# Sum-Enchanted Evenings

## The Fun and Joy of Mathematics

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## LECTURE 10

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**Evening Course, UCD, Autumn 2017**

# Outline

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# Meaning and Content of Mathematics

The word Mathematics comes from Greek $\mu\alpha\theta\eta\mu\alpha$ (máthéma), meaning "knowledge" or "study" or "learning".

It is the study of topics such as

- Quantity (numbers)
- Structure (patterns)
- Space (geometry)
- Change (analysis).

***Reminder:* A square from A4 paper sheets.**

**PUZZLE:**
**Is it possible to form a square out of sheets of A4 sized paper (without them overlapping)?**

**Remember: Ratio of width to height is $1 : \sqrt{2}$.**

# A Square from A4 Paper Sheets

**Let dimensions be: Width = $1$ unit. Height = $\sqrt{2}$ units.**

**Suppose there are $a$ short sides and $b$ long sides along the *lower horizontal edge* of the big square.**

**Then the length of the horizontal edge is**

$$H = a.1 + b.\sqrt{2}$$

**Suppose there are $c$ short sides and $d$ long sides along the *left vertical edge* of the big square.**

**So the length of the vertical edge is**

$$V = c.1 + d\sqrt{2}$$

**Since the region is square, $V = H$ and we must have**

$$a.1 + b.\sqrt{2} = c.1 + d\sqrt{2}$$

**Therefore**

$$\begin{aligned} a + b\sqrt{2} &= c + d\sqrt{2} \\ a - c &= (d - b)\sqrt{2} \\ \left(\frac{a-c}{d-b}\right) &= \sqrt{2} \end{aligned}$$

**But the left side is a ratio of two whole numbers, whereas the right side is irrational.**

**This is impossible. There is no solution!**

***Reductio ad absurdum.***

# Outline

# Alfred Moessner's Conjecture

*Aus den Sitzungsberichten der Bayerischen Akademie der Wissenschaften*
*Mathematisch-naturwissenschaftliche Klasse 1951 Nr. 3*

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Eine Bemerkung über die Potenzen der natürlichen Zahlen

Von Alfred Moessner in Gunzenhausen

Vorgelegt von Herrn O. Perron am 2. März 1951

**A Remark on the Powers of the Natural Numbers**

# Moessner's Construction: n=2

**We start with the sequence of natural numbers:**

| | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | ... |

**Now we delete *every second number* and calculate the sequence of partial sums:**

| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
| 1 | | 4 | | 9 | | 16 | | 25 | | 36 | | 49 | | 64 | |

**The result is the sequence of perfect squares:**

$$1^2 \quad 2^2 \quad 3^2 \quad 4^2 \quad 5^2 \quad 6^2 \quad 7^2 \quad 8^2 \quad \ldots$$

# Moessner's Construction: n=3

**Now we delete *every third number* and calculate the sequence of partial sums.**

**Then we delete *every second number* and calculate the sequence of partial sums:**

| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
| 1 | 3 | | 7 | 12 | | 19 | 27 | | 37 | 48 | | 61 | 75 | | 91 |
| 1 | | | 8 | | | 27 | | | 64 | | | 125 | | | 216 |

**The result is the sequence of perfect cubes:**

$$1^3 \quad 2^3 \quad 3^3 \quad 4^3 \quad 5^3 \quad 6^3 \quad \ldots$$

# Moessner's Construction: n=4

**The Moessner Construction also works for larger n:**

| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
| 1 | 3 | 6 | | 11 | 17 | 24 | | 33 | 43 | 54 | | 67 | 81 | 96 | |
| 1 | 4 | | | 15 | 32 | | | 65 | 108 | | | 175 | 256 | | |
| 1 | | | | 16 | | | | 81 | | | | 256 | | | |

**The result is the sequence of fourth powers:**

$$1^4 \quad 2^4 \quad 3^4 \quad 4^4 \quad \ldots$$

# Moessner’s Construction for n!

**We begin by striking out the *triangular numbers*, $\{1, 3, 6, 10, 15, 21, \ldots\}$ and form partial sums.**

**Next, we delete the final entry in each group and form partial sums. This process is repeated indefinitely:**

| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
| | 2 | | 6 | 11 | | 18 | 26 | 35 | | 46 | 58 | 71 | 85 | | 101 |
| | | | 6 | | | 24 | 50 | | | 96 | 154 | 225 | | | 326 |
| | | | | | | 24 | | | | 120 | 274 | | | | 600 |
| | | | | | | | | | | 120 | | | | | 720 |

**This yields the *factorial numbers:***

$$1! \quad 2! \quad 3! \quad 4! \quad 5! \quad 6! \quad \ldots$$

# Beautiful Math

**The beauty of maths?**
***What do mathematicians think?***

**VIDEO: Beautiful Maths, available at**

**`http://momath.org/home/beautifulmath/`**

**Video by James Tanton**

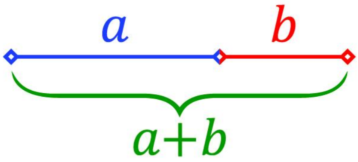
**Try to disregard the antipodean exuberance!**

# Outline

# Golden Rectangle in Your Pocket


**Aspect ratio is about $\phi = \frac{1+\sqrt{5}}{2} \approx 1.618$.**

# Geometric Ratio: $a+b$ is to $a$ as $a$ is to $b$.




$$\left[\frac{\text{Short Bit}}{\text{Long Bit}}\right] = \left[\frac{\text{Long Bit}}{\text{Full Line}}\right] \qquad \text{or} \qquad \frac{b}{a} = \frac{a}{a+b}$$

**Let the blue segment be $a = 1$ and the whole line $\phi$.**

**Then $b = \phi - 1$ and we have**

$$\frac{\phi - 1}{1} = \frac{1}{\phi}$$

$$\phi - 1 = \frac{1}{\phi}$$

**This means $\phi$ solves a quadratic equation:**

$$\phi^2 - \phi - 1 = 0$$

**Recall the two solutions of a quadratic equation**

$$ax^2 + bx + c = 0 \qquad \textbf{are} \qquad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

**In the present case, this means that the roots are**

$$\phi = \frac{1 \pm \sqrt{1+4}}{2}$$

**We take the *positive root,* giving**

$$\phi = \frac{1+\sqrt{5}}{2} \approx 1.618$$

**This is the golden ratio.**

# Golden Rectangle

**Ratio of breath to height is $\phi = \frac{1+\sqrt{5}}{2}$.**

# Golden Rectangle in Your Pocket


**Aspect ratio is about $\phi = \frac{1+\sqrt{5}}{2} \approx 1.618$.**

# Terminology

- Golden Ratio. Golden Number. Golden Mean.
- Golden Proportion. Golden Cut.
- Golden Section. Medial Section.
- *Divine Proportion.* Divine Section.
- Extreme and Mean Ratio.
- Various Other Terms.

# Fibonacci Numbers

**The Fibonacci sequence is the sequence**

$$\{0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \dots\}$$

**where *each number is the sum of the previous two.***

**The Fibonacci numbers obey a recurrence relation**

$$F_{n+1} = F_n + F_{n-1}$$

**with the *starting values* $F_0 = 0$ *and* $F_1 = 1$*.***

**Can we *solve this recurrence relation for all* $F_n$*?***

# Fibonacci Numbers

**The *recurrence relation* is**

$$F_{n+1} = F_n + F_{n-1}$$

**We assume that the solution is of the form $F_n = k\phi^n$, where we have to find $\phi$ (this is called an *Ansatz*).**

**Substitute this solution into the recurrence relation:**

$$k\phi^{n+1} = k\phi^n + k\phi^{n-1}$$

**Divide by $k\phi^{n-1}$ to get the quadratic equation**

$$\phi^2 = \phi + 1 \qquad \text{or} \qquad \phi^2 - \phi - 1 = 0$$

**This is the quadratic we got for the golden number.**

# Fibonacci Numbers

We found that $F_n = k\phi^n$ where $\phi$ is a root of

$$\phi^2 - \phi - 1 = 0$$

The two roots are

$$\frac{1+\sqrt{5}}{2} \qquad \text{and} \qquad \frac{1-\sqrt{5}}{2}$$

Then the full solution for the Fibonacci numbers is

$$F_n = \frac{1}{\sqrt{5}}\left[\frac{1+\sqrt{5}}{2}\right]^n - \frac{1}{\sqrt{5}}\left[\frac{1-\sqrt{5}}{2}\right]^n$$

Check that the conditions $F_0 = 0$ and $F_1 = 1$ are true.

# Fibonacci Numbers

$$F_n = \frac{1}{\sqrt{5}}\left[\frac{1+\sqrt{5}}{2}\right]^n - \frac{1}{\sqrt{5}}\left[\frac{1-\sqrt{5}}{2}\right]^n$$

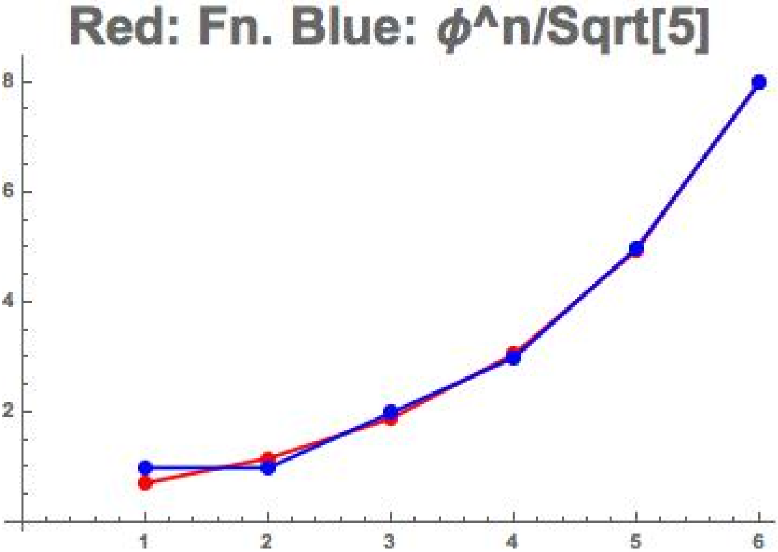
**The first term in square brackets is greater than 1, so the powers *grow rapidly with* $n$.**

**The second term in square brackets is less than 1, so the powers *become small rapidly with* $n$.**

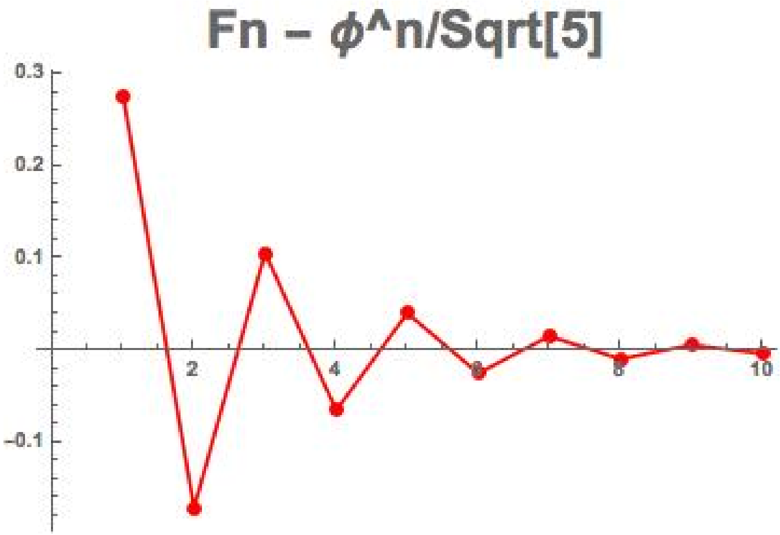
**So, we ignore the second term and write**

$$F_n \approx \frac{1}{\sqrt{5}}\left[\frac{1+\sqrt{5}}{2}\right]^n \qquad \textbf{or} \qquad F_n \approx \frac{\phi^n}{\sqrt{5}}$$

# Approximation to $F_n$



# Oscillating Error of Approximation



# Ratio $F_n / F_{n-1}$

$$F_n \approx \frac{\phi^n}{\sqrt{5}} \implies \frac{F_n}{F_{n-1}} \approx \phi$$

**Let's consider the sequence of ratios of terms**

$$\frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \frac{21}{13}, \frac{34}{21}, \dots$$

**The ratios get closer and closer to $\phi$:**

$$\frac{F_{n+1}}{F_n} \to \phi \quad \textbf{as} \quad n \to \infty$$

# Continued Fraction for $\phi$

$$\phi^2 - \phi - 1 = 0 \implies \phi = 1 + \frac{1}{\phi}$$

**Now use the equation to replace $\phi$ on the right:**

$$\phi = 1 + \frac{1}{\phi} = 1 + \frac{1}{1 + \frac{1}{\phi}} = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{\phi}}}$$

**Eventually**

$$\phi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}}$$

# Continued Root for $\phi$

$$\phi^2 - \phi - 1 = 0 \implies \phi = \sqrt{1+\phi}$$

**Now use the equation to replace $\phi$ on the right:**

$$\phi = \sqrt{1+\phi} = \sqrt{1+\sqrt{1+\phi}} = \sqrt{1+\sqrt{1+\sqrt{1+\phi}}}$$

**Eventually**

$$\phi = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\cdots}}}}}$$

# Fibonacci Numbers in Nature

**Look at post**

**Sunflowers and Fibonacci: Models of Efficiency**

**on the *ThatsMaths* blog.**

# Outline

# Carl Friedrich Gauss (1777–1855)

# Carl Friedrich Gauss (1777–1855)

A German mathematician who made profound contributions to many fields of mathematics:

- Number theory
- Algebra
- Statistics
- Analysis
- Differential geometry
- Geodesy & Geophysics
- Mechanics & Electrostatics
- Astronomy

Gauss is regarded as one of the greatest mathematicians of all time.

# Gauss Outsmarts his Teacher

**Gauss was a genius. He was known as**

***The Prince of Mathematicians.***

**When very young, Gauss outsmarted his teacher.**

**I can now reveal a fact unknown to historians:**

**The teacher got his own back. Ho! ho! ho!**

# Gauss Outsmarts his Teacher

Gauss's school teacher tasked the class:

- Add up all the whole numbers from 1 to 100.

Gauss solved the problem in a flash.

He wrote the correct answer,

$$5,050$$

on his slate and handed it to the teacher.

How did Gauss do it?

First, Gauss wrote the numbers in a row:

$$1 \quad 2 \quad 3 \quad \dots \quad 98 \quad 99 \quad 100$$

Next he wrote them again, *in reverse order:*

$$\begin{array}{ccccccc} 1 & 2 & 3 & \dots & 98 & 99 & 100 \\ 100 & 99 & 98 & \dots & 3 & 2 & 1 \end{array}$$

Then he added the two rows, column by column:

$$\begin{array}{ccccccc} 1 & 2 & 3 & \dots & 98 & 99 & 100 \\ 100 & 99 & 98 & \dots & 3 & 2 & 1 \\ \_\_\_ & \_\_\_ & \_\_\_ & \_\_\_ & \_\_\_ & \_\_\_ & \_\_\_ \\ 101 & 101 & 101 & \dots & 101 & 101 & 101 \end{array}$$

Clearly, the total for the two rows is 10,100.

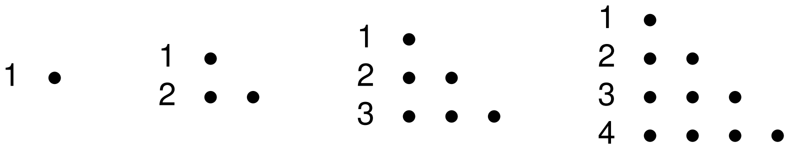
But every number from 1 to 100 is counted twice.

$$\therefore \; \mathbf{1 + 2 + 3 + \cdots + 98 + 99 + 100 = 5{,}050}$$

# Triangular Numbers

**Gauss had calculated the 100-th triangular number.**

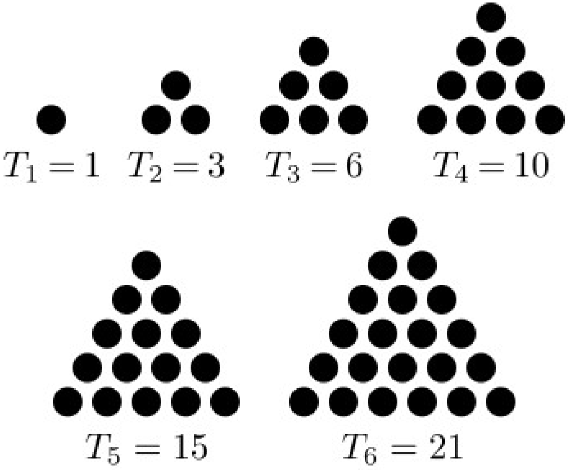
**Let us take a geometrical look at the sums of the first few natural numbers:**



**We see that the sums can be arranged as triangles.**

# Triangular Numbers

**The first few *triangular numbers* are** $\{1, 3, 6, 10, 15, 21\}$**.**



**Let's look at the 10th triangular number.**

**For** $n = 10$ **the pattern is:**

| | |
|---|---|
| 1 | • |
| 2 | • • |
| 3 | • • • |
| 4 | • • • • |
| 5 | • • • • • |
| 6 | • • • • • • |
| 7 | • • • • • • • |
| 8 | • • • • • • • • |
| 9 | • • • • • • • • • |
| 10 | • • • • • • • • • • |

**How do we compute its value? Gauss's method!**

It is easy to show that the $n$-th triangular number is

$$T_n = (1 + 2 + 3 + \cdots + n) = \tfrac{1}{2}n(n+1)$$

We do just as Gauss did, and list the numbers twice:

$$\begin{array}{cccccc} 1 & 2 & 3 & \dots & n-1 & n \\ n & n-1 & n-2 & \dots & 2 & 1 \\ --- & --- & --- & \dots & --- & --- \\ n+1 & n+1 & n+1 & \dots & n+1 & n+1 \end{array}$$

There are $n$ columns, each with total $n+1$.

So the grand total is $n \times (n+1)$.

Each number has been counted twice, so

$$T_n = \tfrac{1}{2}n(n+1)$$

Let's check this for Gauss's problem of $n = 100$:

$$T_{100} = 1 + 2 + 3 + \cdots + 100 = \frac{100 \times 101}{2} = 5,050$$

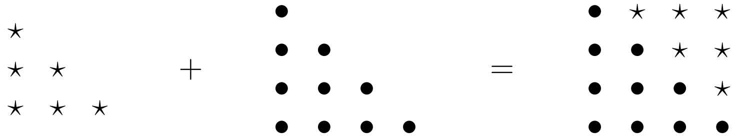
Gauss's approach was to look at the problem from a new angle.

Such *lateral thinking* is very common in mathematics:

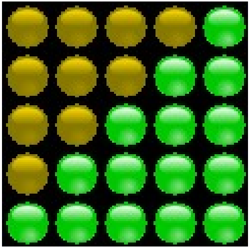
Problems that look difficult can sometimes be solved easily when tackled from a different angle.

# Two Triangles Make a Square

**A nice property of *consecutive* triangular numbers:**



$$T_3 + T_4 = 6 + 10 = 16 = 4^2$$



# Triangular Numbers

**We have seen, by means of geometry that the sum of two consecutive triangular numbers is a square.**

**Now let us prove this algebraically:**

$$\begin{aligned} T_n + T_{n+1} &= \tfrac{1}{2}n(n+1) + \tfrac{1}{2}(n+1)(n+2) \\ &= \tfrac{1}{2}(n+1)[n+(n+2)] \\ &= \tfrac{1}{2}(n+1)[2(n+1)] \\ &= (n+1)^2 \end{aligned}$$

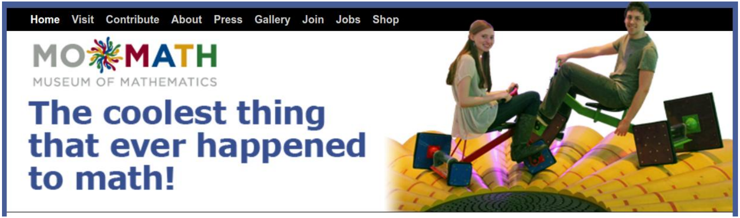
**The result is *a perfect square.***

# Puzzle

**What is the sum of all the numbers from 1 up to 100 and back down again?**

**The answer is in the video coming up now.**

# A Video from the Museum of Mathematics




**VIDEO: Beautiful Maths, available at**

**`http://momath.org/home/beautifulmath/`**

**Video by James Tanton**

# Gauss Outsmarted by his Teacher

The teacher thought that he would have a half-hour of peace and quiet while the pupils grappled with the problem of adding up the first 100 numbers.

He was annoyed when Gauss came up almost immediately with the correct answer 5,050.

So, he said:

"Oh, you zink you are zo zmart!
Zo, multiply ze first 100 numbers."

EXERCISE: Zink about that!

# A Lateral Thinking Puzzle

- Jill is 23 years younger than her father.
- What age was she when she was half his age?

**Hint: Be Smart**
**There is no need for tricky algebra.**

**Thank you**