Sum-Enchanted Evenings

The Fun and Joy of Mathematics

•

LECTURE 7

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School of Mathematics & Statistics
University College Dublin

Evening Course, UCD, Autumn 2018

Outline

Outline

Meaning and Content of Mathematics

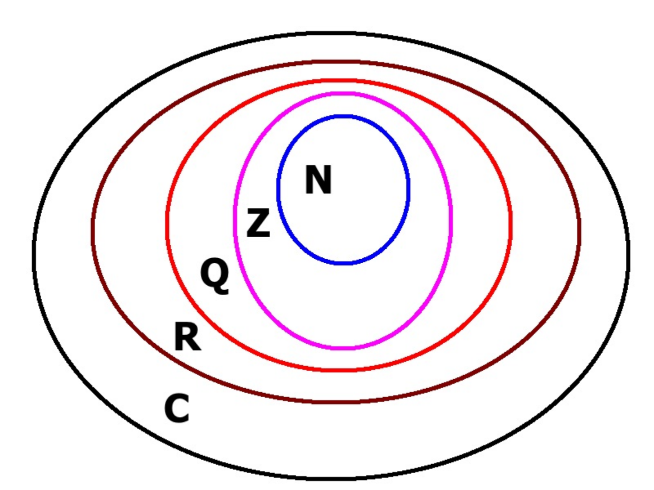
The word Mathematics comes from Greek $\mu\alpha\theta\eta\mu\alpha$ (máthéma), meaning “knowledge” or “study” or “learning”.

It is the study of topics such as

- Quantity (numbers)
- Structure (patterns)
- Space (geometry)
- Change (analysis).

Outline

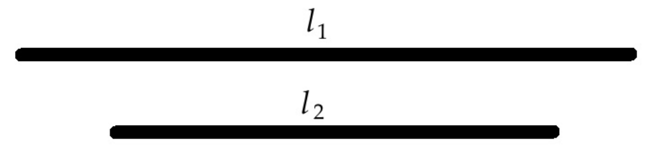
The Hierarchy of Numbers



$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$$

Incommensurability

Suppose we have two line segments



Can we find a unit of measurement such that both lines are a whole number of units?

Can they be co-measured? Are they commensurable?

Are ℓ_1 and ℓ_2 commensurable?

If so, let the unit of measurement be λ.

Then

$$\begin{aligned} \ell_1 &= m\lambda\,, \quad m \in \mathbb{N} \\ \ell_2 &= n\lambda\,, \quad n \in \mathbb{N} \end{aligned}$$

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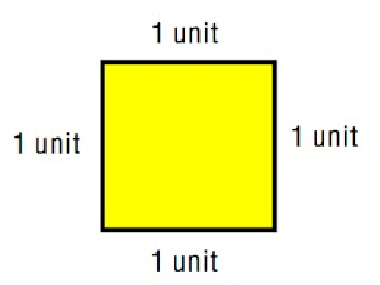
$$\frac{\ell_1}{\ell_2} = \frac{m\lambda}{n\lambda} = \frac{m}{n}$$

If not, then ℓ_1 and ℓ_2 are incommensurable.

Irrational Numbers

If the side of a square is of length 1, then the diagonal has length $\sqrt{2}$ (by the Theorem of Pythagoras).



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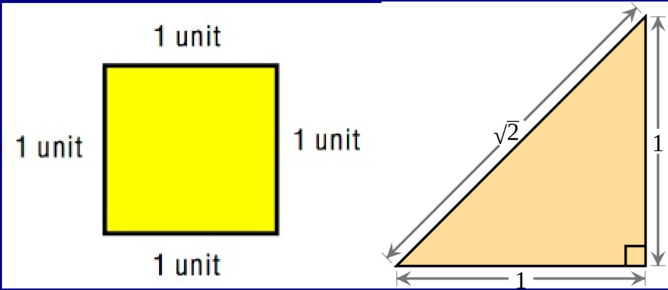

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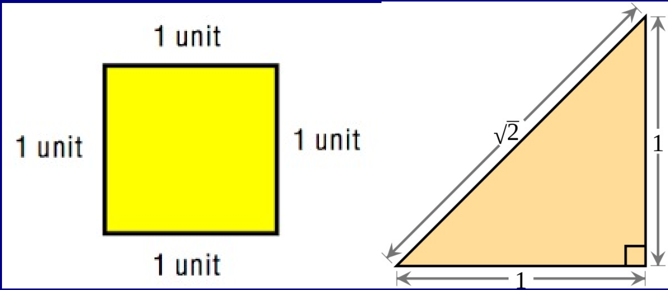

The ratio between the diagonal and the side is:

$$\frac{\text{Diagonal}}{\text{Side Length}} = \sqrt{2}$$

Irrationality of $\sqrt{2}$

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1. Whole numbers
2. Ratios of whole numbers

There were no other numbers.

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For example, suppose $p = 42$ and $q = 30$. Then

$$\frac{p}{q} = \frac{42}{30} = \frac{7 \times 6}{5 \times 6} = \frac{7}{5}$$

Remarks on *Reductio ad Absurdum.*

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Sherlock Holmes:

"How often have I said to you that when you have eliminated the impossible, whatever remains, however improbable, must be the truth?"

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Now square both sides of the equation $\sqrt{2} = p/q$:

$$2 = \frac{p}{q} \times \frac{p}{q} = \frac{p^2}{q^2} \qquad \text{or} \qquad p^2 = 2q^2$$

This means that p^2 is even. Therefore, **p is even.**

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But this means that q^2 is even. So, q is even.

Both p and q are even. This is a contradiction.

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It is not a ratio of whole numbers.

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By *reductio ad absurdum,* $\sqrt{2}$ is irrational.

It is not a ratio of whole numbers.

To the Pythagoreans, $\sqrt{2}$ was not a number.

$\kappa\rho\iota\sigma\eta$ $\quad$ $\kappa\alpha\tau\alpha\sigma\tau\rho$**O**$\phi\eta$**!**

$\sqrt{2}$ and the Development of Mathematics

The discovery of irrational quantities had a dramatic effect on the development of mathematics.

Legend has it that the discoveror of this fact was thrown from a ship and drowned.

The result was that focus now fell on geometry, and arithmetic or number theory was neglected.

The problems were not resolved for many centuries.

Outline

Distraction 6: Slicing a Pizza

Cut the pizza using only straight cuts.

There should be exactly one piece of pepperoni on each slice of pizza.

Minimum number of cuts?

Abstract Formulation

Let us pretend we are pure mathematicians.

Problem:
If the plane is cut by n lines,
how many regions are formed?

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n Lines	k Regions
0	1
1	2
2	?
3	?
4	?

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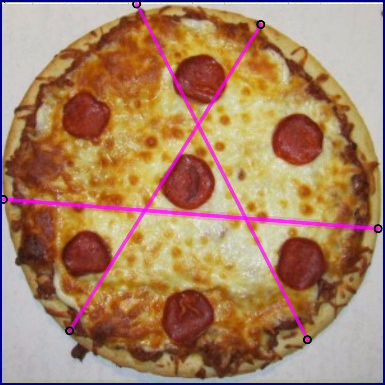


Try This For Fun

Problem:

How many regions are formed by n cuts?

n Lines	k Regions
0	1
1	2
2	?
3	?
4	?
5	?
6	?

Complete this table. Can you find a general formula?

Outline

The Real Numbers

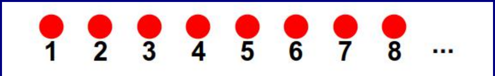
We need to be able to assign a number to a line of any length.

The Pythagoreans found that no number known to them gave the diagonal of a unit square.

It is as if there are gaps in the number system.

We look at the rational numbers and show how to complete them: how to fill in the gaps.

The set $\mathbb{N}$ is infinite, but each element is isolated.



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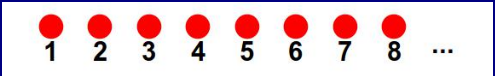

The set $\mathbb{Q}$ is infinite and also dense:
between any two rationals there is another rational.

PROOF: Let $r_1 = p_1/q_1$ and $r_2 = p_2/q_2$ be rationals.

$$\bar{r} = \tfrac{1}{2}(r_1 + r_2) = \frac{1}{2}\left(\frac{p_1}{q_1} + \frac{p_2}{q_2}\right) = \frac{p_1 q_2 + q_1 p_2}{2 q_1 q_2}$$

is another rational between them: $r_1 < \bar{r} < r_2$.

Although $\mathbb{Q}$ is dense, there are gaps.
The line of rationals is discontinuous.

We complete it—filling in the gaps—by defining the limit of any sequence of rationals as a real number.

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WARNING:
We are glossing over a number of fundamental ideas of mathematical analysis:

- What is an infinite sequence?
- What is the limit of a sequence?

To give a particular example, we know that

$$\sqrt{2} = 1.41421356\ldots$$

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In terms of fractions, this is the sequence

$$\left\{\frac{1}{1}, \frac{14}{10}, \frac{141}{100}, \frac{1414}{1000}, \frac{14142}{10000}, \frac{141421}{100000}, \frac{1414213}{1000000}, \ldots\right\}$$

These rational numbers get closer and closer to $\sqrt{2}$.

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EXERCISE:
Construct a sequence in $\mathbb{Q}$ that tends to π.

The Real Number Line

The set of **Real Numbers**, $\mathbb{R}$, contains all the rational numbers in $\mathbb{Q}$ and also all the limits of sequences of rationals [technically, all 'Cauchy sequences'].

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- Every point on the number line corresponds to a real number.
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PHYSICS: There are unknown aspects of the microscopic structure of spacetime! These go beyond our 'Universe of Discourse'.

Now we have the chain of sets:

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We can also consider the prime numbers $\mathbb{P}$. They are subset of the natural numbers, so

$$\mathbb{P} \subset \mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$$

Outline

The Greek Alphabet, Part 5

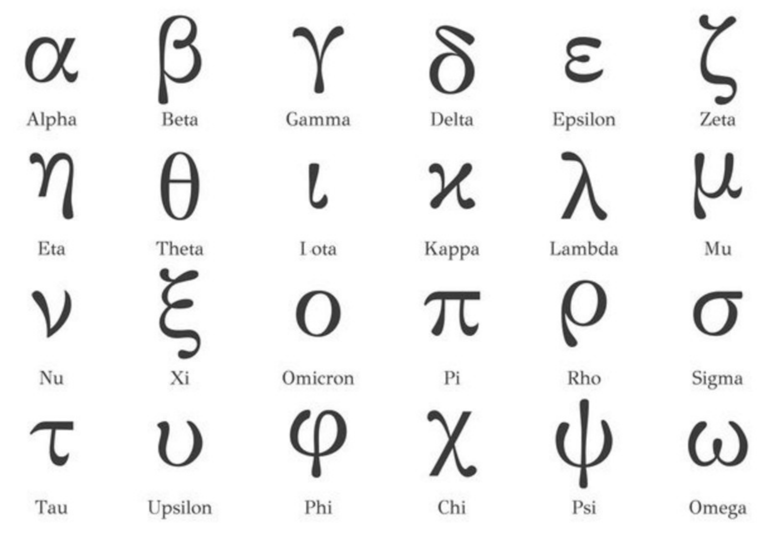


Figure : 24 beautiful letters

The Full Alphabet

α	β	γ	δ	ϵ	ζ
A	B	Γ	Δ	E	Z
η	θ	ι	κ	λ	μ
H	Θ	I	K	Λ	M
ν	ξ	o	π	ρ	σ
N	Ξ	O	Π	P	Σ
τ	υ	ϕ	χ	ψ	ω
T	Υ	Φ	X	Ψ	Ω

The Full Monty

Letter	Name	Sound	
		Ancient[5]	Modern[6]
Α α	alpha, άλφα	[a] [aː]	[a]
Β β	beta, βήτα	[b]	[v]
Γ γ	gamma, γάμμα	[g], [ŋ][7]	[ɣ] ~ [ʝ], [ŋ][8] ~ [ɲ][9]
Δ δ	delta, δέλτα	[d]	[ð]
Ε ε	epsilon, έψιλον	[e]	[e]
Ζ ζ	zeta, ζήτα	[zd][A]	[z]
Η η	eta, ήτα	[ɛː]	[i]
Θ θ	theta, θήτα	[t^h]	[θ]
Ι ι	iota, ιώτα	[i] [iː]	[i], [ʝ],[10] [ɲ][11]
Κ κ	kappa, κάππα	[k]	[k] ~ [c]
Λ λ	lambda, λάμδα	[l]	[l]
Μ μ	mu, μυ	[m]	[m]

Letter	Name	Sound	
		Ancient[5]	Modern[6]
Ν ν	nu, νυ	[n]	[n]
Ξ ξ	xi, ξι	[ks]	[ks]
Ο ο	omicron, όμικρον	[o]	[o]
Π π	pi, πι	[p]	[p]
Ρ ρ	rho, ρώ	[r]	[r]
Σ σ/ς[13]	sigma, σίγμα	[s]	[s]
Τ τ	tau, ταυ	[t]	[t]
Υ υ	upsilon, ύψιλον	[y] [yː]	[i]
Φ φ	phi, φι	[p^h]	[f]
Χ χ	chi, χι	[k^h]	[x] ~ [ç]
Ψ ψ	psi, ψι	[ps]	[ps]
Ω ω	omega, ωμέγα	[ɔː]	[o]

Figure : Wikipedia: “Greek Alphabet”

A Few Greek Words With Large Letters

’ΕΛΛΑΣ
ΠΛΑΤΟΝ
ΑΚΡΟΠΟΛΙΣ

ΑΡΙΣΤΟΤΕΛΗΣ
ΠΥΘΑΓΌΡΑΣ
ΣΟΦΟΚΛΗΣ

A Few Greek Words With Large Letters

’ΕΛΛΑΣ	HELLAS: ΈΛΛΑΣ
ΠΛΑΤΟΝ	PLATO: ΠΛΑΤΟΝ
ΑΚΡΟΠΟΛΙΣ	ACROPOLIS: ΑΚΡΟΠΟΛΙΣ
ΑΡΙΣΤΟΤΕΛΗΣ	ARISTOTLE: ΑΡΙΣΤΟΤΕΛΗΣ
ΠϒΘΑΓΌΡΑΣ	PYTHAGORAS: ΠϒΘΑΓΌΡΑΣ
ΣΟΦΟΚΛΗΣ	SOPHOCLES: ΣΟΦΟΚΛΗΣ

Robinson’s Anemometer on East Pier

Figure : Inscription on Church in Sean McDermott Street

I asked Cosetta Cadau, Department of Classics Trinity College Dublin about this inscription.

Here is how she replied:

I asked Cosetta Cadau, Department of Classics Trinity College Dublin about this inscription.

Here is how she replied:

The text is not complete (the last word is cut), but what I can read is

ΜΟΝΩ ΣΟΦΩ ΘΕΩ

ΣΩΤΗΡΙ ΗΜΩΝ

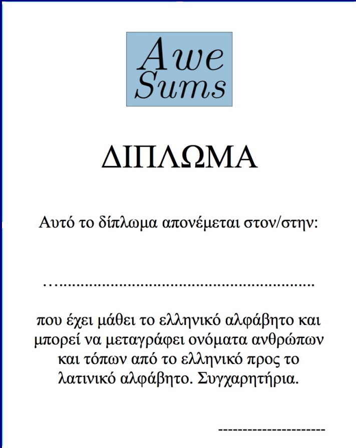
which can be translated as

To God, Our Only Saviour

End of Greek 105

Collect Your Diploma

Your Diploma


Awe Sums

ΔΙΠΛΩΜΑ

Αυτό το δίπλωμα απονέμεται στον/στην:

……………………………………………………

που έχει μάθει το ελληνικό αλφάβητο και μπορεί να μεταγράφει ονόματα ανθρώπων και τόπων από το ελληνικό προς το λατινικό αλφάβητο. Συγχαρητήρια.

This diploma is awarded to (=== NAME ===) who has learned the Greek alphabet and who can transliterate names of people and places from the Greek to the Roman alphabet.

Congratulations.

Outline

Pascal’s Triangle

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1

Combinatorial Symbol

$$\binom{n}{r}$$ "***n*** **choose** ***r***"

This symbol represents the number of combinations of *r* objects selected from a set of *n* objects.

Pascal's Triangle: Combinations

$$\binom{0}{0}$$

$$\binom{1}{0} \quad \binom{1}{1}$$

$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$

$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$

$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$

Pascal's Triangle

Pascal's triangle is a triangular array of the binomial coefficients.

It is named after French mathematician Blaise Pascal.

It was studied centuries before him in:

- India (Pingala, C2BC)
- Persia (Omar Khayyam, C11AD)
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Draw Pascal's triangle on the board.

Pascal's Triangle

The rows of Pascal's triangle are numbered starting with row n = 0 at the top (0-th row).

The entries in each row are numbered from the left beginning with k = 0.

The triangle is easily constructed:

- A single entry 1 in row 0.
- Add numbers above for each new row.

The entry in the nth row and k-th column of Pascal's triangle is denoted $\binom{n}{k}$.

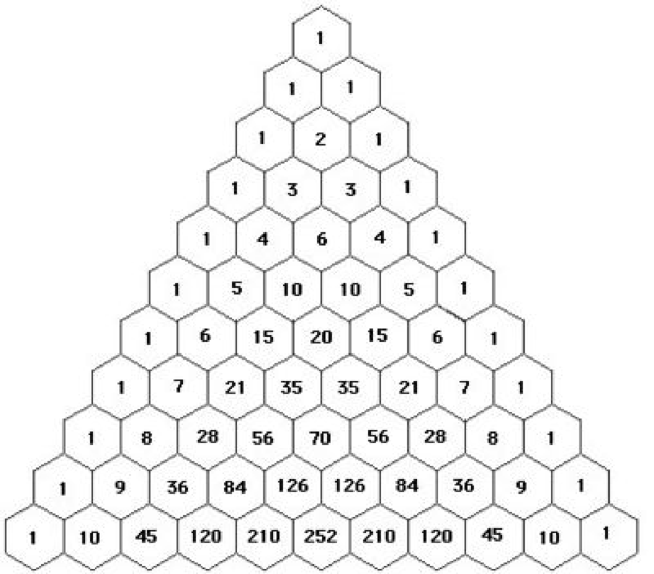
The entry in the topmost row is $\binom{0}{0} = 1$.

Pascal's Identity

The construction of the triangle may be written:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

This relationship is known as Pascal's Identity.


1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1

Pascal’s Triangle & Fibonacci Numbers.

1							
1	1						
1	2	1					
1	3	3	1				
1	4	6	4	1			
1	5	10	10	5	1		
1	6	15	20	15	6	1	
1	7	21	35	35	21	7	1

Figure : Pascal’s Triangle and Fibonacci Numbers

Where are the Fibonacci Numbers hiding here?

Sierpinski's Gasket

Sierpinski Gasket: usual construction by trisection.

Sierpinski's Gasket is constructed by starting with an equilateral triangle, and successively removing the central triangle at each scale.

Sierpinski's Gasket at Stage 6

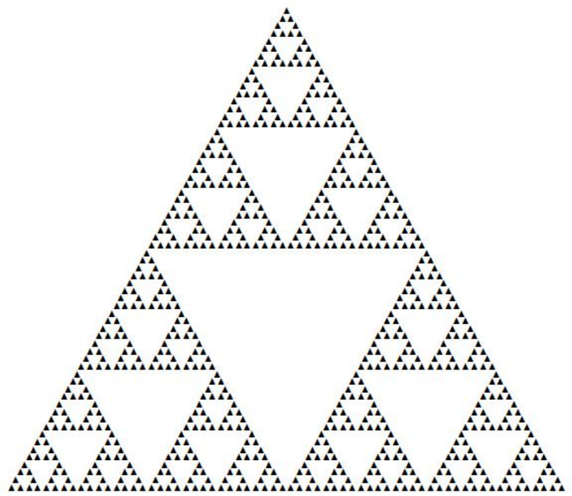


Figure : Result after 6 subdivisions

Sierpinski’s Gasket in Pascal’s Triangle

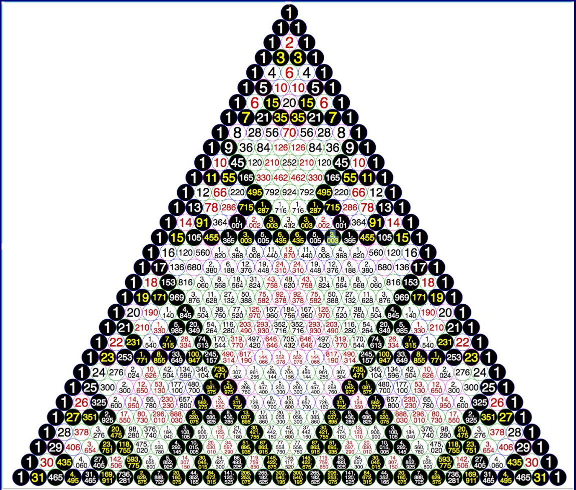


Figure : Odd numbers are in black

Remember Walking in Manhattan?

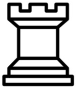
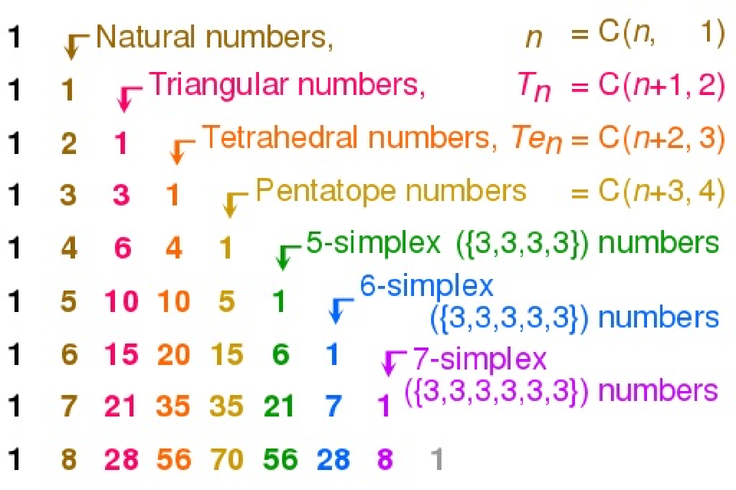
	1	1	1
1	2	3	4
1	3	6	10
1	4	10	20

Figure : Number of routes for a rook in chess.

Geometric Numbers in Pascal's Triangle



Outline

Numerical Weather Prediction

Outline of a talk on NWP
given at UCC, March 2018.

`~/Dropbox/TALKS/NWP-UCC/NWP-UCC.pdf`

`https://maths.ucd.ie/~plynch/Talks/`

Thank you