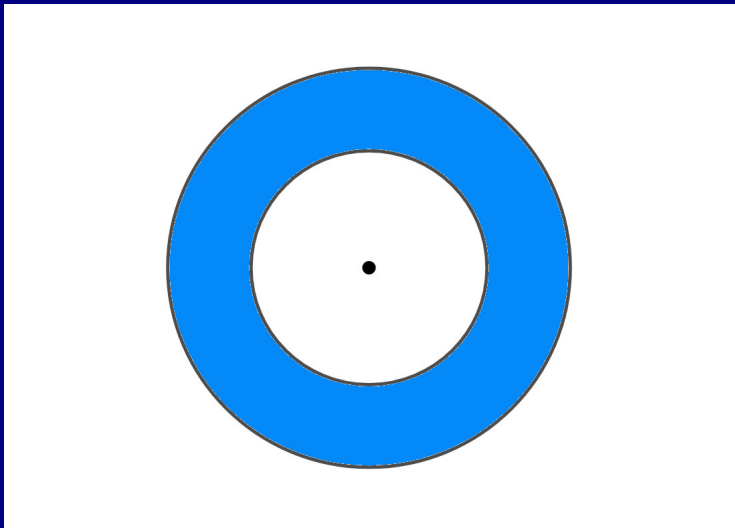
Discrete Mathematics Level 2



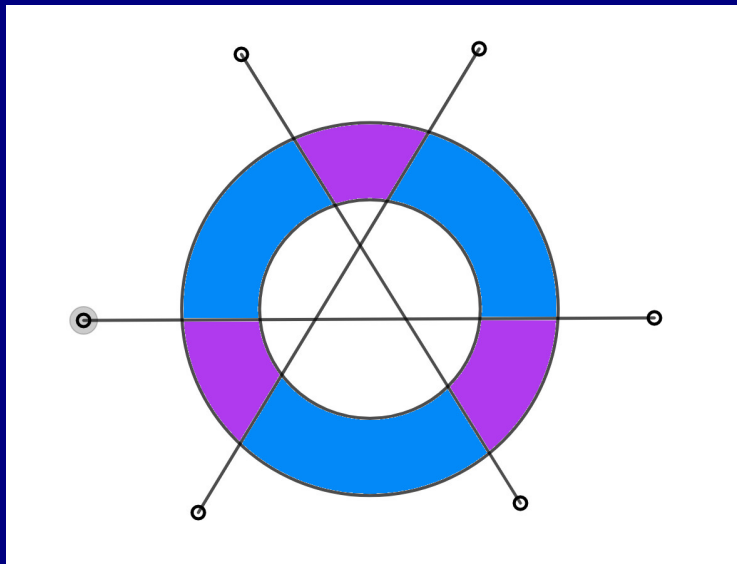
What is the maximum number of pieces into which a ring can be cut by 3 straight lines?  
In the image above, the ring is cut into 6 pieces by 3 lines.



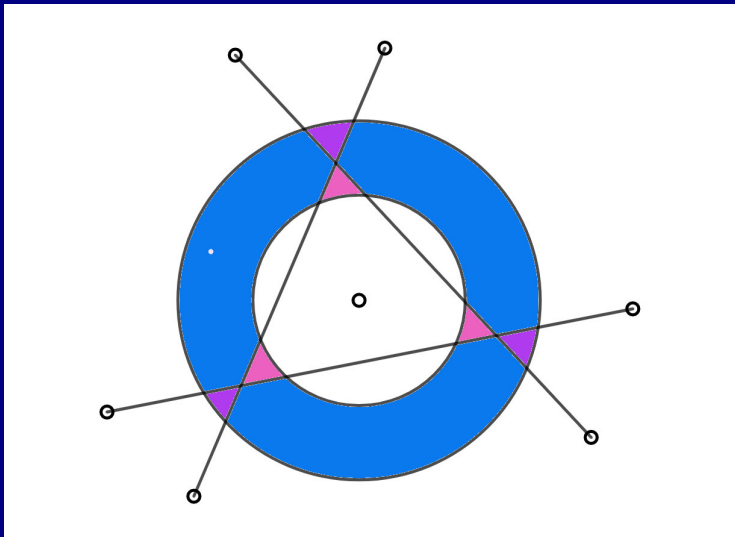
# Distraction 6A: Slicing a (Flat) Doughnut



# Distraction 6A: Slicing a (Flat) Doughnut



# Distraction 6A: Slicing a (Flat) Doughnut



# Distraction 6A: Brilliant Website

The screenshot shows the Brilliant.org website interface. At the top, there is a navigation bar with the Brilliant logo, links for 'Courses', 'Practice', and 'Community', a search bar, and a 'Go Premium' button. Below the navigation bar, the 'Courses' section is displayed. On the left, there is a sidebar with 'Recent' and 'Recommended' categories, listing 'Math', 'Science', and 'Computer Science'. The main content area is divided into two sections: 'Recent' and 'Recommended - Popular in the last month'. The 'Recent' section features three course cards: 'Ace the AMC' (purple background with a bar chart), 'Group Theory' (teal background with a Rubik's cube), and 'Classical Mechanics' (dark grey background with a rocket). The 'Recommended - Popular in the last month' section features four course cards: 'Logic' (purple background with a silhouette of a head), 'Computer Science Fundamentals' (red background with a tree diagram), 'Artificial Neural Networks' (blue background with a brain diagram), and 'Mathematical Fundamentals' (orange background with a clock face). Each card includes a title, a brief description, and a small graphic.

**BRILLIANT** Courses Practice Community  Go Premium

## Courses

**Recent**

Recommended

Math

Science

Computer Science

### Recent

#### Ace the AMC

Guided training for the strategies needed to excel in AMC 10 and 12.

#### Group Theory

Explore groups through symmetries, applications, and problems.

#### Classical Mechanics

Hardcore training for the aspiring physicist.

### Recommended - Popular in the last month

#### Logic

Stretch your analytic muscles with knights, knaves, logic gates, and more!

#### Computer Science Fundamentals

The fundamental toolkit for the aspiring computer scientist or programmer.

#### Artificial Neural Networks

A quick dive into a cutting-edge computational method for learning.

#### Mathematical Fundamentals

The essential tools for mastering algebra, probability, and logic!

<https://brilliant.org/>





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**Distraction 7: Plus Magazine**

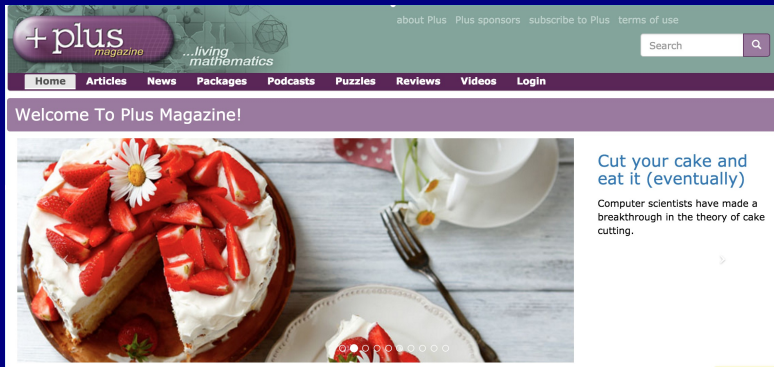
Astronomy II

Distraction 8: Sum by Inspection

Carl Friedrich Gauss



# Distraction 7: Plus Magazine



The screenshot shows the homepage of Plus Magazine. At the top left is the logo '+ plus magazine' and the tagline '...living mathematics'. To the right are links for 'about Plus', 'Plus sponsors', 'subscribe to Plus', and 'terms of use'. A search bar is located on the right side of the header. Below the header is a navigation menu with links for 'Home', 'Articles', 'News', 'Packages', 'Podcasts', 'Puzzles', 'Reviews', 'Videos', and 'Login'. The main content area features a large image of a strawberry cake with a slice cut out and served on a plate. To the right of the image is a text block with the headline 'Cut your cake and eat it (eventually)' and a sub-headline 'Computer scientists have made a breakthrough in the theory of cake cutting.' Below the image is a row of small circular icons.

**PLUS: The Mathematics e-zine**

<https://plus.maths.org/>



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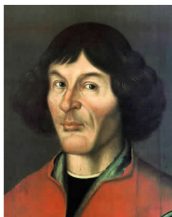
Carl Friedrich Gauss



# The Scientific Revolution

## INTRODUCTION

This week, we will look at developments in the sixteenth and seventeenth centuries.



Nicolaus Copernicus  
1473 – 1543



Tycho Brahe  
1546 – 1601



Johannes Kepler  
1571 – 1630



Galileo Galilei  
1564 – 1642

Figure from [mathigon.org](http://mathigon.org)



# The Heliocentric Model

In 1543, **Nicolaus Copernicus** (1473–1543) published *“On the Revolutions of the Celestial Spheres”*.

He explained that the Sun is at the centre of the universe and that the Earth and planets move around it in circular orbits.



# The Heliocentric Model

In 1543, **Nicolaus Copernicus** (1473–1543) published *“On the Revolutions of the Celestial Spheres”*.

He explained that the Sun is at the centre of the universe and that the Earth and planets move around it in circular orbits.

Danish astronomer **Tycho Brahe** (1546–1601) made very accurate observations of the movements of the planets, and developed his own model of the solar system.



# Johannes Kepler (1571–1630)

**Johannes Kepler (1571–1630) succeeded Brahe as imperial mathematician.**

**After many years of struggling, Kepler succeeded in formulating his **three Laws of Planetary Motion**.**

**Kepler's Laws describe the solar system much as we know it to be true today.**



# Kepler's Laws

- ▶ **The planets move on elliptical orbits, with the Sun at one of the two foci.**  
This explains why the Sun appears larger at some times of the year and smaller at others.
- ▶ **A line joining the planet and the Sun sweeps out equal areas in equal times.**  
This means that a planet moves faster when close to the Sun, and slower when further away.
- ▶ **The square of the orbital period is proportional to the cube of the mean radius of the orbit.**  
This law allows us to find the orbital time of a planet if we know the size of the orbit.

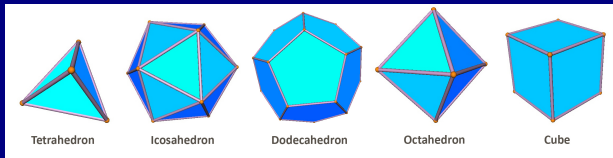




# The *Mysterium Cosmographicum*

There were **six known planets** in Kepler's time:  
Mercury, Venus, Earth, Mars, Jupiter, Saturn.

There are precisely **five platonic solids**:



This gave Kepler an extraordinary idea!

<https://thatsmaths.com/2016/10/13/>

[\keplers-magnificent-mysterium-cosmographicum/](#)



# Galileo Galelii (1564–1630)

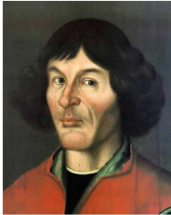
Galileo introduced the **telescope** to astronomy, and made some dramatic discoveries.

He observed the four large moons of Jupiter **revolving around that planet.**

He established the laws of inertia that underlie Newton's dynamical laws.



# Four Remarkable Scientists



Nicolaus Copernicus  
1473 – 1543



Tycho Brahe  
1546 – 1601



Johannes Kepler  
1571 – 1630



Galileo Galilei  
1564 – 1642

Figure from [mathigon.org](http://mathigon.org)



# Isaac Newton (1642–1727)

In 1687, Isaac Newton published the **Principia Mathematica**. He established the mathematical foundations of dynamics.

Between any two masses there is a force:

$$F = \frac{GMm}{r^2}$$

This is the **force of gravity** and gravity is what makes the planets move around the Sun.

Newton's Laws imply and explain Kepler's laws.



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Carl Friedrich Gauss



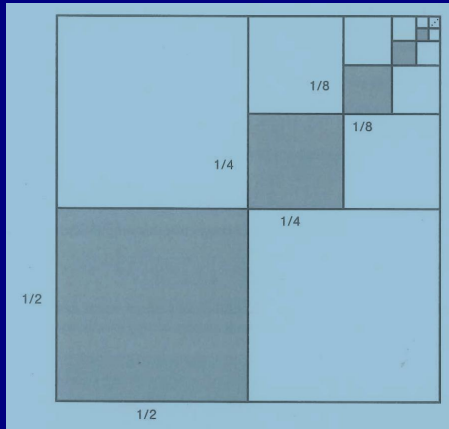
# Distraction 8: Sum by Inspection

Can you guess the sum of this series:

$$\left(\frac{1}{2}\right)^2 + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{8}\right)^2 + \left(\frac{1}{16}\right)^2 + \dots$$



# Distraction 8: Sum by Inspection



We will find the shaded area without calculation



# Proof by Inspection

Look at the figure in two different ways

At each scale, we have three squares the same size, and we keep one of them (black) and omit the others.

So, the area of the shaded squares is  $\frac{1}{3}$ .





# Proof by Inspection

Look at the figure in two different ways

At each scale, we have three squares the same size, and we keep one of them (black) and omit the others.

So, the area of the shaded squares is  $\frac{1}{3}$ .

However, it is also given by the series

$$\left(\frac{1}{2}\right)^2 + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{8}\right)^2 + \left(\frac{1}{16}\right)^2 + \dots$$

Therefore we can sum the series:

$$\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \dots = \frac{1}{3}$$



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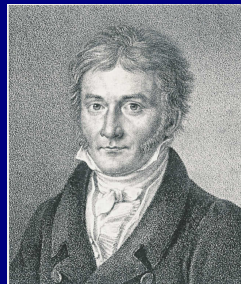
# Carl Friedrich Gauss (1777–1855)



# Carl Friedrich Gauss (1777–1855)

**A German mathematician who made profound contributions to many fields of mathematics:**

- ▶ **Number theory**
- ▶ **Algebra**
- ▶ **Statistics**
- ▶ **Analysis**
- ▶ **Differential geometry**
- ▶ **Geodesy & Geophysics**
- ▶ **Mechanics & Electrostatics**
- ▶ **Astronomy**



**One of the greatest mathematicians of all time.**



# Gauss Outsmarts his Teacher

Gauss was a genius. He was known as  
**The Prince of Mathematicians.**



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Gauss was a genius. He was known as

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When very young, Gauss outsmarted his teacher.



# Gauss Outsmarts his Teacher

Gauss was a genius. He was known as

**The Prince of Mathematicians.**

When very young, Gauss outsmarted his teacher.

I can now reveal a fact **unknown to historians:**

**The teacher got his own back. Ho! ho! ho!**



# Gauss Outsmarts his Teacher

**Gauss's school teacher tasked the class:**

- ▶ **Add up all the whole numbers from 1 to 100.**





# Gauss Outsmarts his Teacher

Gauss's school teacher tasked the class:

- ▶ Add up all the whole numbers from 1 to 100.

Gauss solved the problem in a flash.

He wrote the correct answer,

**5,050**

on his slate and handed it to the teacher.



# Gauss Outsmarts his Teacher

Gauss's school teacher tasked the class:

- ▶ Add up all the whole numbers from 1 to 100.

Gauss solved the problem in a flash.

He wrote the correct answer,

**5,050**

on his slate and handed it to the teacher.

How did Gauss do it?



**First, Gauss wrote the numbers in a row:**

1 2 3 ... 98 99 100



First, Gauss wrote the numbers in a row:

1 2 3 ... 98 99 100

Next he wrote them again, **in reverse order**:

1 2 3 ... 98 99 100  
100 99 98 ... 3 2 1



**First, Gauss wrote the numbers in a row:**

1 2 3 ... 98 99 100

**Next he wrote them again, in reverse order:**

1 2 3 ... 98 99 100  
100 99 98 ... 3 2 1

**Then he added the two rows, column by column:**

1	2	3	...	98	99	100
100	99	98	...	3	2	1
-----						
101	101	101	...	101	101	101

**Clearly, the total for the two rows is 10,100.**



First, Gauss wrote the numbers in a row:

1 2 3 ... 98 99 100

Next he wrote them again, in reverse order:

1 2 3 ... 98 99 100  
100 99 98 ... 3 2 1

Then he added the two rows, column by column:

1	2	3	...	98	99	100
100	99	98	...	3	2	1
-----						
101	101	101	...	101	101	101

Clearly, the total for the two rows is 10,100.

But every number from 1 to 100 is counted twice.

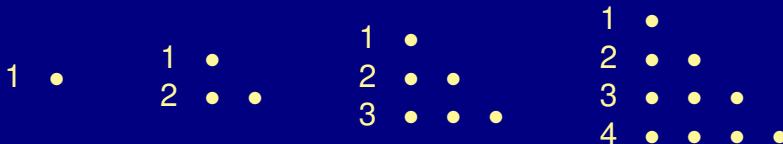
$$\therefore 1 + 2 + 3 + \dots + 98 + 99 + 100 = 5,050$$



# Triangular Numbers

Gauss had calculated the 100-th **triangular number**.

Let us take a geometrical look at the sums of the first few natural numbers:

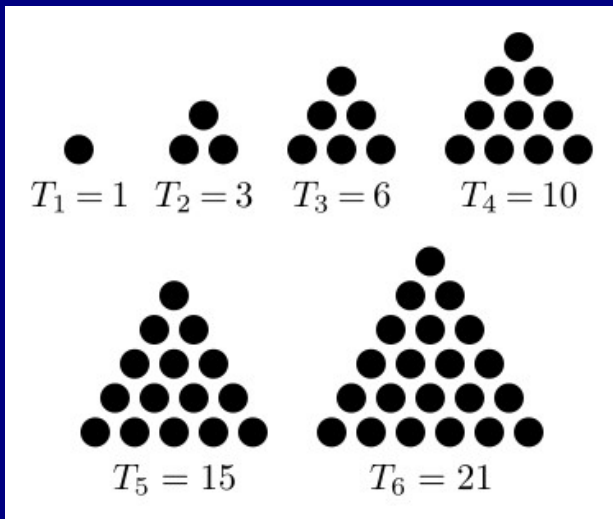


We see that the sums can be arranged as triangles.



# Triangular Numbers

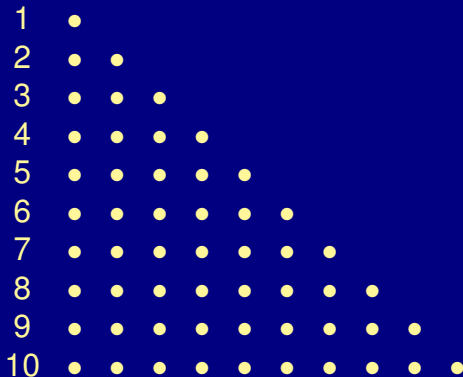
The first few **triangular numbers** are  $\{1, 3, 6, 10, 15, 21\}$ .





Let's look at the 10th triangular number.

For  $n = 10$  the pattern is:



How do we compute its value? Gauss's method!



It is easy to show that the  $n$ -th triangular number is

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We do just as Gauss did, and list the numbers twice:

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Each number has been counted twice, so

$$T_n = \frac{1}{2}n(n + 1)$$



Let's check this for Gauss's problem of  $n = 100$ :

$$T_{100} = 1 + 2 + 3 + \cdots + 100 = \frac{100 \times 101}{2} = 5,050$$



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Gauss's approach was to look at the problem from a new angle.

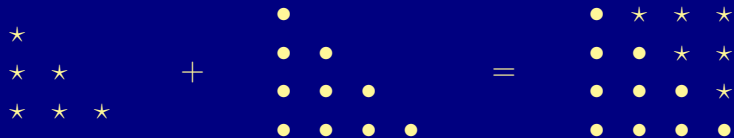
Such *lateral thinking* is very common in mathematics:

Problems that look difficult can sometimes be solved easily when tackled from a different angle.



# Two Triangles Make a Square

A nice property of *consecutive* triangular numbers:



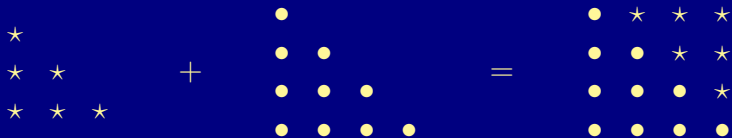
$$T_3 + T_4 = 6 + 10 = 16 = 4^2$$



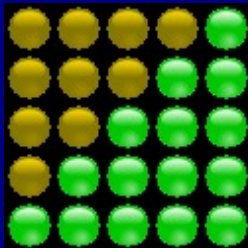


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The result is **a perfect square**.



# Puzzle

What is the sum of all the numbers  
from 1 up to 100 and back down again?



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What is the sum of all the numbers  
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The answer is in the video coming up now.



# A Video from the Museum of Mathematics



**VIDEO: Beautiful Maths, available at**

**<http://momath.org/home/beautifulmath/>**

**Video by James Tanton**



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**EXERCISE: Zink about that!**



# A Lateral Thinking Puzzle

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- ▶ What age was she when she was half his age?

Let Jill's age be  $J$ . Let her father's age be  $F$ .

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**Hint: Be Smart**  
**There is no need for tricky algebra.**



Thank you

