

Sum-Enchanted Evenings

The Fun and Joy of Mathematics



LECTURE 7

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**School of Mathematics & Statistics
University College Dublin**

Evening Course, UCD, Autumn 2018



Outline

Introduction

Irrational Numbers

Distraction 6: Slicing a Pizza

The Real Number Line

Greek 5

Pascal's Triangle

Numerical Weather Prediction



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Meaning and Content of Mathematics

The word **Mathematics** comes from Greek *μαθημα* (*máthéma*), meaning “knowledge” or “study” or “learning”.

It is the study of topics such as

- ▶ Quantity (numbers)
- ▶ Structure (patterns)
- ▶ Space (geometry)
- ▶ Change (analysis).



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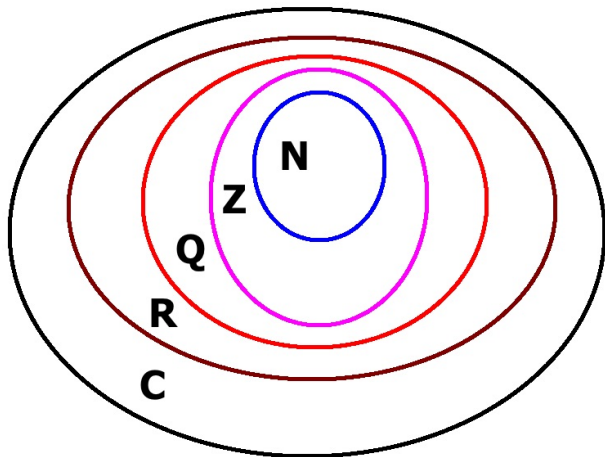
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The Hierarchy of Numbers

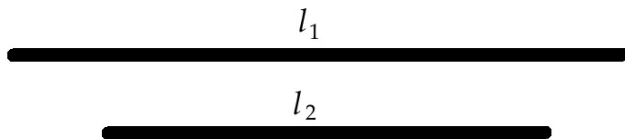


$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$



Incommensurability

Suppose we have two line segments



Can we find a unit of measurement such that *both lines are a whole number of units?*

Can they be co-measured? Are they commensurable?



Are l_1 and l_2 commensurable?

If so, let the unit of measurement be λ .

Then

$$l_1 = m\lambda, \quad m \in \mathbb{N}$$

$$l_2 = n\lambda, \quad n \in \mathbb{N}$$

Therefore

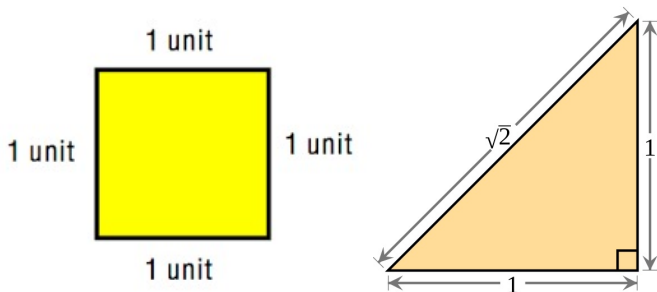
$$\frac{l_1}{l_2} = \frac{m\lambda}{n\lambda} = \frac{m}{n}$$

If not, then l_1 and l_2 are incommensurable.



Irrational Numbers

If the side of a square is of length 1, then the diagonal has length $\sqrt{2}$ (by the Theorem of Pythagoras).



The ratio between the diagonal and the side is:

$$\frac{\text{Diagonal}}{\text{Side Length}} = \sqrt{2}$$



Irrationality of $\sqrt{2}$

For the Pythagoreans, numbers were of two types:

1. *Whole numbers*
2. *Ratios of whole numbers*

There were no other numbers.

Let's suppose that $\sqrt{2}$ is a ratio of whole numbers:

$$\sqrt{2} = \frac{p}{q}$$

We can suppose that p and q have no common factors. Otherwise, we just cancel them out.

For example, suppose $p = 42$ and $q = 30$. Then

$$\frac{p}{q} = \frac{42}{30} = \frac{7 \times 6}{5 \times 6} = \frac{7}{5}$$



Remarks on *Reductio ad Absurdum*.

Sherlock Holmes:

“How often have I said to you that when you have eliminated the impossible, whatever remains, however improbable, must be the truth?”

The Sign of the Four (1890)

The Sign of the Four (1890)



We say that p and q are relatively prime if they have no common factors.

In particular, p and q cannot both be even numbers.

Now square both sides of the equation $\sqrt{2} = p/q$:

$$2 = \frac{p}{q} \times \frac{p}{q} = \frac{p^2}{q^2} \quad \text{or} \quad p^2 = 2q^2$$

This means that p^2 is even. Therefore, p is even.

Let $p = 2r$ where r is another whole number. Then

$$p^2 = (2r)^2 = 4r^2 = 2q^2 \quad \text{or} \quad 2r^2 = q^2$$

But this means that q^2 is even. So, q is even.



Both p and q are even. This is a contradiction.

The supposition was that $\sqrt{2}$ is a ratio of two integers that have no common factors:

$$\sqrt{2} = \frac{p}{q}$$

This assumption has led to a contradiction.

By *reductio ad absurdum*, $\sqrt{2}$ is irrational.

It is *not a ratio of whole numbers*.

To the Pythagoreans, $\sqrt{2}$ was not a number.

κρῖση καταστροφὴ!



$\sqrt{2}$ and the Development of Mathematics

The discovery of irrational quantities had a dramatic effect on the development of mathematics.

Legend has it that the discoveror of this fact was thrown from a ship and drowned.

The result was that focus now fell on geometry, and arithmetic or number theory was neglected.

The problems were not resolved for many centuries.



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Distraction 6: Slicing a Pizza



Cut the pizza using only straight cuts.

There should be exactly one piece of pepperoni on each slice of pizza.

Minimum number of cuts?



Abstract Formulation

Let us pretend we are pure mathematicians.

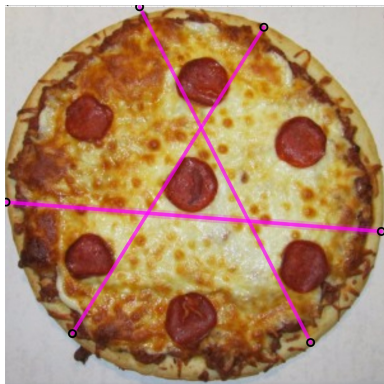
Problem:

If the plane is cut by n lines,
how many regions are formed?

n Lines	k Regions
0	1
1	2
2	?
3	?
4	?



Distraction 6: Slicing a Pizza



Try This For Fun

Problem:

How many regions are formed by n cuts?

n Lines	k Regions
0	1
1	2
2	?
3	?
4	?
5	?
6	?

Complete this table. Can you find a general formula?



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The Real Numbers

We need to be able to assign a number to a line of any length.

The Pythagoreans found that no number known to them gave the diagonal of a unit square.

It is as if there are gaps in the number system.

We look at the rational numbers and show how to complete them: how to fill in the gaps.



The set \mathbb{N} is infinite, but each element is isolated.



The set \mathbb{Q} is infinite and also dense:
between any two rationals there is another rational.

PROOF: Let $r_1 = p_1/q_1$ and $r_2 = p_2/q_2$ be rationals.

$$\bar{r} = \frac{1}{2}(r_1 + r_2) = \frac{1}{2} \left(\frac{p_1}{q_1} + \frac{p_2}{q_2} \right) = \frac{p_1 q_2 + q_1 p_2}{2q_1 q_2}$$

is another rational between them: $r_1 < \bar{r} < r_2$.





Although \mathbb{Q} is dense, there are gaps.
The line of rationals is discontinuous.

We complete it—filling in the gaps—by *defining* the limit of any sequence of rationals as a real number.

WARNING:

We are glossing over a number of fundamental ideas of mathematical analysis:

- ▶ What is an infinite sequence?
- ▶ What is the limit of a sequence?



To give a particular example, we know that

$$\sqrt{2} = 1.41421356\dots$$

We construct a sequence of rational numbers

$$\{1, 1.4, 1.41, 1.414, 1.4142, 1.41421, 1.414213, \dots\}$$

In terms of fractions, this is the sequence

$$\left\{ \frac{1}{1}, \frac{14}{10}, \frac{141}{100}, \frac{1414}{1000}, \frac{14142}{10000}, \frac{141421}{100000}, \frac{1414213}{1000000}, \dots \right\}$$

These rational numbers get *closer and closer to* $\sqrt{2}$.

EXERCISE:

Construct a sequence in \mathbb{Q} that tends to π .



The Real Number Line

The set of Real Numbers, \mathbb{R} , contains all the rational numbers in \mathbb{Q} and also all the limits of sequences of rationals [technically, all ‘Cauchy sequences’].

We may assume that

- ▶ Every point on the number line corresponds to a real number.
- ▶ Every real number corresponds to a point on the number line.

PHYSICS: There are unknown aspects of the microscopic structure of spacetime!
These go beyond our ‘Universe of Discourse’.



Now we have the chain of sets:

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$$

**We can also consider the prime numbers \mathbb{P} .
They are subset of the natural numbers, so**

$$\mathbb{P} \subset \mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$$



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The Greek Alphabet, Part 5

α	β	γ	δ	ϵ	ζ
Alpha	Beta	Gamma	Delta	Epsilon	Zeta
η	θ	ι	κ	λ	μ
Eta	Theta	Iota	Kappa	Lambda	Mu
ν	ξ	\omicron	π	ρ	σ
Nu	Xi	Omicron	Pi	Rho	Sigma
τ	υ	ϕ	χ	ψ	ω
Tau	Upsilon	Phi	Chi	Psi	Omega

Figure : 24 beautiful letters



The Full Alphabet

α	β	γ	δ	ϵ	ζ
A	B	Γ	Δ	E	Z
η	θ	ι	κ	λ	μ
H	Θ	I	K	Λ	M
ν	ξ	O	π	ρ	σ
N	Ξ	O	Π	P	Σ
τ	υ	ϕ	χ	ψ	ω
T	Υ	Φ	X	Ψ	Ω



The Full Monty

Letter	Name	Sound	
		Ancient ^[5]	Modern ^[6]
Α α	alpha, άλφα	[a] [a:]	[a]
Β β	beta, βήτα	[b]	[v]
Γ γ	gamma, γάμμα	[g], [ŋ] ^[7]	[ɣ] ~ [j], [ŋ] ^[8] ~ [ŋ] ^[9]
Δ δ	delta, δέλτα	[d]	[ð]
Ε ε	epsilon, έψιλον	[e]	[e]
Ζ ζ	zeta, ζήτα	[zd] ^A	[z]
Η η	eta, ήτα	[ɛ:]	[i]
Θ θ	theta, θήτα	[tʰ]	[θ]
Ι ι	iota, ιώτα	[i] [i:]	[i], [j], ^[10] [ɪ] ^[11]
Κ κ	kappa, κάππα	[k]	[k] ~ [c]
Λ λ	lambda, λάμδα	[l]	[l]
Μ μ	mu, μυ	[m]	[m]

Letter	Name	Sound	
		Ancient ^[5]	Modern ^[6]
Ν ν	nu, νυ	[n]	[n]
Ξ ξ	xi, ξι	[ks]	[ks]
Ο ο	omicron, όμικρον	[o]	[o]
Π π	pi, πι	[p]	[p]
Ρ ρ	rho, ρώ	[r]	[r]
Σ σ/ς ^[13]	sigma, σίγμα	[s]	[s]
Τ τ	tau, ταυ	[t]	[t]
Υ υ	upsilon, ύψιλον	[y] [y:]	[i]
Φ φ	phi, φι	[pʰ]	[f]
Χ χ	chi, χι	[kʰ]	[x] ~ [ç]
Ψ ψ	psi, ψι	[ps]	[ps]
Ω ω	omega, ωμέγα	[ɔ:]	[o]

Figure : Wikipedia: “Greek Alphabet”



A Few Greek Words With Large Letters

ἙΛΛΑΣ
ΠΛΑΤΟΝ
ΑΚΡΟΠΟΛΙΣ

HELLAS: ἙΛΛΑΣ
PLATO: ΠΛΑΤΟΝ
ACROPOLIS: ΑΚΡΟΠΟΛΙΣ

ΑΡΙΣΤΟΤΕΛΗΣ
ΠΥΘΑΓΟΡΑΣ
ΣΟΦΟΚΛΗΣ

ARISTOTLE: ΑΡΙΣΤΟΤΕΛΗΣ
PYTHAGORAS: ΠΥΘΑΓΟΡΑΣ
SOPHOCLES: ΣΟΦΟΚΛΗΣ



Robinson's Anemometer on East Pier





Figure : Inscription on Church in Sean McDermott Street



I asked *Cosetta Cadau*, Department of Classics
Trinity College Dublin about this inscription.

Here is how she replied:



*The text is not complete
(the last word is cut), but
what I can read is*

ΜΟΝΩ ΣΟΦΩ ΘΕΩ

ΣΩΤΗΡΙ ΗΜΩΝ

*which can be translated as
To God, Our Only Saviour*



End of Greek 105

Collect Your Diploma



Your Diploma



ΔΙΠΛΩΜΑ

Αυτό το δίπλωμα απονέμεται στον/στην:

.....

που έχει μάθει το ελληνικό αλφάβητο και μπορεί να μεταγράφει ονόματα ανθρώπων και τόπων από το ελληνικό προς το λατινικό αλφάβητο. Συγχαρητήρια.

**This diploma is awarded to
(=== NAME ===)
who has learned the Greek
alphabet and who can
transliterate names of
people and places
from the Greek to the
Roman alphabet.**

Congratulations.



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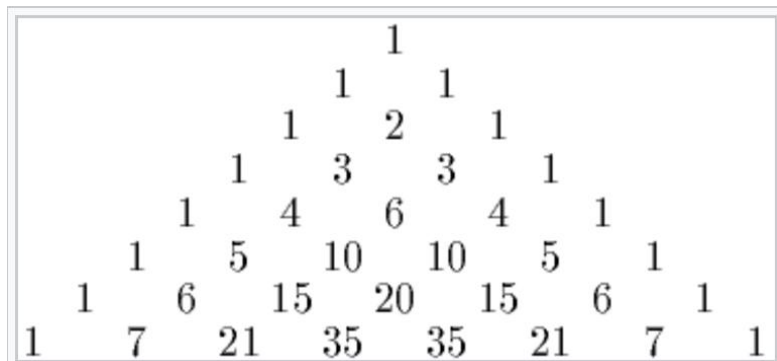
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Pascal's Triangle

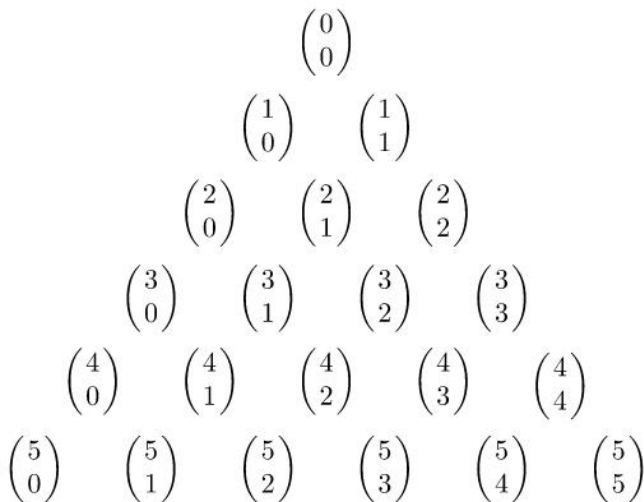


Combinatorial Symbol

$$\binom{n}{r} \quad \text{“}n \text{ choose } r\text{”}$$

This symbol represents the number of combinations of r objects selected from a set of n objects.

Pascal's Triangle: Combinations



Pascal's Triangle

Pascal's triangle is a triangular array of the binomial coefficients.

It is named after French mathematician *Blaise Pascal*.

It was studied centuries before him in:

- ▶ India (Pingala, C2BC)
- ▶ Persia (Omar Khayyam, C11AD)
- ▶ China (Yang Hui, C13AD).

Pascal's *Traité du triangle arithmétique* (Treatise on Arithmetical Triangle) was published in 1665.

Draw Pascal's triangle on the board.



Pascal's Triangle

The rows of Pascal's triangle are numbered starting with row $n = 0$ at the top (0-th row).

The entries in each row are numbered from the left beginning with $k = 0$.

The triangle is easily constructed:

- ▶ A single entry 1 in row 0.
- ▶ Add numbers above for each new row.

The entry in the n th row and k -th column of Pascal's triangle is denoted $\binom{n}{k}$.

The entry in the topmost row is $\binom{0}{0} = 1$.

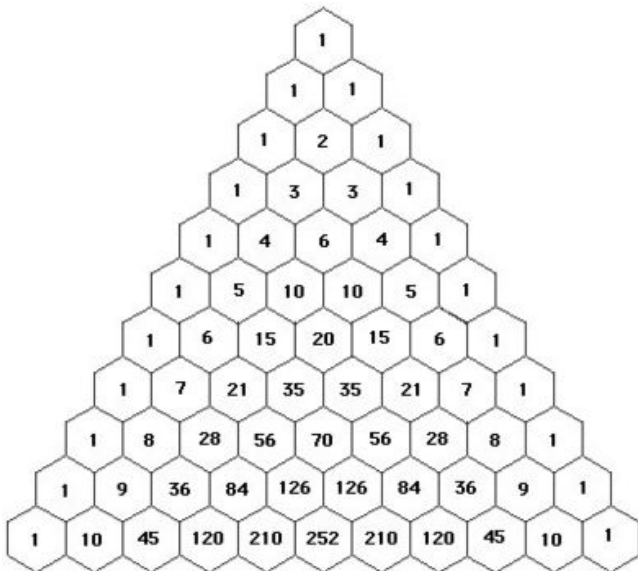


Pascal's Identity

The construction of the triangle may be written:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

This relationship is known as **Pascal's Identity**.



Pascal's Triangle & Fibonacci Numbers.

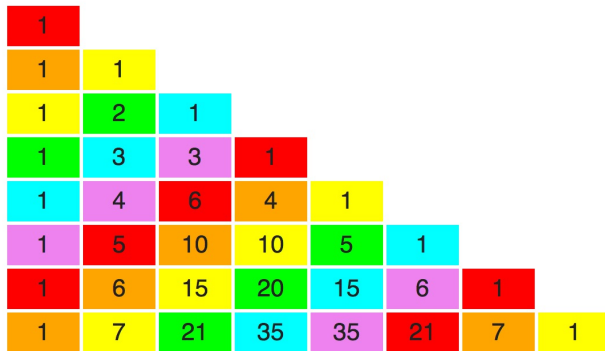
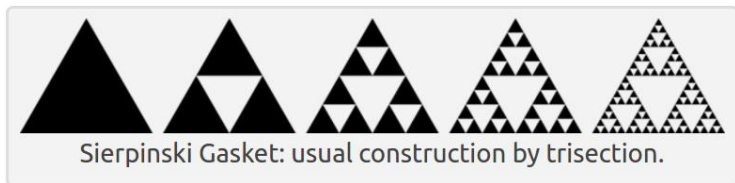


Figure : Pascal's Triangle and Fibonacci Numbers

Where are the Fibonacci Numbers hiding here?



Sierpinski's Gasket



Sierpinski's Gasket is constructed by starting with an equilateral triangle, and successively removing the central triangle at each scale.



Sierpinski's Gasket at Stage 6

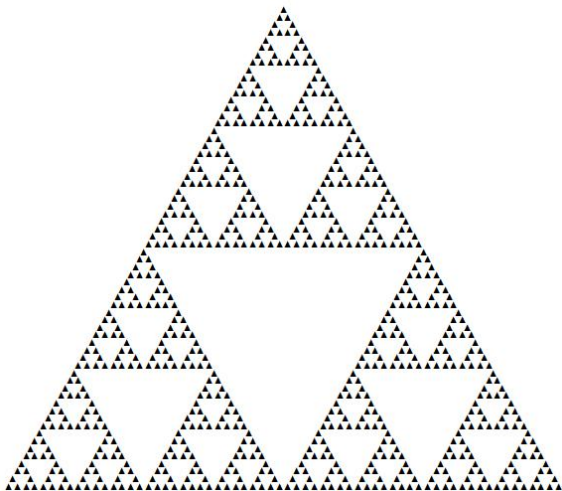


Figure : Result after 6 subdivisions



Sierpinski's Gasket in Pascal's Triangle

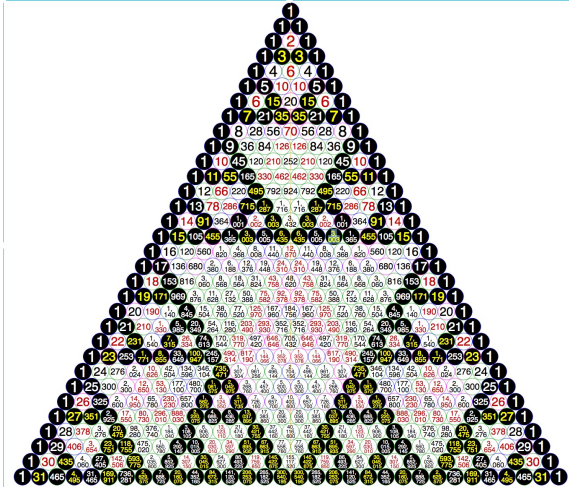


Figure : Odd numbers are in black



Remember Walking in Manhattan?


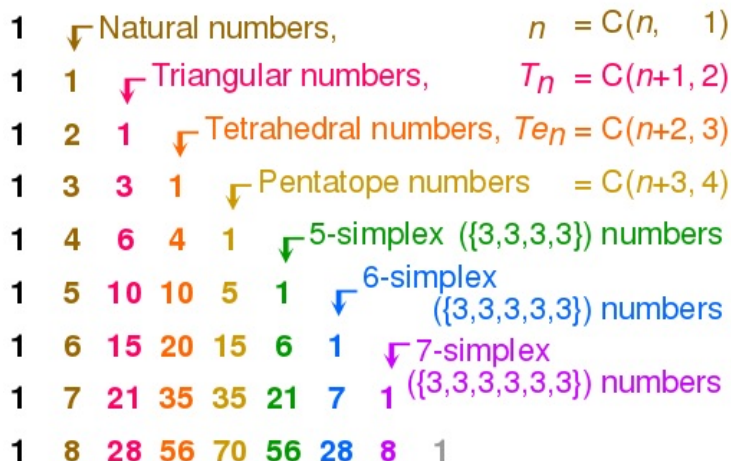
	1	1	1
1	2	3	4
1	3	6	10
1	4	10	20

Figure : Number of routes for a rook in chess.



Geometric Numbers in Pascal's Triangle



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**Outline of a talk on NWP
given at UCC, March 2018.**

`~/Dropbox/TALKS/NWP-UCC/NWP-UCC.pdf`

`https://maths.ucd.ie/~plynch/Talks/`



Thank you

