Sum-Enchanted Evenings

The Fun and Joy of Mathematics

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LECTURE 3

Peter Lynch
School of Mathematics & Statistics
University College Dublin

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Outline

Outline

Meaning and Content of Mathematics

The word Mathematics comes from Greek $\mu\alpha\theta\eta\mu\alpha$ (máthéma), meaning "knowledge" or "study" or "learning".

It is the study of topics such as

- Quantity (numbers)
- Structure (patterns)
- Space (geometry)
- Change (analysis).

Outline

Three Circles in a Plane

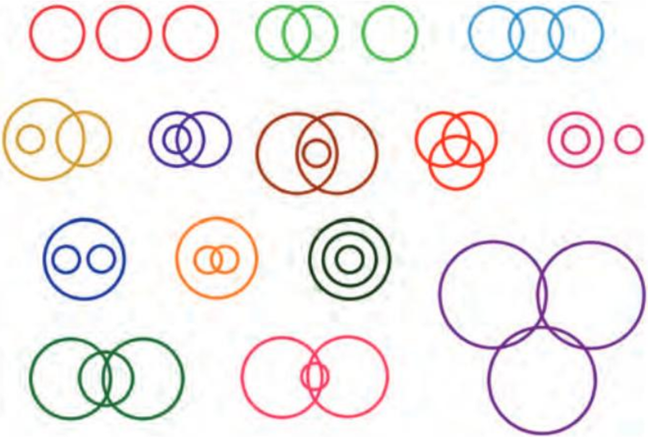
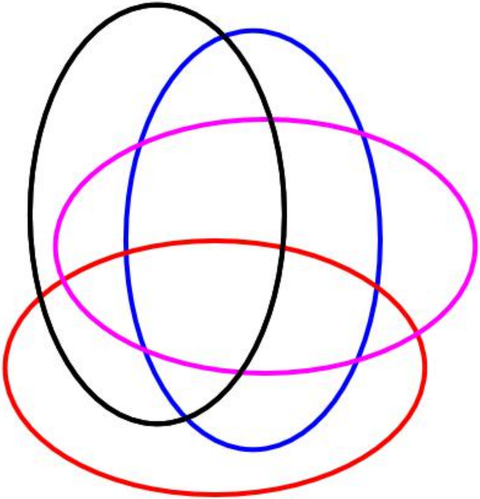
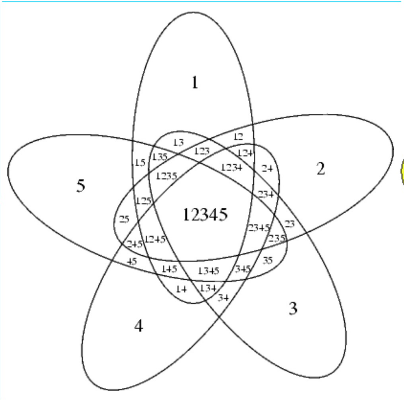


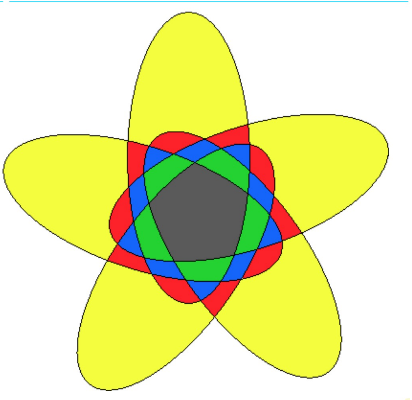
Figure 6. The fourteen ways to draw three circles in the affine plane.

Venn Diagram for 4 Sets

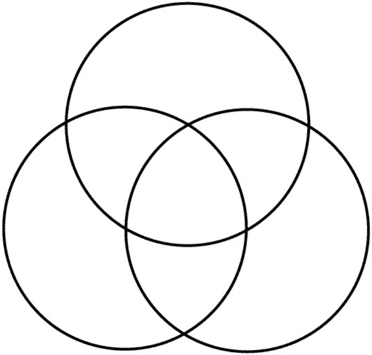


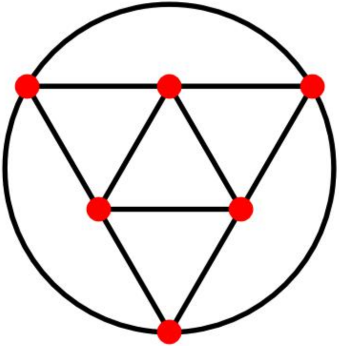
Venn Diagram for 5 Sets





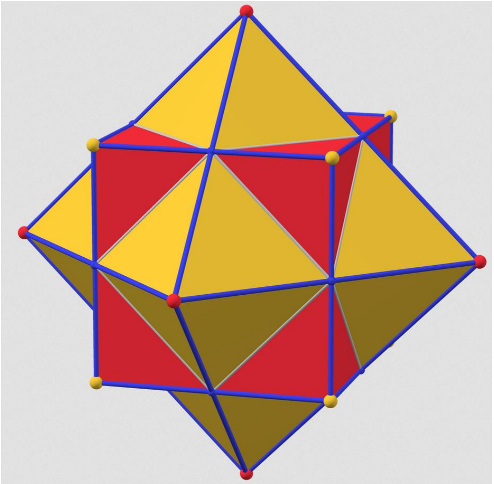
Venn Diagram as a Graph



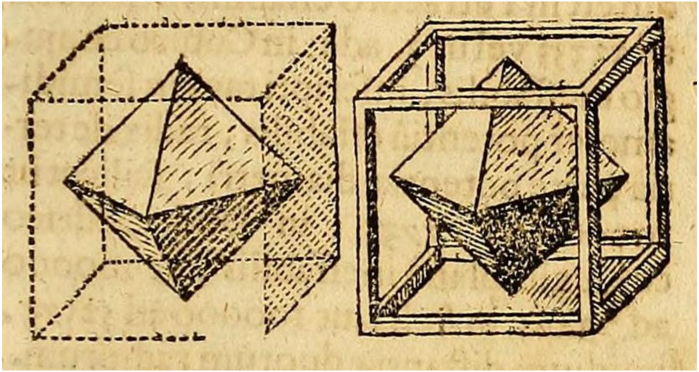


Graph is equivalent to an octahedron

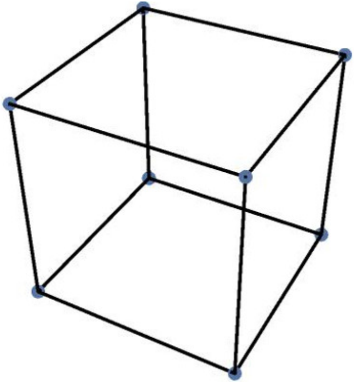
Cube and Octahedron are Duals

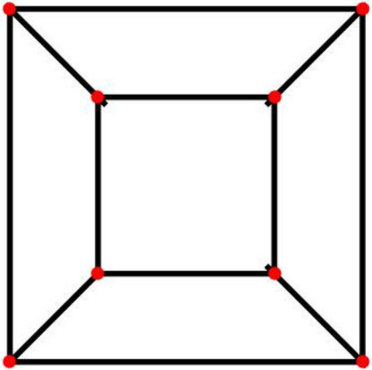


From Kepler's *Harmonices Mundi*



Venn3 Dual as a Cube





See blog post

Venn Again's Awake

on my mathematical blog `thatsmaths.com`

There is No Largest Number

Children often express bemusement at the idea that there is no largest number.

Given any number, 1 can be added to it to give a larger number.

But the implication that there is no limit to this process is perplexing.

The concept of infinity has exercised the greatest minds throughout the history of human thought.

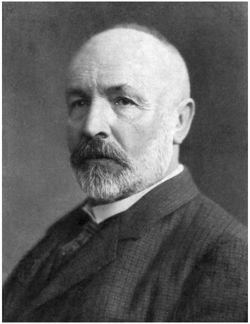
Degrees of Infinity

In the late 19th century, Georg Cantor showed that there are different degrees of infinity.

In fact, there is an infinite hierarchy of infinities.

Cantor brought into prominence several paradoxical results that had a profound impact on the development of logic and of mathematics.

Georg Cantor (1845–1918)



Cantor discovered many remarkable properties of infinite sets.

Cardinality

Finite Sets have a finite number of elements.

Example: The Counties of Ireland form a finite set.

Counties $=$ {Antrim, Armagh, . . . , Wexford, Wicklow}

For a finite set A, the *cardinality* of A is:

The number of elements in A

One-to-one Correspondence

A particular number, say 5, is associated with all the sets having five elements.

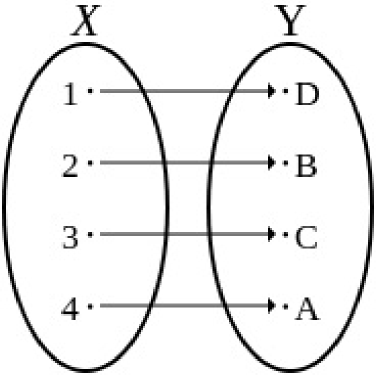
For any two of these sets, we can find a 1-to-1 correspondence between the elements of the two sets.

The number 5 is called the cardinality of these sets.

Generalizing this:

Any two sets are the same size (or cardinality) if there is a 1-to-1 correspondence between them.

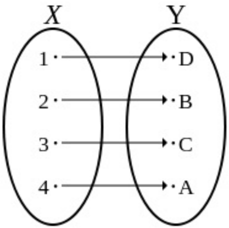
One-to-one Correspondence



Equality of Set Size: 1-1 Correspondence

How do we show that two sets are the same size?

For finite sets, this is straightforward counting.



For infinite sets, we must find a 1-1 correspondence.

Cardinality

The number of elements in a set is called the cardinality of the set.

Cardinality of a set A is written in various ways:

$$|\mathbf{A}| \qquad \|\mathbf{A}\| \qquad \mathrm{card}(\mathbf{A}) \qquad \#(A)$$

For example

$$\#\{\textbf{Irish Counties}\} = 32$$

The Empty Set

We call the set with *no elements* the empty set.

It is denoted by a special symbol

$$\varnothing = \{ \ \}$$

Clearly

$$\#\{ \ \} = 0 \,.$$

We could have a philosophical discussion about the empty set. Is it related to a perfect vacuum?

The Greeks regarded the vacuum as an impossibility.

The Natural Numbers $\mathbb{N}$

The *counting numbers* (positive whole numbers) are

1 2 3 4 5 6 7 8 . . . .

They are also called the *Natural Numbers*.

The set of natural numbers is denoted $\mathbb{N}$.

This is our first infinite set.

We use a special symbol to denote its cardinality:

$$\#(\mathbb{N}) = \aleph_0$$

$$\aleph_0$$

The Power Set

For any set, we can form a new one, the Power Set.

The Power Set is the *set of all subsets of* A.

Suppose the set A has just two elements:

$$\mathbf{A} = \{a, b\}$$

Here are the subsets of A:

$$\{\ \} \qquad \{a\} \qquad \{b\} \qquad \{a, b\}$$

The *power set* is

$$\mathrm{P}[\mathbf{A}] = \Big\{ \{\ \}, \{a\}, \{b\}, \{a, b\} \Big\}$$

Cantor's Theorem

Cantor's theorem states that, for any set A, the power set of A has a strictly greater cardinality than A itself.

$$\#[\mathrm{P}(\mathbf{A})] > \#[A]$$

This holds for both finite and infinite sets.

It means that, for every cardinal number, there is a greater cardinal number.

One-to-one Correspondence

Now we consider sets are infinite:
take all the natural numbers,

$$\mathbb{N} = \{1, 2, 3, ...\}$$

as one set and all the even numbers

$$\mathbb{E} = \{2, 4, 6, ...\}$$

as the other.

By associating each number $n \in \mathbb{N}$ with $2n \in \mathbb{E}$,
we have a perfect 1-to-1 correspondence.

By Cantor's argument, the two sets are the same size:

$$\#[\mathbb{N}] = \#[\mathbb{E}]$$

Again,

$$\#[\mathbb{N}] = \#[\mathbb{E}]$$

But this is *paradoxical:* The set of natural numbers contains all the even numbers

$$\mathbb{E} \subset \mathbb{N}$$

and also all the odd ones.

In an intuitive sense, $\mathbb{N}$ is larger than $\mathbb{E}$.

The same paradoxical result had been deduced by *Galileo* some 250 years earlier.

Cantor carried these ideas much further:

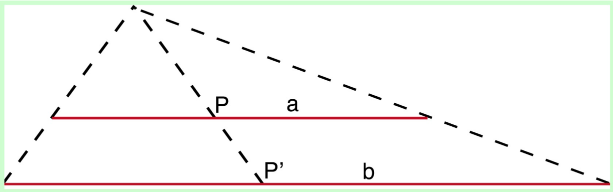
The set of all the real numbers has a degree of infinity, or cardinality, greater than the counting numbers:

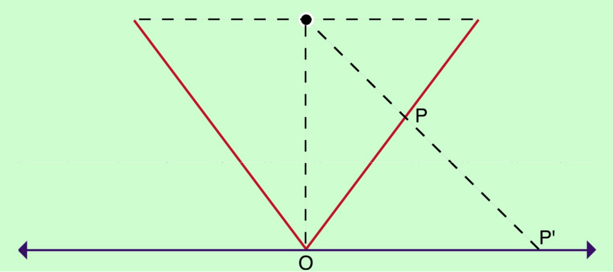
$$\#[\mathbb{R}] > \#[\mathbb{N}]$$

Cantor showed this using an ingenious approach called the diagonal argument.

This is a fascinating technique, but we will not give details here.

How Many Points on a Line?





There is a 1-1 map between $(-1, +1)$ and $\mathbb{R}$.

Review: Infinities Without Limit

For any set A, the power set P(A) is
the collection of all the subsets of A.

Cantor proved P(A) has cardinality greater than A.

For finite sets, this is obvious;
for infinite ones, it was startling.

The result is now known as Cantor's Theorem,
and Cantor used his diagonal argument in proving it.

He thus developed an entire hierarchy
of transfinite cardinal numbers.

Outline

Enigmas of Infinity

Zeno of Elea devised several paradoxes involving infinity.

He argued that one cannot travel from A to B: to do so, one must first travel half the distance, then half of the remaining half, then half the remainder, and so on.

He concluded that motion is logically impossible.

Zeno was misled by his belief that the sum of finite quantities must grow without limit as more are added.

Enigmas of Infinity

Systematic mathematical study of infinite sets began around 1875 when Georg Cantor developed a theory of transfinite numbers.

He reasoned that the method of comparing the sizes of finite sets could be carried over to infinite ones.

If two finite sets, for example the cards in a deck and the weeks in a year, can be matched up one to one they must have the same number of elements.

Bijections

Mathematicians call a 1:1 correspondence a bijection.

Cantor used this approach to compare infinite sets: if there is a bijection between them, two sets are said to be *the same size.*

Cantor built an entire theory of infinity on this idea.

Hilbert's Hotel

We will look at a fantasy devised by David Hilbert.

We could call it a Gedankenexperiment

It was introduced in 1924 in a lecture
Über das Unendliche.

Hilbert's Hotel

Hilbert's Grand Hotel

Leading German mathematician David Hilbert constructed *an amusing metaphor* to illustrate the surprising and counter-intuitive properties of infinity.

He imagined a hotel with an infinite number of rooms.

Even with the hotel full, there is always room to accommodate an extra guest.

Simply move guest 1 to room 2, guest 2 to room 3 and so on, thereby vacating the first room.

Indeed, an infinite number of new arrivals could be accommodated: for all rooms n, move the guest in room n to room 2n, and magically all the odd-numbered room become vacant.

Indeed, a countably infinite number of busses, each with a countably infinite number of passengers, can be accommodated.

Video:
`https://www.youtube.com/watch?v=Uj3_KqkI9Zo&t=191s`

`http://world.mathigon.org/resources/Infinity/Miss_Marple.mp4`

Outline

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Infinitesimals

Infinite quantities are unboundedly large.

At the opposite extreme, *infinitesimals are infinitely small quantities.*

An infinitesimal is smaller than any finite number, yet greater than zero.

Infinitesimals

Infinitesimals were used by *Leibniz* and *Newton* in formulating differential and integral calculus.

They were the cause of great controversy that raged for centuries.

Bishop George Berkeley

Born	12 March 1685 County Kilkenny, Ireland
Died	14 January 1753 (aged 67) Oxford, England
Nationality	Irish
Alma mater	Trinity College, Dublin
Era	18th century philosophy
Region	Western philosophy
School	Subjective idealism (phenomenalism) Empiricism Foundationalism[1] Conceptualism[2] Indirect realism[3]
Main interests	Christianity, metaphysics, epistemology, language, mathematics, perception

Bishop George Berkeley

In his satirical critique on the foundations of mathematics, Irish bishop George Berkeley described infinitesimals as *the ghosts of departed quantities.*

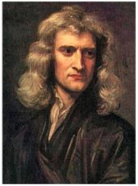
Berkeley's witty polemic was justified: the foundations of mathematical analysis were unsound.

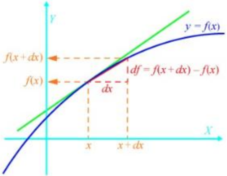
The problems were resolved only by a rigorous theory of limits, devised around 1820 by Augustin-Louis Cauchy and Karl Weierstrass.

Berkeley on Calculus

Calculus

Leibniz and Newton (1680's): Compute derivatives by dividing infinitesimals.





Lord Berkeley: "They are neither finite quantities, nor quantities infinitely small, nor yet nothing. May we not call them the ghosts of departed quantities?"

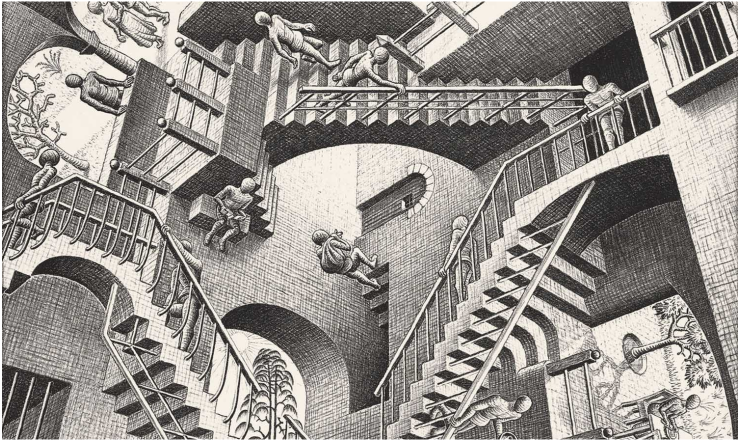
Infinity in Perspective

Vanishing points used in perspective art correspond to mathematical points at infinity.

They allow artists to render forms and distances realistically.

The dutch artist M. C. Escher was a genius at exploiting the concept of infinity and its paradoxes.

M. C. Escher



Infinities Everywhere

Mathematicians deal with infinity on a daily basis. The concept of infinity is essential in analysis (calculus) and set theory.

Zoom in on the *arc of a circle.* It approximates a line segment, but never quite gets there.

Its length would have to be infinitesimal before we could truly call it straight.

Archimedes used this idea in his calculation of π. "A circle is a line under a microscope."

Outline

Music and Mathematics

Music and Mathematics:
Symmetry and Symbiosis

Part 1

Thank you