

Outline

Introduction

Euler's Gem

Distraction 6A: Slicing a Pizza (Again)

Distraction 7: Plus Magazine

Astronomy II

Distraction 8: Sum by Inspection

Carl Friedrich Gauss



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Meaning and Content of Mathematics

The word **Mathematics** comes from Greek *μαθημα* (*máthéma*), meaning “knowledge” or “study” or “learning”.

It is the study of topics such as

- ▶ Quantity (numbers)
- ▶ Structure (patterns)
- ▶ Space (geometry)
- ▶ Change (analysis).



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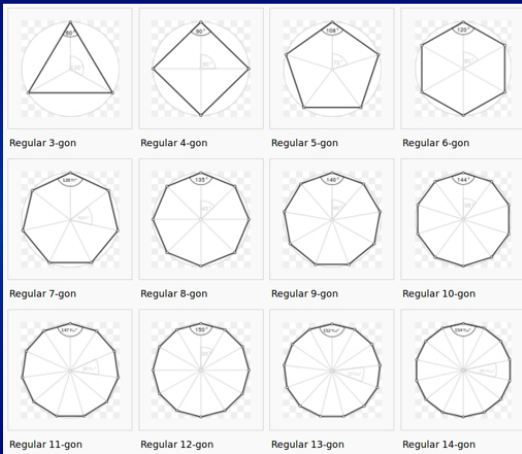


Euler's polyhedron formula.






Carving up the globe.



Regular Polygons



The Platonic Solids (polyhedra)

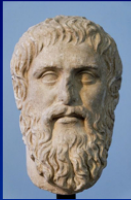
Tetrahedron (four faces)	Cube or hexahedron (six faces)	Octahedron (eight faces)	Dodecahedron (twelve faces)	Icosahedron (twenty faces)
				

These five regular polyhedra were discovered in ancient Greece, perhaps by **Pythagoras**.

Plato used them as models of the universe.

They are analysed in Book XIII of **Euclid's Elements**.



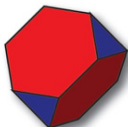


There are only five **Platonic** solids.

But **Archimedes** found, using different types of polygons, that he could construct 13 new solids.



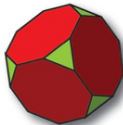
The Thirteen Archimedean Solids



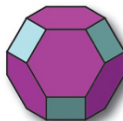
TRUNCATED TETRAHEDRON



CUBOCTAHEDRON



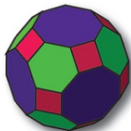
TRUNCATED CUBE



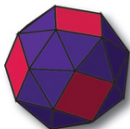
TRUNCATED OCTAHEDRON



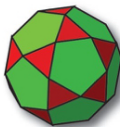
RHOMBICUBOCTAHEDRON



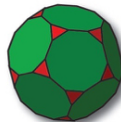
TRUNCATED CUBOCTAHEDRON



SNUB CUBE



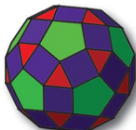
ICOSIDODECAHEDRON



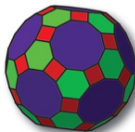
TRUNCATED DODECAHEDRON



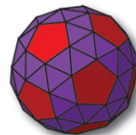
TRUNCATED ICOSAHEDRON



RHOMBICOSIDODECAHEDRON



TRUNCATED ICOSIDODECAHEDRON



SNUB DODECAHEDRON

Check $V - E + F$ for the Truncated Cube



Euler's Polyhedron Formula

The great Swiss mathematician, **Leonard Euler**, noticed that, for all (convex) polyhedra,

$$V - E + F = 2$$

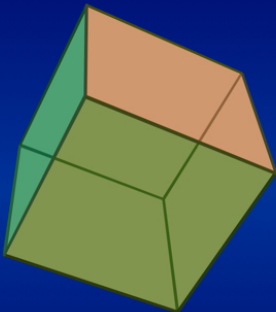
where

- **V** = Number of vertices
- **E** = Number of edges
- **F** = Number of faces

Mnemonic: Very Easy Formula



For example, a Cube



Number of vertices: $V = 8$

Number of edges: $E = 12$

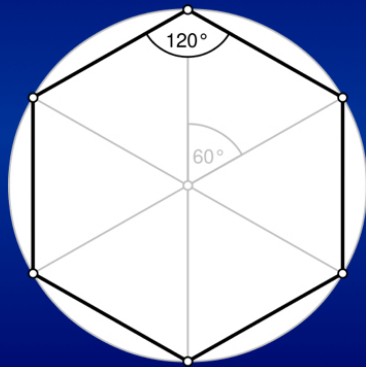
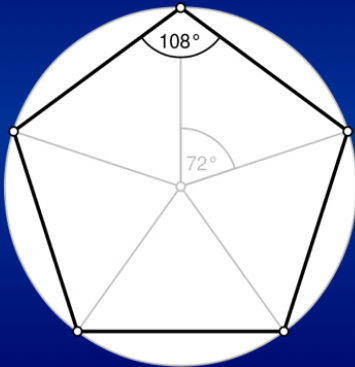
Number of faces: $F = 6$

$$(V - E + F) = (8 - 12 + 6) = 2$$

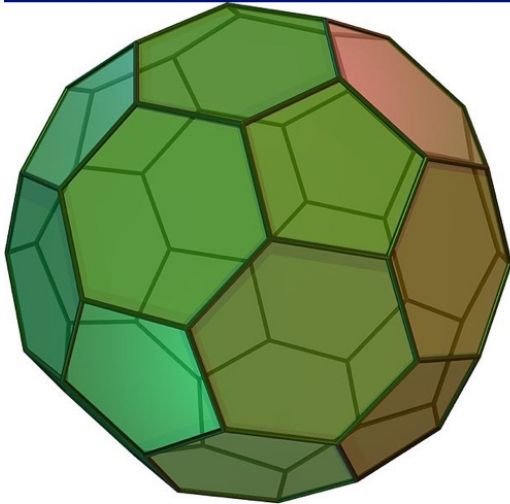
Mnemonic: Very Easy Formula



Pentagons and Hexagons



The Truncated Icosahedron



**An Archimedean solid
with
pentagonal and
hexagonal faces.**



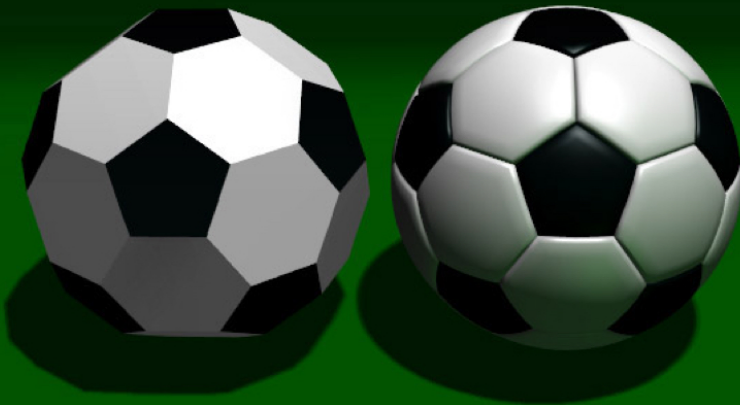
The Truncated Icosahedron



Where have
you seen this
before?



The Truncated Icosahedron





The "**Buckyball**", introduced at the 1970 World Cup Finals in Mexico.

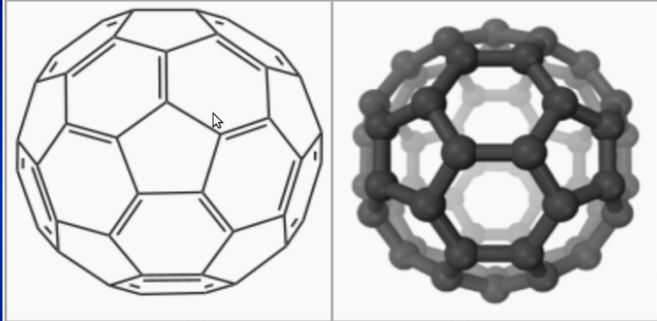
It has 32 panels: 20 hexagons and 12 pentagons.



**A Geodesic Dome designed by the American architect
Richard Buckminster "Bucky" Fuller.**



Buckminsterfullerene



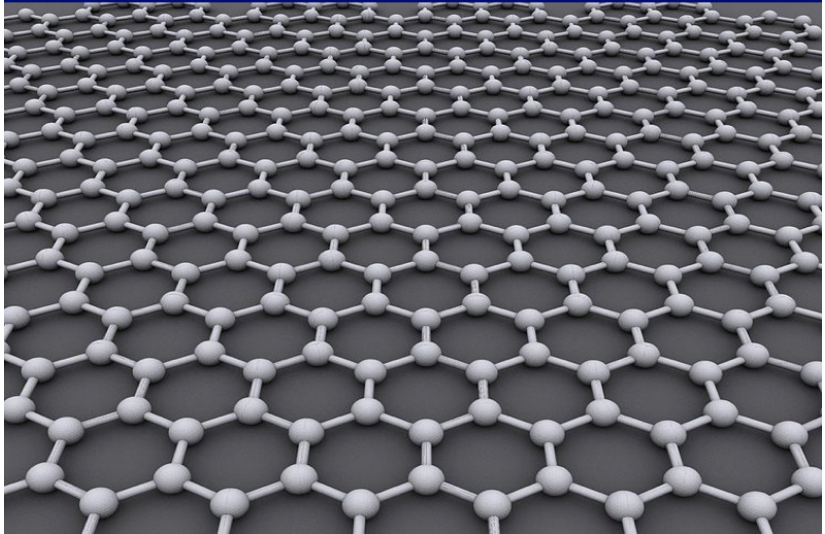
Buckminsterfullerene is a molecule with formula C_{60}

It was first synthesized in 1985.

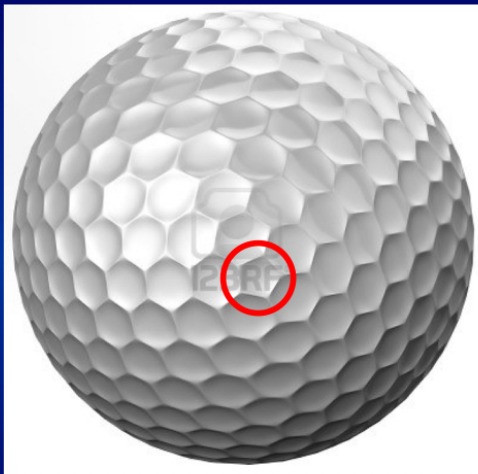


Graphene

A hexagonal pattern of carbon one atom thick



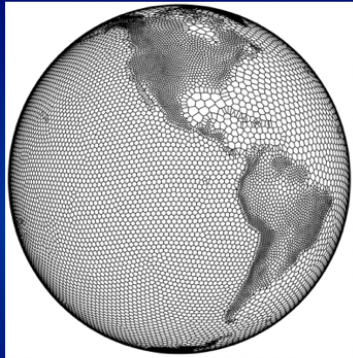




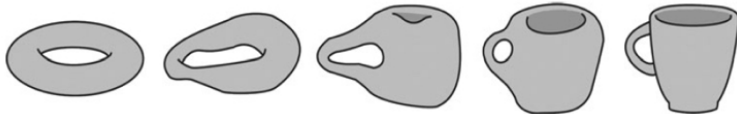
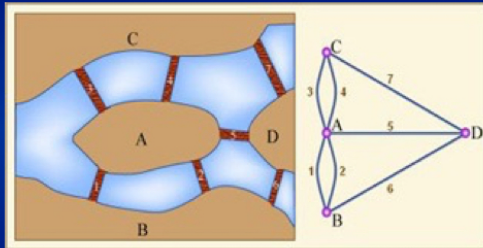
Euler's Polyhedron Formula

$$V - E + F = 2$$

still holds.

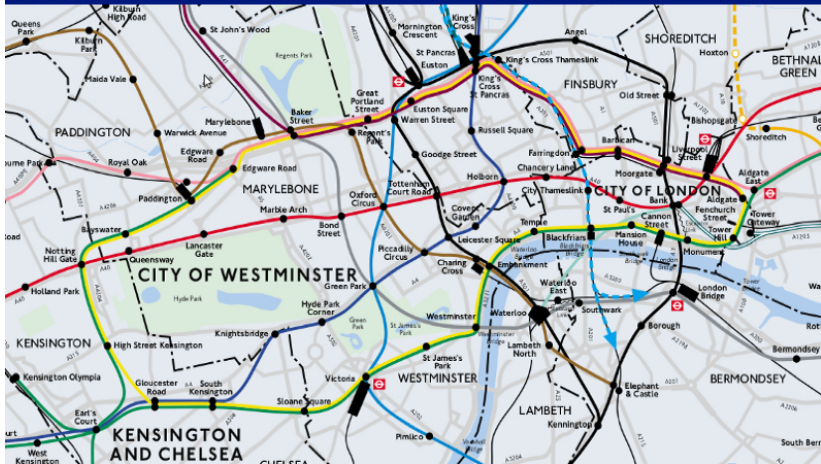


Topology is often called Rubber Sheet Geometry



Topology and the London Underground

Topographical Map



Topology and the London Underground

Topological Map



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Distraction 6A: Slicing a Pizza (Again)



Cut the pizza using only straight cuts.

There should be exactly one piece of pepperoni on each slice of pizza.

Minimum number of cuts?



Abstract Formulation

Last Week's Problem:

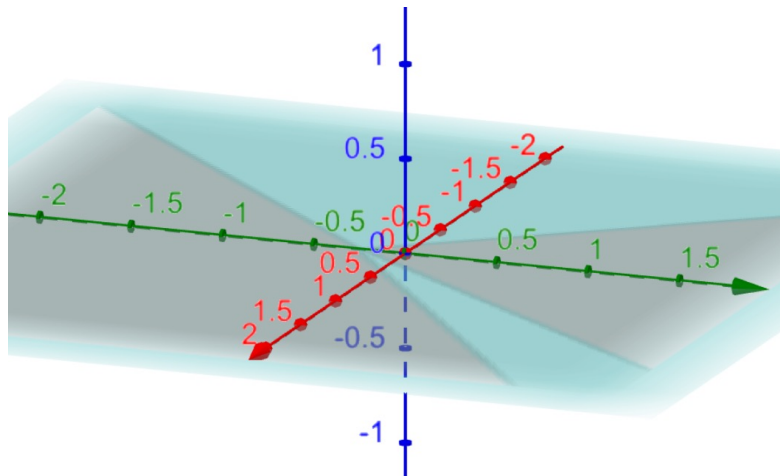
Plane cut by n lines. How many regions are formed?

Cuts	Segments (1D)	Regions (2D)	Solids (3D)
0	1	1	1
1	2	2	2
2	3	4	4
3	4	7	8
4	5	11	15
5	6	16	26
6	7	22	42

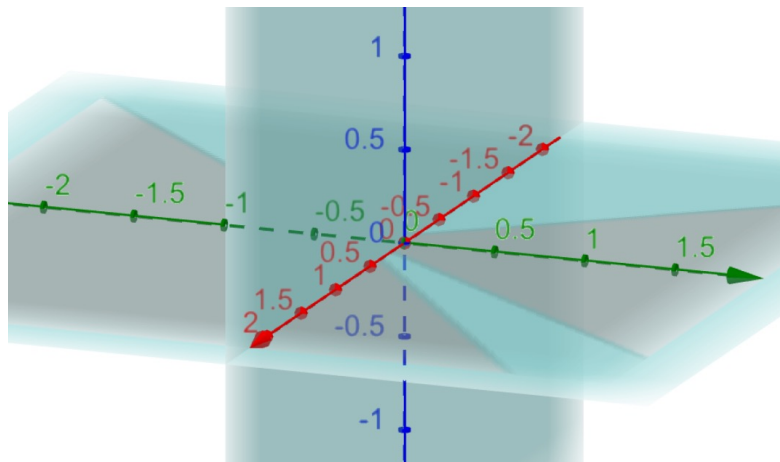
What is the pattern here?



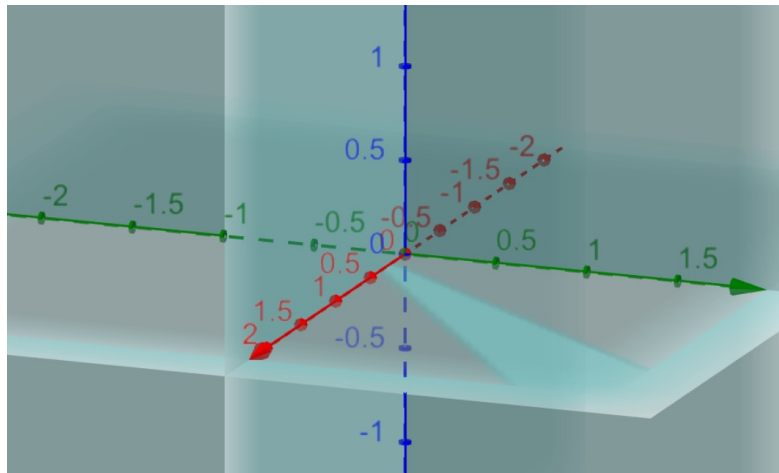
Distraction 6A: Slicing a Pizza



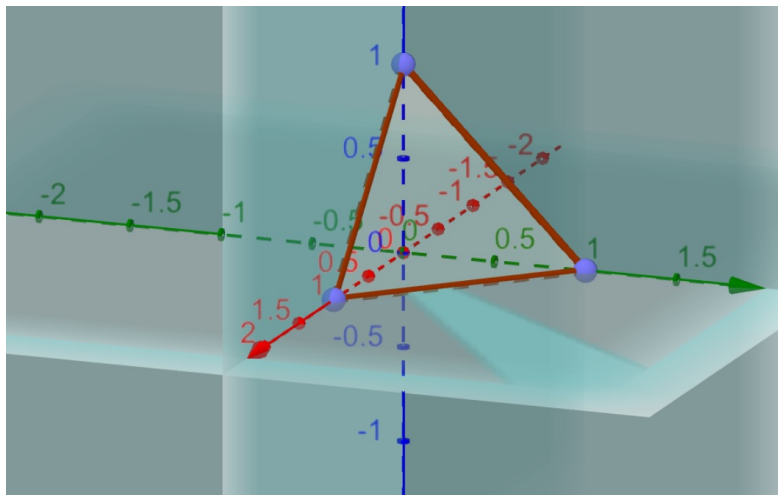
Distraction 6A: Slicing a Pizza



Distraction 6A: Slicing a Pizza



Distraction 6A: Slicing a Pizza



Cutting Lines, Planes and Spaces

Cuts	Segments (1D)	Regions (2D)	Solids (3D)
0	1	1	1
1	2	2	2
2	3	4	4
3	4	7	8
4	5	11	15
5	6	16	26
6	7	22	42

**There is a pattern here.
It is reminiscent of Pascal's Triangle.**



Distraction 6A: Doughnut-Slicing Problem



BRILLIANT

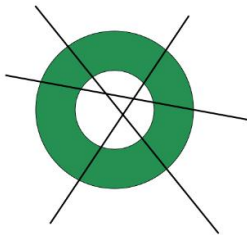
Courses

Practice

Community

Cutting an Annulus

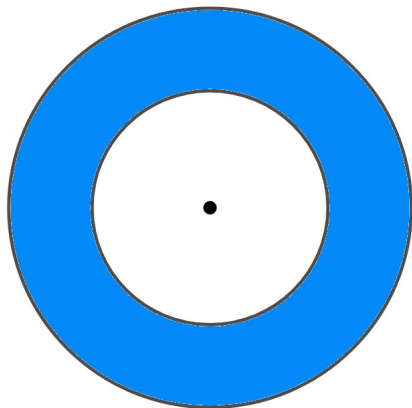
Discrete Mathematics Level 2



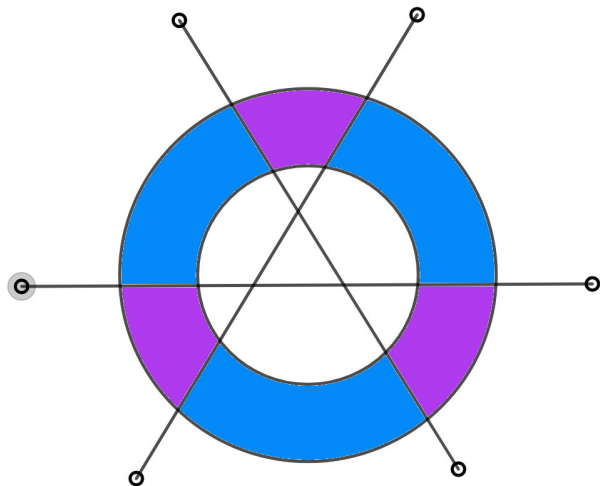
What is the maximum number of pieces into which a ring can be cut by 3 straight lines?
In the image above, the ring is cut into 6 pieces by 3 lines.



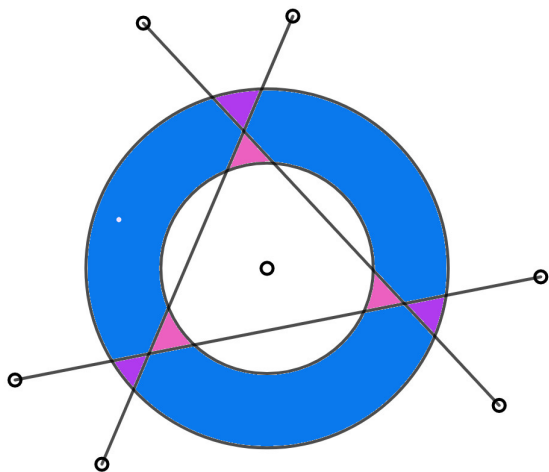
Distraction 6A: Slicing a (Flat) Doughnut



Distraction 6A: Slicing a (Flat) Doughnut



Distraction 6A: Slicing a (Flat) Doughnut



Distraction 6A: Brilliant Website

The screenshot shows the Brilliant website interface. At the top, there is a navigation bar with the Brilliant logo, links for 'Courses', 'Practice', and 'Community', a search bar, and a 'Go Premium' button. Below the navigation bar, the 'Courses' section is displayed. On the left, there is a sidebar with 'Recent' and 'Recommended' sections, listing 'Math', 'Science', and 'Computer Science'. The main content area features two rows of course cards. The 'Recent' row includes 'Ace the AMC' (purple card with a bar chart), 'Group Theory' (teal card with a Rubik's cube), and 'Classical Mechanics' (dark grey card with a rocket). The 'Recommended - Popular in the last month' row includes 'Logic' (purple card with a brain silhouette), 'Computer Science Fundamentals' (red card with a number grid), 'Artificial Neural Networks' (blue card with a brain diagram), and 'Mathematical Fundamentals' (orange card with a clock face and mathematical symbols).

BRILLIANT Courses Practice Community [Go Premium](#)

Courses

Recent

Recommended

Math

Science

Computer Science

Recent

- Ace the AMC**
Guided training for the strategies needed to excel in AMC 10 and 12.
- Group Theory**
Explore groups through symmetries, applications, and problems.
- Classical Mechanics**
Hardcore training for the aspiring physicist.

Recommended - Popular in the last month

- Logic**
Stretch your analytic muscles with knights, knaves, logic gates, and more!
- Computer Science Fundamentals**
The fundamental toolkit for the aspiring computer scientist or programmer.
- Artificial Neural Networks**
A quick dive into a cutting-edge computational method for learning.
- Mathematical Fundamentals**
The essential tools for mastering algebra, probability, and logic.

<https://brilliant.org/>



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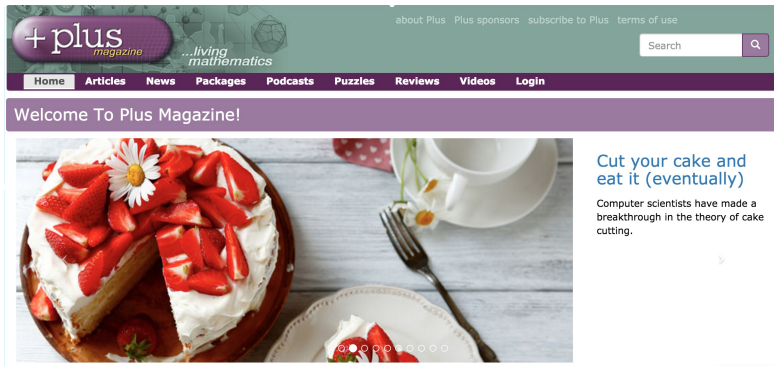
Astronomy II

Distraction 8: Sum by Inspection

Carl Friedrich Gauss



Distraction 7: Plus Magazine



The screenshot shows the homepage of Plus Magazine. At the top left is the logo '+ plus magazine' with the tagline '...living mathematics' and a geometric diagram. To the right are links for 'about Plus', 'Plus sponsors', 'subscribe to Plus', and 'terms of use'. A search bar is located on the right side of the header. Below the header is a navigation menu with links for 'Home', 'Articles', 'News', 'Packages', 'Podcasts', 'Puzzles', 'Reviews', 'Videos', and 'Login'. A purple banner below the menu says 'Welcome To Plus Magazine!'. The main content area features a large image of a strawberry cake with a slice cut out and served on a plate. To the right of the image is the article title 'Cut your cake and eat it (eventually)' and a short paragraph: 'Computer scientists have made a breakthrough in the theory of cake cutting.' Below the image is a series of small circular navigation icons.

PLUS: The Mathematics e-zine
<https://plus.maths.org/>



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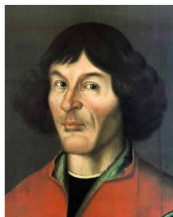
Carl Friedrich Gauss



The Scientific Revolution

INTRODUCTION

This week, we will look at developments in the sixteenth and seventeenth centuries.



Nicolaus Copernicus
1473 – 1543



Tycho Brahe
1546 – 1601



Johannes Kepler
1571 – 1630



Galileo Galilei
1564 – 1642

Figure from mathigon.org



The Heliocentric Model

In 1543, *Nicolaus Copernicus* (1473–1543) published “*On the Revolutions of the Celestial Spheres*”.

He explained that the Sun is at the centre of the universe and that the Earth and planets move around it in circular orbits.

Danish astronomer *Tycho Brahe* (1546–1601) made very accurate observations of the movements of the planets, and developed his own model of the solar system.



Johannes Kepler (1571–1630)

Johannes Kepler (1571–1630) succeeded Brahe as imperial mathematician.

After many years of struggling, Kepler succeeded in formulating his three Laws of Planetary Motion.

Kepler's Laws describe the solar system much as we know it to be true today.



Kepler's Laws

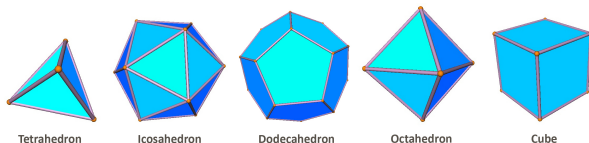
- ▶ **The planets move on elliptical orbits, with the Sun at one of the two foci.**
This explains why the Sun appears larger at some times of the year and smaller at others.
- ▶ **A line joining the planet and the Sun sweeps out equal areas in equal times.**
This means that a planet moves faster when close to the Sun, and slower when further away.
- ▶ **The square of the orbital period is proportional to the cube of the mean radius of the orbit.**
This law allows us to find the orbital time of a planet if we know the size of the orbit.



The *Mysterium Cosmographicum*

There were six known planets in Kepler's time:
Mercury, Venus, Earth, Mars, Jupiter, Saturn.

There are precisely five platonic solids:



This gave Kepler an extraordinary idea!

<https://thatsmaths.com/2016/10/13/>

[\keplers-magnificent-mysterium-cosmographicum/](#)



Galileo Galelii (1564–1630)

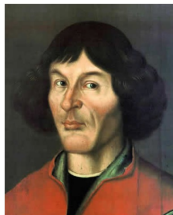
Galileo introduced the *telescope* to astronomy, and made some dramatic discoveries.

He observed the four large moons of Jupiter *revolving around that planet*.

He established the laws of inertia that underlie Newton's dynamical laws.



Four Remarkable Scientists



Nicolaus Copernicus
1473 – 1543



Tycho Brahe
1546 – 1601



Johannes Kepler
1571 – 1630



Galileo Galilei
1564 – 1642

Figure from mathigon.org



Isaac Newton (1642–1727)

In 1687, Isaac Newton published the *Principia Mathematica*. He established the mathematical foundations of dynamics.

Between any two masses there is a force:

$$F = \frac{GMm}{r^2}$$

This is the force of gravity and gravity is what makes the planets move around the Sun.

Newton's Laws imply and explain Kepler's laws.



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Distraction 8: Sum by Inspection

Can you guess the sum of this series:

$$\left(\frac{1}{2}\right)^2 + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{8}\right)^2 + \left(\frac{1}{16}\right)^2 + \dots$$

Proof by Inspection

Look at the figure in two different ways

At each scale, we have three squares the same size, and we keep one of them (black) and omit the others.

So, the area of the shaded squares is $\frac{1}{3}$.

However, it is also given by the series

$$\left(\frac{1}{2}\right)^2 + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{8}\right)^2 + \left(\frac{1}{16}\right)^2 + \dots$$

Therefore we can sum the series:

$$\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \dots = \frac{1}{3}$$



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Carl Friedrich Gauss



Carl Friedrich Gauss (1777–1855)



Carl Friedrich Gauss (1777–1855)

A German mathematician who made profound contributions to many fields of mathematics:

- ▶ Number theory
- ▶ Algebra
- ▶ Statistics
- ▶ Analysis
- ▶ Differential geometry
- ▶ Geodesy & Geophysics
- ▶ Mechanics & Electrostatics
- ▶ Astronomy



One of the greatest mathematicians of all time.



Gauss Outsmarts his Teacher

Gauss was a genius. He was known as

The Prince of Mathematicians.

When very young, Gauss outsmarted his teacher.

I can now reveal a fact unknown to historians:

The teacher got his own back. Ho! ho! ho!



Gauss Outsmarts his Teacher

Gauss's school teacher tasked the class:

- ▶ **Add up all the whole numbers from 1 to 100.**

Gauss solved the problem in a flash.

He wrote the correct answer,

5,050

on his slate and handed it to the teacher.

How did Gauss do it?



First, Gauss wrote the numbers in a row:

1 2 3 ... 98 99 100

Next he wrote them again, *in reverse order*:

1 2 3 ... 98 99 100
100 99 98 ... 3 2 1

Then he added the two rows, column by column:

1	2	3	...	98	99	100
100	99	98	...	3	2	1

101	101	101	...	101	101	101

Clearly, the total for the two rows is 10,100.

But every number from 1 to 100 is counted twice.

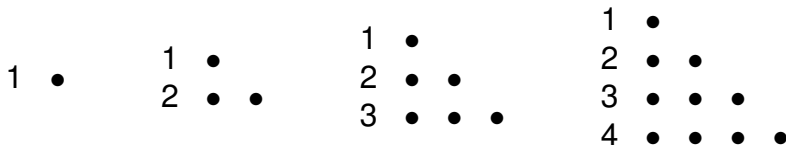
$$\therefore 1 + 2 + 3 + \dots + 98 + 99 + 100 = 5,050$$



Triangular Numbers

Gauss had calculated the 100-th triangular number.

Let us take a geometrical look at the sums of the first few natural numbers:

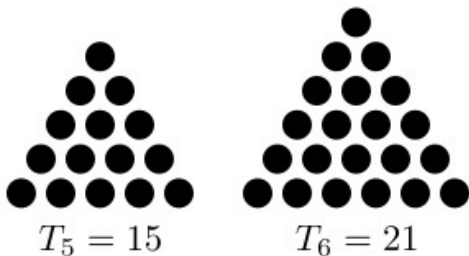
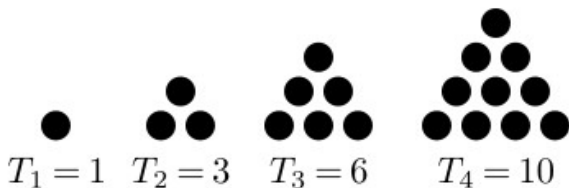


We see that the sums can be arranged as triangles.



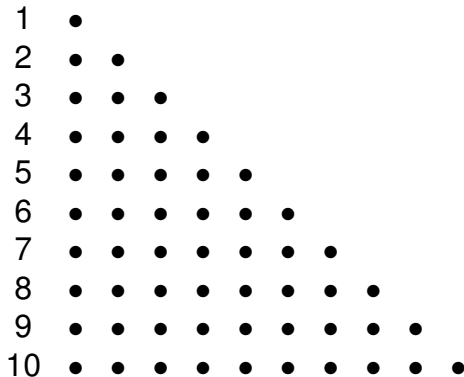
Triangular Numbers

The first few *triangular numbers* are $\{1, 3, 6, 10, 15, 21\}$.



Let's look at the 10th triangular number.

For $n = 10$ the pattern is:



How do we compute its value? Gauss's method!



It is easy to show that the n -th triangular number is

$$T_n = (1 + 2 + 3 + \dots + n) = \frac{1}{2}n(n + 1)$$

We do just as Gauss did, and list the numbers twice:

1	2	3	...	$n - 1$	n
n	$n - 1$	$n - 2$...	2	1
---	---	---	...	---	---
$n + 1$	$n + 1$	$n + 1$...	$n + 1$	$n + 1$

There are n columns, each with total $n + 1$.

So the grand total is $n \times (n + 1)$.

Each number has been counted twice, so

$$T_n = \frac{1}{2}n(n + 1)$$



Let's check this for Gauss's problem of $n = 100$:

$$T_{100} = 1 + 2 + 3 + \dots + 100 = \frac{100 \times 101}{2} = 5,050$$

Gauss's approach was to look at the problem from a new angle.

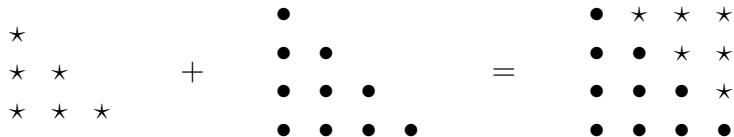
Such *lateral thinking* is very common in mathematics:

Problems that look difficult can sometimes be solved easily when tackled from a different angle.

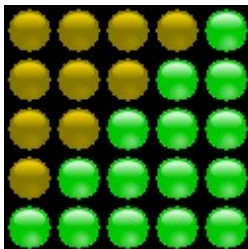


Two Triangles Make a Square

A nice property of *consecutive* triangular numbers:



$$T_3 + T_4 = 6 + 10 = 16 = 4^2$$



Triangular Numbers

We have seen, by means of geometry that the sum of two consecutive triangular numbers is a square.

Now let us prove this algebraically:

$$\begin{aligned}T_n + T_{n+1} &= \frac{1}{2}n(n+1) + \frac{1}{2}(n+1)(n+2) \\ &= \frac{1}{2}(n+1)[n + (n+2)] \\ &= \frac{1}{2}(n+1)[2(n+1)] \\ &= (n+1)^2\end{aligned}$$

The result is *a perfect square*.



Puzzle

**What is the sum of all the numbers
from 1 up to 100 and back down again?**

The answer is in the video coming up now.

A Video from the Museum of Mathematics



VIDEO: Beautiful Maths, available at

<http://momath.org/home/beautifulmath/>

Video by James Tanton



Gauss Outsmarted by his Teacher

The teacher thought that he would have a half-hour of peace and quiet while the pupils grappled with the problem of adding up the first 100 numbers.

He was annoyed when Gauss came up almost immediately with the correct answer 5,050.

So, he said:

“Oh, you zink you are zo zmart!
Zo, multiply ze first 100 numbers.”

EXERCISE: Zink about that!



A Lateral Thinking Puzzle

- ▶ Jill is 23 years younger than her father.
- ▶ What age was she when she was half his age?

Let Jill's age be J . Let her father's age be F .

$$F - J = 23$$

Hint: Be Smart
There is no need for tricky algebra.



Thank you

