Sum-Enchanted Evenings

The Fun and Joy of Mathematics

•

LECTURE 5

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Evening Course, UCD, Autumn 2018

Outline

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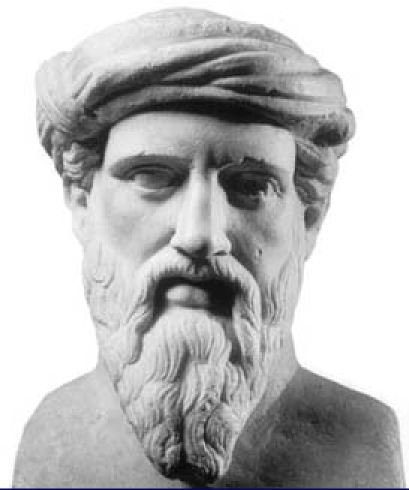
Meaning and Content of Mathematics

The word Mathematics comes from Greek $\mu\alpha\theta\eta\mu\alpha$ (máthéma), meaning “knowledge” or “study” or “learning”.

It is the study of topics such as

- Quantity (numbers)
- Structure (patterns)
- Space (geometry)
- Change (analysis).

Outline



The Thallasic Age

The period from 800 BC to AD 800.

$\Theta\alpha\lambda\alpha\sigma\sigma\alpha$ — the Sea.

The Thallasic Age

The period from 800 BC to AD 800.

$\Theta\alpha\lambda\alpha\sigma\sigma\alpha$ — **the Sea.**

- The first Olympic Games in 776 BC
- Homer and Hesiod lived around 700 BC
- Greek mathematics began to thrive
- First two major figures: Thales and Pythagoras.

Pythagoras (c. 570–495 BC)

Pythagoras was

- Born on the island of Samos (off Turkey).
- Philosopher, mystic, prophet and religious leader.
- Contemporary with Confucius and Lao-Tzu.

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Words philosophy (love of learning) and mathematics (that which is learned) attributed to Pythagoras.

May have been first person to imagine that natural phenomena can be understood through mathematics.

Pythagoras (c. 570–495 BC)

- No contemporary documents
- Myth, legend and tradition
- Second or third hand accounts
  often written centuries later
- Aristotle's biography no longer extant.

Hardly any statement about Pythagoras uncontested.

Difficult to separate history from myth and legend.

Pythagoras (c. 570–495 BC)

- Travelled to Egypt, Babylon and perhaps India
- Mathematics, astronomy and religious lore
- Theorem on right-angled triangles
- Result known to Babylonians 1000 years earlier
- No record of a proof by Pythagoras survives.

The Pythagoreans

Around 530 BC Pythagoras moved to Croton in Magna Graecia (now Southern Italy).

He established an organization or school (philosophical / religious / political).

Both men and women were members of “The Pythagoreans”

Adherents were very secretive:
Bound by an oath of allegiance

Led lives of temperance; observed strict moral codes.

Pythagorean Women

“Women were given equal opportunity to study as Pythagoreans, and learned practical domestic skills in addition to philosophy.

“Women were held to be different from men, sometimes in positive ways.

“The priestess, philosopher and mathematician Themistoclea is regarded as Pythagoras’ teacher; Theano, Damo and Melissa as female disciples.”

From the Wikipedia article: The Pythagoreans.

Pythagorean Quotes

- “I was Euphorbus at the siege of Troy.”
- “In anger, refrain from both speech and action.”
- “Educate the children and it won’t
  be necessary to punish the men.”
- “Abstain from beans!”

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- "Abstain from beans!"

- "There is geometry in the humming of the strings, There is music in the spacing of the spheres."
- "Number rules the universe."

Harmony & Discord

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Stringed instruments produce harmonious sounds when string lengths are ratios of small numbers.

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the principles of musical harmony.

Stringed instruments produce harmonious sounds
when string lengths are ratios of small numbers.

Extended this idea to the heavens: planets emit
sounds according to their speed of movement

Concept of the "harmony of the spheres".

Johannes Kepler: Harmonices Mundi

All is Number

The motto of the Pythagoreans: *All is Number.*

All natural phenomena in the universe can be expressed using whole numbers or ratios of them.

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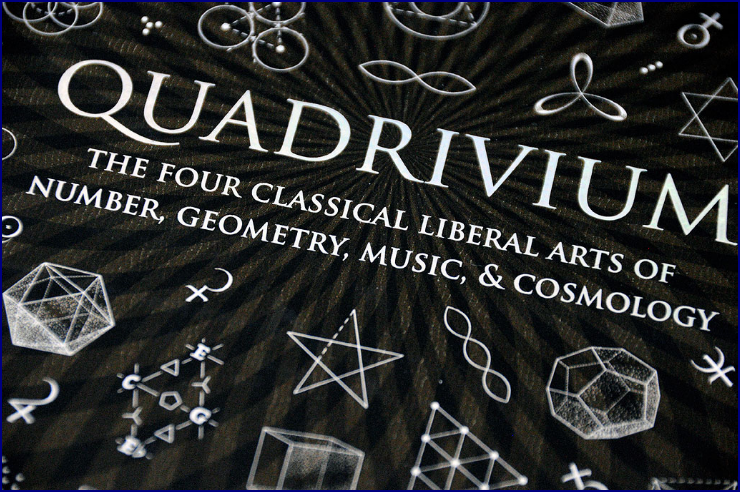
Modern physics holds that, at its deepest level, the universe is mathematical in nature.

This view is a topic of current serious discussion (*The Mathematical Universe*, by Max Tegmark).

Outline

The Quadrivium



The Quadrivium

The Quadrivium originated with the Pythagoreans around 500 BC.

The Pythagoreans' quest was to find the eternal laws of the Universe, and they organized their studies into the scheme later known as the Quadrivium.

It comprised four disciplines:

- Arithmetic
- Geometry
- Music
- Astronomy

The Quadrivium

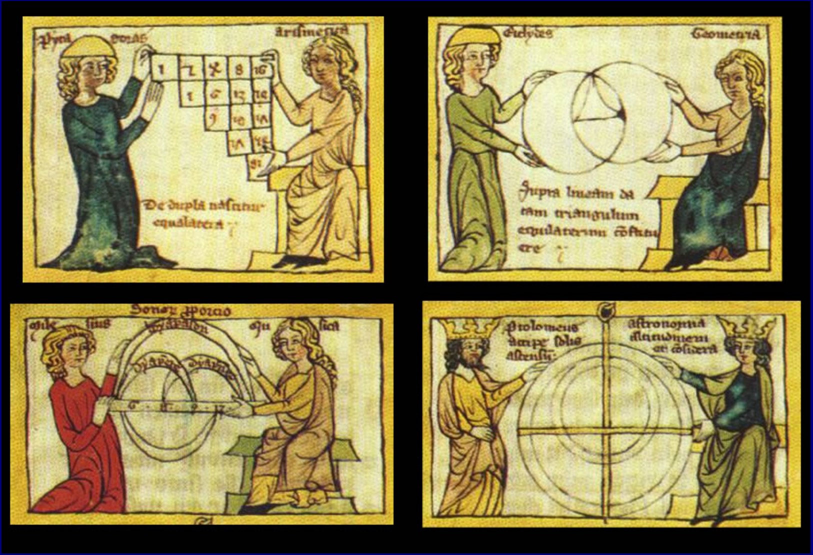
First comes Arithmetic, concerned with the infinite linear array of numbers.

Moving beyond the line to the plane and 3D space, we have Geometry.

The third discipline is Music, which is an application of the science of numbers.

Fourth comes Astronomy, the application of Geometry to the world of space.

The Quadrivium



Static/Dynamic. Pure/Applied

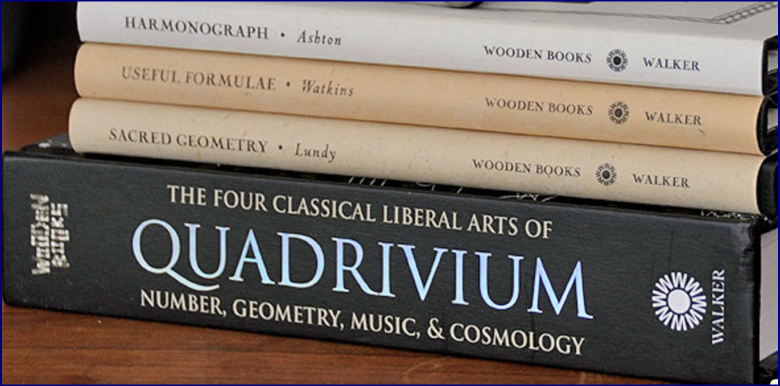
- Arithmetic (static number)
- Music (moving number)
- Geometry (measurement of static Earth)
- Astronomy (measurement of moving Heavens)

Arithmetic represents numbers at rest,
Geometry is magnitudes at rest,

Music is numbers in motion and
Astronomy is geometry in motion.

The first two are pure in nature,
while the last two are applied.

The Quadrivium



For the Greeks, **Mathematics** embraced all four areas.

The Pythagoreans

Pythagoras distinguished between
quantity and magnitude.

Objects that can be counted
yield quantities or numbers.

Substances that are measured provide magnitudes.

Thus, cattle are counted whereas milk is measured.

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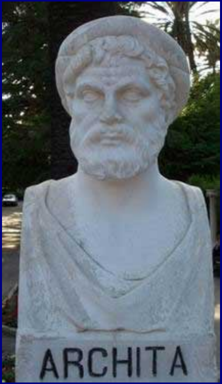
Thus, cattle are counted whereas milk is measured.

Arithmetic studies quantities or numbers and
Music involves the relationship between numbers
and their evolution in time.

Geometry deals with magnitudes, and
Astronomy with their distribution in space.

Archytas (428–350 BC): *ΑΡΧΥΤΑΣ*



Αρχυτας.

Born in Tarentum, son of Hestiaeus.

Mathematician and philosopher.

Pythagorean, student of Philolaus.

Provided a solution for the Delian problem of doubling the cube.

Said to have tutored Plato in mathematics(?)

Archytas (428–350 BC)

Archytas lived in Tarentum (now in Southern Italy).

One of the last scholars of the Pythagorean School and was a good friend of Plato.

The designation of the four disciplines of the Quadrivium was ascribed to Archytas.

His views were to dominate pedagogical thought for over two millennia.

Partly due to Archytas, mathematics has played a prominent role in education ever since.

Plato’s Academy

According to Plato, mathematical knowledge was essential for an understanding of the Universe. The curriculum was outlined in Plato’s *Republic.*

Inscription over the entrance to Plato’s Academy:

"Let None But Geometers Enter Here".

This indicated that the Quadrivium was a prerequisite for the study of philosophy in ancient Greece.

Boethius (AD 480–524)

The Western Roman Empire was in many ways static for centuries.

No new mathematics between the conquest of Greece and the fall of the Roman Empire in AD 476.

Boethius, the 6th century Roman philosopher, was one of the last great scholars of antiquity.

The organization of the Quadrivium was formalized by Boethius.

It was the mainstay of the medieval monastic system of education.

The Quadrivium

Typus Arithmeticae

A woodcut from the book *Margarita Philosophica,* by Gregor Reisch, Freiburg, 1503.

The central figure is Dame Arithmetic, watching a competition between Boethius, using pen and Hindu-Arabic numerals, and Pythagoras, using a counting board or *tabula*.

She looks favourably toward Boethius.

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But how did Boetius know about Hindu-Arabic numerals?

The Liberal Arts

The seven liberal arts comprised
the Trivium and the Quadrivium.

The Trivium was centred on three arts of language:

- Grammar: proper structure of language.
- Logic: for arriving at the truth.
- Rhetoric: the beautiful use of language.

Aim of the Trivium: Goodness, Truth and Beauty.

Aristotle traced the origin of the Trivium back to Zeno.

The Ultimate Goal

The goal of studying the Quadrivium was
an insight into the nature of reality,
an understanding of the Universe.

The Quadrivium offered the seeker of wisdom
an understanding of the integral nature of
the Universe and the role of humankind within it.

That is our aim in Sum-enchanted Evenings!

Outline

The Greek Alphabet, Part 3

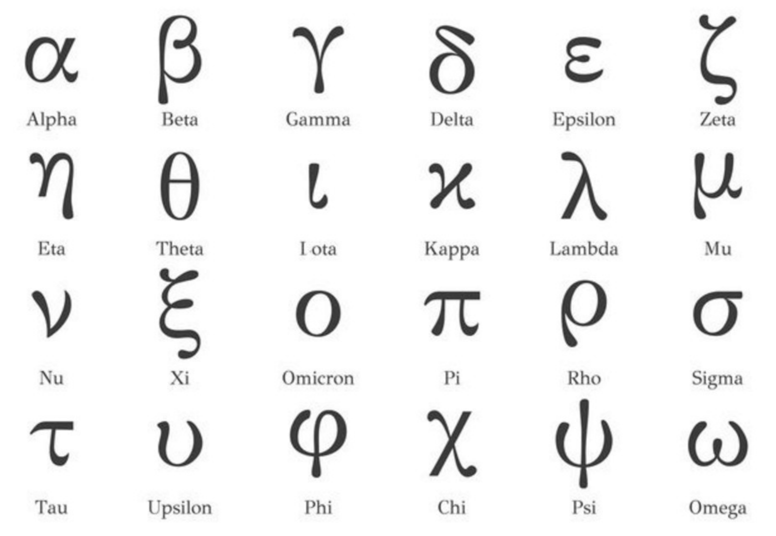


Figure : 24 beautiful letters

The Next Six Letters

We will consider the third group of six letters.

ν	ξ	o	π	ρ	σ
N	Ξ	O	Π	P	Σ

Let us focus first on the small letters
and come back to the big ones later.

$\nu \quad \xi \quad o \quad \pi \quad \rho \quad \sigma$

Nu (ν) is in Planck's formula: $E = h\nu$.
Then ν is the frequency of a photon of light.

Xi (ξ) is the Greek X, as in $\kappa\lambda\iota\mu\alpha\xi$ or $\mathrm{K\Lambda IMAX}$.

Omicron: Think of Oh-Micron, small Oh (not OMG).
Is there a large O, or Oh-Mega ?

Pi (π) is already very familiar to you all.

Rho (ρ) is Greek R, used for density.

Sigma (σ) is the Greek S. At the end of a word it is written ς.

Now we know eighteen letters. We're 75% done!

A Few Greek Words (for practice)

$\kappa\lambda\iota\mu\alpha\xi$

$\delta\rho\alpha\mu\alpha$

$\nu\epsilon\kappa\tau\alpha\rho$

$\kappa\omega\lambda o\nu$

$\kappa o\sigma\mu o\varsigma$

A Few Greek Words (for practice)

κλιμαξ
δραμα
νεκταρ
κωλον
κοσμος

Climax: κλιμαξ
Drama: δραμα
Nectar: νεκταρ
Colon: κωλον
Cosmos: κοσμος


Κέντρο
Center
Λιμάνι
Port


Μουσειο Ακρόπολης
Acropolis Museum


Αθήνα
Δελφοί
400M


ΤΑΞΙ

End of Greek 103

Outline

Theorem of Pythagoras

The Theorem of Pythagoras is of fundamental importance in Euclidean geometry

It encapsulates the structure of space.

In the BBC series, The Ascent of Man, Jacob Bronowski said

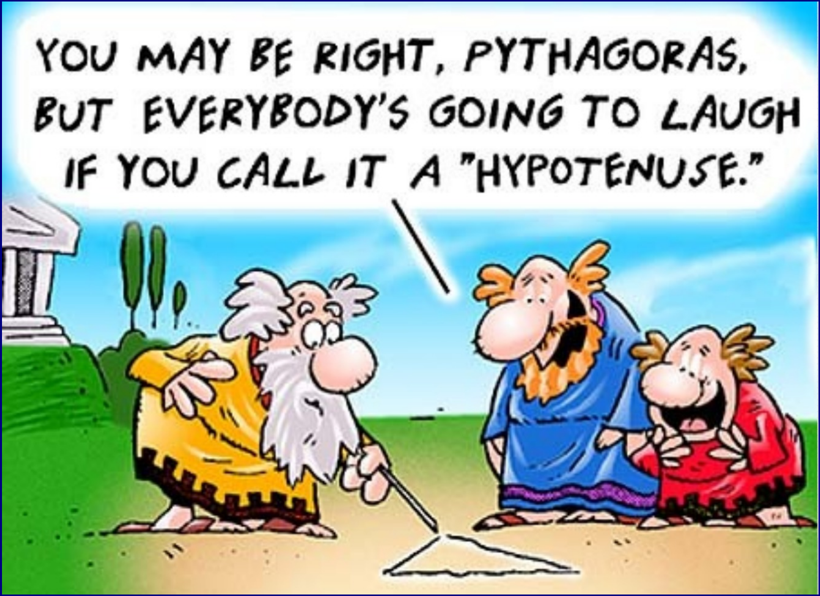
"The theorem of Pythagoras remains the most important single theorem in mathematics."

Theorem of Pythagoras

YouTube video with Danny Kaye

Google search for
"Danny Kaye Hypotenuse"

https://www.youtube.com/watch?v=oeRCsAGQVy8


YOU MAY BE RIGHT, PYTHAGORAS, BUT EVERYBODY'S GOING TO LAUGH IF YOU CALL IT A "HYPOTENUSE."

Hypotenuse

The side of a right triangle opposite to the right angle.

1570s, from Late Latin hypotenusa, from Greek hypoteinousa "stretching under" (the right angle).

Fem. present participle of hypoteinein, from hypo- "under" + teinein "to stretch"

From Online Etymology Dictionary: `http://www.etymonline.com/`

Mathigon.org

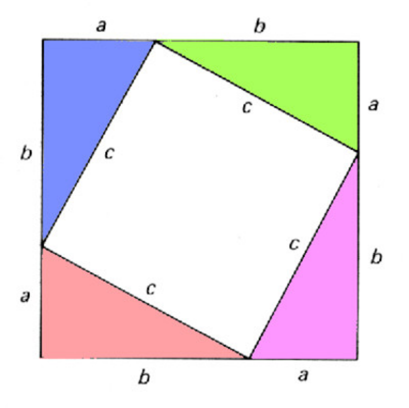
Mathigon.org video on Proofs without Formulas:

- What is the sum of the angles in a triangle?
- What is the sum of the angles in a polygon?
- What is the area of a triangle?
- How does Pythagoras' Theorem work?

In the video below, these and other important concepts are explained in only two minutes using nothing but graphics.

https://youtu.be/IUCK8bk0xPo

Proof without Formulae



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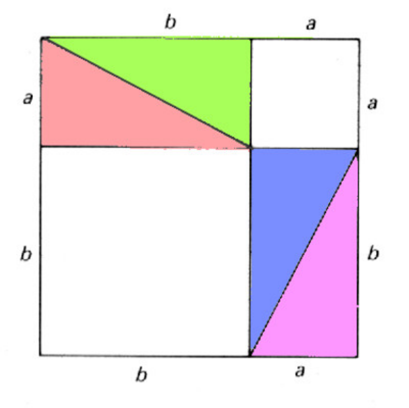

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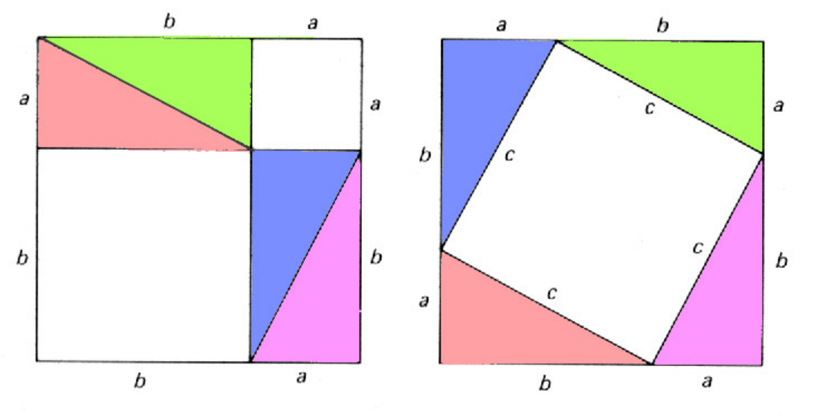

$$a^2 + b^2 = c^2$$

Why is this Important / Interesting?

Squares on the sides of triangles don’t seem much.

But the theorem gives us distances.

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If one point is at $(0, 0)$ and another at (x, y), the theorem gives the distance:

$$r^2 = x^2 + y^2 \qquad \textbf{or} \qquad r = \sqrt{x^2 + y^2}$$

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This tells us about the structure of space.

I should expand on this topic (e.g., SAR)

Outline

Set Theory Puzzle

In a small Canadian village, everyone speaks either English or French, or both.

80% of the people speak French

60% of the people speak English

What percentage speak both English and French?

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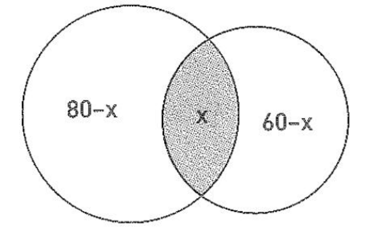
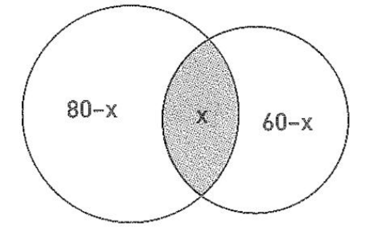
Answer next week!

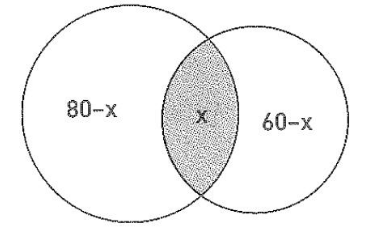

80–x
x
60–x



$$(80 - x) + x + (60 - x) = 100\,.$$

Therefore

$$140 - x = 100 \qquad \textbf{or} \qquad x = 40\,.$$



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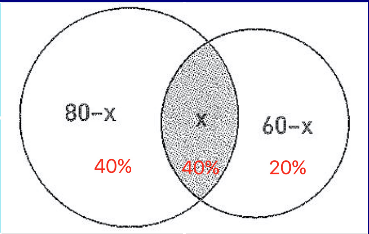

Outline

The Unary System

The simplest numeral system is the unary system.

Each natural number is represented by
a corresponding number of symbols.

If the symbol is “ | ” , the number seven
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Tally marks represent one such system,
which is still in common use.

The unary system is only useful for small numbers.

The unary notation can be abbreviated,
with new symbols for certain values.

Sign-Value Notation

The **five-bar gate** system groups 5 strokes together.

Normally, distinct symbols are used for powers of 10.

If “ | ” stands for one, “ Λ ” for ten and “ ϒ ” for 100, then the number **123** becomes ϒ ΛΛ | | |

Sign-Value Notation

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There is no need for a symbol for zero.

This is called **sign-value notation.**

Ancient Egyptian numerals were of this type.

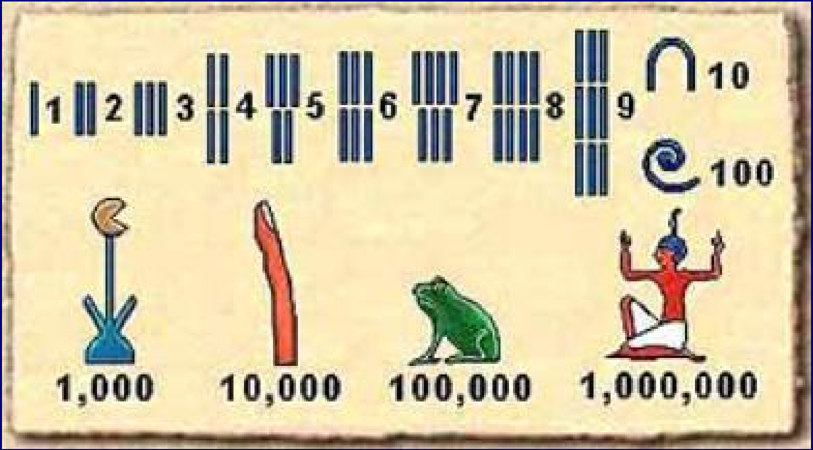
Roman numerals were a modification of this idea.

Egypyian Numerals

Value	1	10	100	1,000	10,000	100,000	1 million, or many
Hieroglyph						 or 	
Description	Single stroke	Heel bone	Coil of rope	Water lily (also called Lotus)	Bent Finger	Tadpole or Frog	Man with both hands raised, perhaps Heh.[3]

Figure : From Wikipedia page `https://en.wikipedia.org/wiki/Egyptian_numerals`

Egypyian Numerals



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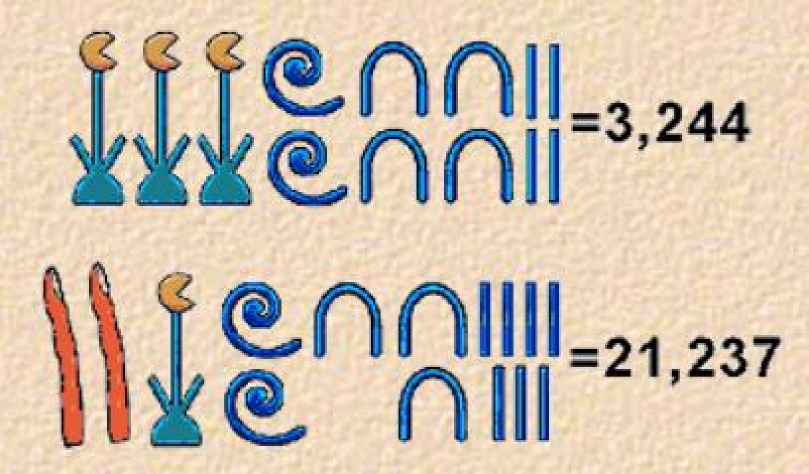
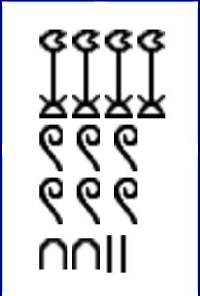
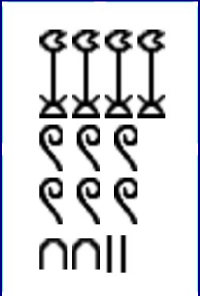




The arrangement of symbols is not important.

What number is this?



The arrangement of symbols is not important.

What number is this?

This pattern represents

4622.

Hebrew Numerals

Hebrew Number Values

א	Aleph -	1	ל	Lamed -	30
ב	Beth -	2	מ	Mem -	40
ג	Gimel -	3	נ	Nun -	50
ד	Daleth -	4	ס	Samekh -	60
ה	Heh -	5	ע	Ayin -	70
ו	Vav -	6	פ	Peh -	80
ז	Zain -	7	צ	Tzaddi -	90
ח	Cheth -	8	ק	Qoph -	100
ט	Teth -	9	ר	Resh -	200
י	Yod -	10	ש	Shin -	300
כ	Kaph -	20	ת	Tau -	400

The 22 letters of the Hebrew alphabet were used also as numerals.

Each letter corresponded to a numerical value.

Greek Numerals

	Units	Tens	Hundreds
1	α alpha	ι iota	ρ rho
2	β beta	κ kappa	σ sigma
3	γ gamma	λ lambda	τ tau
4	δ delta	μ mu	υ upsilon
5	ε epsilon	ν nu	φ phi
6	ϝ digamma	ξ xi	χ chi
7	ζ zeta	ο omicron	ψ psi
8	η eta	π pi	ω omega
9	θ theta	ϙ koppa	ϡ sampi

The 24 letters of the Greek alphabet had corresponding numerical values.

They were supplemented by three additional letters, which are now archaic.

$$\sigma\mu\gamma = ?$$

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$$\sigma\mu\gamma = ?$$

$$243 = \sigma\mu\gamma$$

Greek Numerals

Arabic number	1	2	3	4	5	6	7	8	9
Greek number	α	β	γ	δ	ε	F	ζ	η	θ
Greek name	alpha	beta	gamma	delta	epsilon	digamma	zeta	eta	theta
Sound	a	b	g	d	short e		z	long e	th
Arabic number	10	20	30	40	50	60	70	80	90
Greek number	ι	κ	λ	μ	ν	ξ	ο	π	ϙ
Greek name	iota	kappa	lambda	mu	nu	xi	omicron	pi	koppa
Sound	i	k/c	l	m	n	x	short o	p	
Arabic number	100	200	300	400	500	600	700	800	900
Greek number	ρ	σ	τ	υ	φ	χ	ψ	ω	ϡ
Greek name	rho	sigma	tau	upsilon	phi	chi	psi	omega	sampi
Sound	r	s	t	u	f/ph	ch	ps	long o	

Outline

Topology: a Major Branch of Mathematics

Topology is all about continuity and connectivity, but the meaning of that will appear later.

We will look at a few aspects of Topology.

- The Bridges of Königsberg
- Doughnuts and Coffee-cups
- Knots and Links
- Nodes and Edges: Graphs
- The Möbius Band

In this lecture, we study The Bridges of Königsberg.

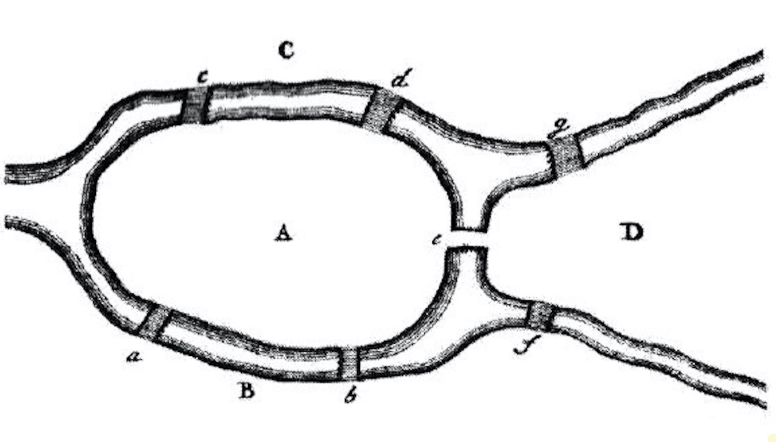
The Bridges of Königsberg

One of the earliest topological puzzles was studied by the renowned Swiss mathematician Leonard Euler.

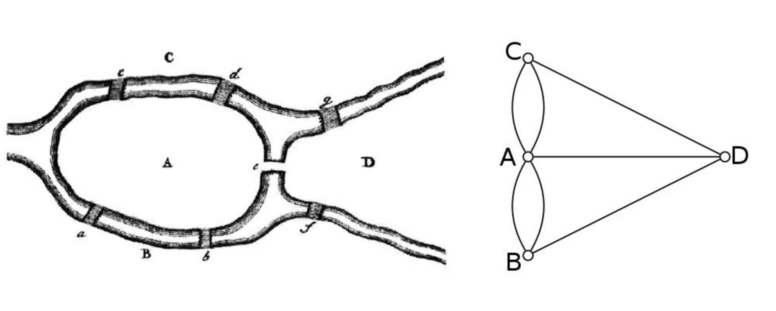
It is called 'The Seven Bridges of Königsberg'.

The goal is to find a route through that city, crossing each of seven bridges exactly once.

The Bridges of Königsberg



The Bridges of Königsberg



Euler reduced the problem to its essentials, removing all extraneous details.

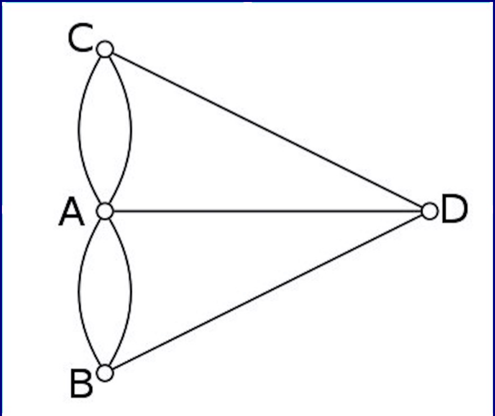
He replaced the map above by the graph on the right.

A simple argument showed that no journey that crosses each bridge exactly once is possible.

Except at the termini of the route, each path arriving at a vertex must have a corresponding path leaving it.

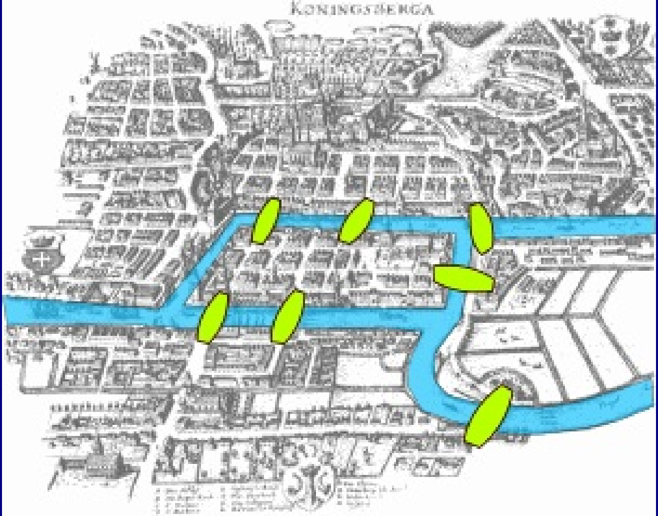
Only two vertices with an odd number of edges are possible for a solution to exist.

The Bridges of Königsberg

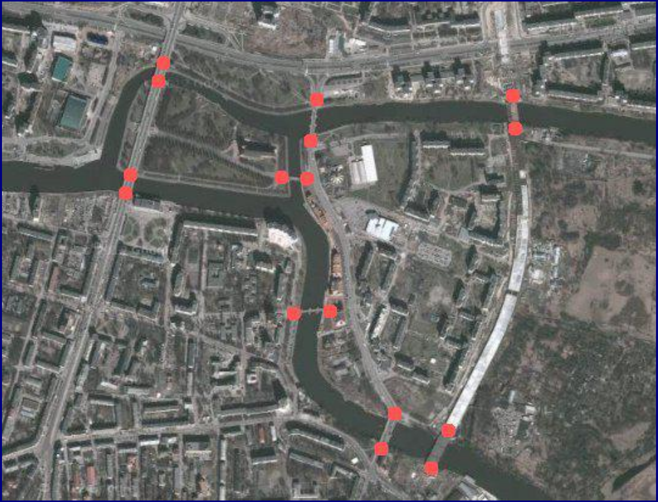


Exercise: Draw the diagram with *A*, *B*, *C* and *D* arranged clockwise at the corners of a square.

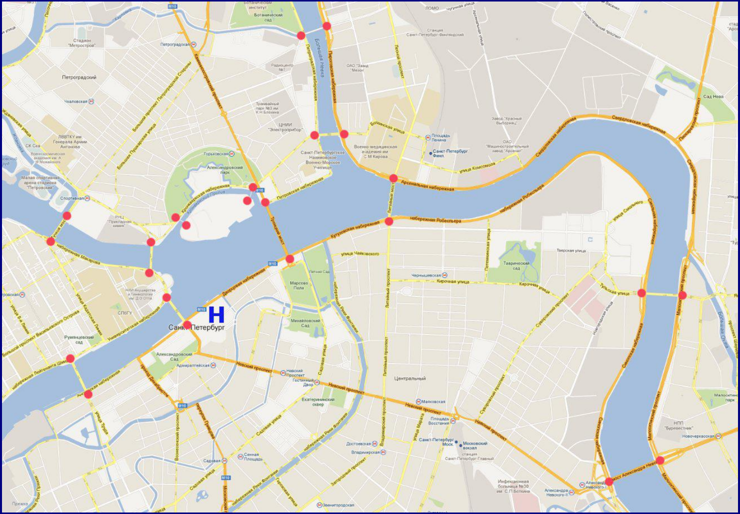
The Bridges of Königsberg



Königsberg Today



The Bridges of St Petersburg



The Bridges of St Petersburg

Euler spend much of his life in St Petersburg,
a city with many rivers, canals and bridges.

Did he think about another problem like the
Königsberg Bridges problem while there?

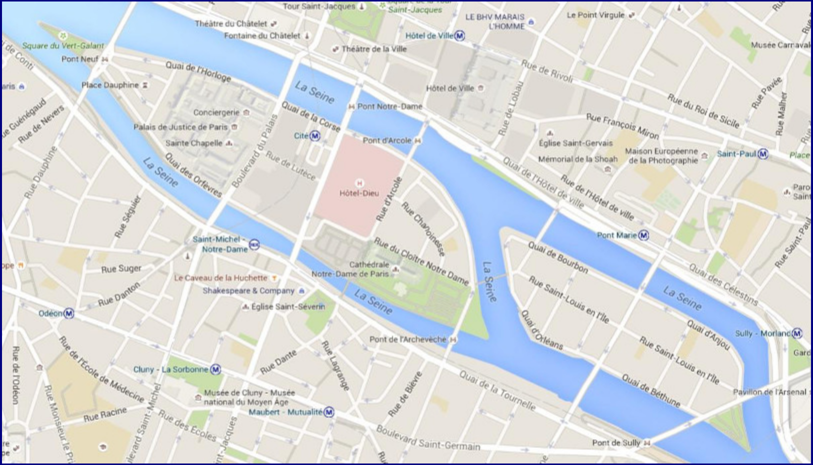
The map of central St Petersburg has twelve bridges.

An Euler cycle is a route that crosses all bridges
exactly once and returns to the starting point?

Is there an Euler cycle starting at the Hermitage
(marked "H" on the map)?

The Bridges of Paris

Cue romantic music



The Bridges of Paris

In central Paris, two small islands, Île de la Cité and Île Saint-Louis, are linked to the Left and Right Banks of the Seine and to each other.

The number of bridges for each land-mass are:

- Left Bank: 7 bridges
- Right Bank: 7 bridges
- Île de la Cité: 10 bridges
- Île Saint-Louis: 6 bridges

The total is 30. How many bridges are there?

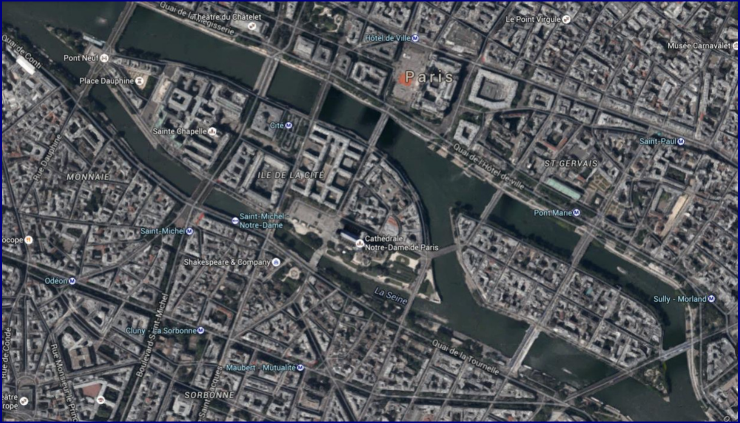
The Bridges of Paris

The Bridges of Paris

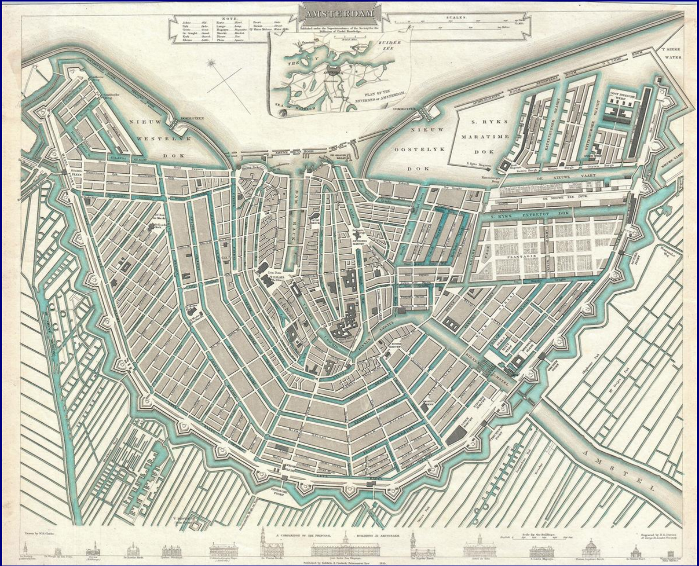
1. Starting from Saint-Michel on the Left Bank, walk continuously so as to cross each bridge once.
2. Start at Saint-Michel but find a closed route that ends back at the starting point.
3. Start at Notre-Dame Cathedral, on Île de la Cité, and cross each bridge exactly once.
4. Find a closed route that crosses each bridge once and arrives back at Notre-Dame.

Try these puzzles yourself. Use logic, not brute force!

The Bridges of Paris



The Bridges of Amsterdam



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Seven Bridges of Königsberg

From Wikipedia, the free encyclopedia

Coordinates: 54°42′12″N 20°30′56″E

This article is about an abstract problem. For the historical group of bridges in the city once known as Königsberg, and those of them that still exist, see § Present state of the bridges.

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The **Seven Bridges of Königsberg** is a historically notable problem in mathematics. Its negative resolution by Leonhard Euler in 1736 laid the foundations of graph theory and prefigured the idea of topology.[1]

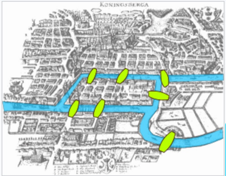
The city of Königsberg in Prussia (now Kaliningrad, Russia) was set on both sides of the Pregel River, and included two large islands which were connected to each other, or to the two mainland portions of the city, by seven bridges. The problem was to devise a walk through the city that would cross each of those bridges once and only once.

By way of specifying the logical task unambiguously, solutions involving either

1. reaching an island or mainland bank other than via one of the bridges, or
2. accessing any bridge without crossing to its other end

are explicitly unacceptable.

Euler proved that the problem has no solution. The difficulty he faced was the development of a suitable technique of analysis, and of subsequent tests that established this assertion with mathematical rigor.



Map of Königsberg in Euler's time showing the actual layout of the seven bridges, highlighting the river Pregel and the bridges

Thank you