

Sum-Enchanted Evenings

The Fun and Joy of Mathematics



LECTURE 9

Peter Lynch

**School of Mathematics & Statistics
University College Dublin**

Evening Course, UCD, Autumn 2017



Outline

Introduction

History of Astronomy II

Applications of Maths

Symmetry and Group Theory

Topology II

Distraction 4: A4 Paper Sheets



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Meaning and Content of Mathematics

The word **Mathematics** comes from Greek *μαθημα* (*máthéma*), meaning “knowledge” or “study” or “learning”.

It is the study of topics such as

- ▶ Quantity (numbers)
- ▶ Structure (patterns)
- ▶ Space (geometry)
- ▶ Change (analysis).



Reminder: Going upstairs in 1's and 2's.



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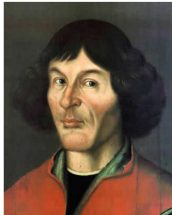
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The Scientific Revolution

INTRODUCTION

This week, we will look at developments in the sixteenth and seventeenth centuries.



Nicolaus Copernicus
1473 – 1543



Tycho Brahe
1546 – 1601



Johannes Kepler
1571 – 1630



Galileo Galilei
1564 – 1642

Figure from mathigon.org



The Heliocentric Model

In 1543, **Nicolaus Copernicus** (1473–1543) published *“On the Revolutions of the Celestial Spheres”*.

He explained that the Sun is at the centre of the universe and that the Earth and planets move around it in circular orbits.



The Heliocentric Model

In 1543, **Nicolaus Copernicus** (1473–1543) published *“On the Revolutions of the Celestial Spheres”*.

He explained that the Sun is at the centre of the universe and that the Earth and planets move around it in circular orbits.

Danish astronomer **Tycho Brahe** (1546–1601) made very accurate observations of the movements of the planets, and developed his own model of the solar system.



Johannes Kepler (1571–1630)

Johannes Kepler (1571–1630) succeeded Brahe as imperial mathematician.

After many years of struggling, Kepler had formulated his three Laws of Planetary Motion.

Kepler's Laws describe the solar system as we know it to be true today.



Kepler's Laws

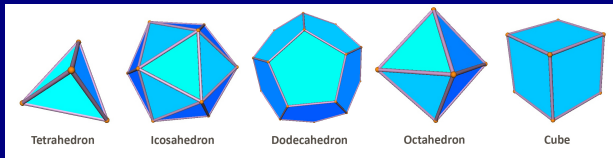
- ▶ **The planets move on elliptical orbits, with the Sun at one of the two foci.**
This explains why the Sun appears larger at some times of the year and smaller at others.
- ▶ **A line joining the planet and the Sun sweeps out equal areas in equal times.**
This means that a planet moves faster when close to the Sun, and slower when further away.
- ▶ **The square of the orbital period is proportional to the cube of the mean radius of the orbit.**
This law allows us to find the orbital time of a planet if we know the size of the orbit.



The *Mysterium Cosmographicum*

There were **six known planets** in Kepler's time:
Mercury, Venus, Earth, Mars, Jupiter, Saturn.

There are precisely **five platonic solids**:



This gave Kepler an extraordinary idea!

<https://thatsmaths.com/2016/10/13/>

[\keplers-magnificent-mysterium-cosmographicum/](#)



Galileo Galelii (1564–1630)

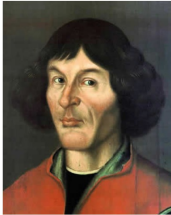
Galileo introduced the **telescope** to astronomy, and made some dramatic discoveries.

He observed the four large moons of Jupiter **revolving around that planet.**

He established the laws of inertia that underlie Newton's dynamical laws.



Four Remarkable Scientists



Nicolaus Copernicus
1473 – 1543



Tycho Brahe
1546 – 1601



Johannes Kepler
1571 – 1630



Galileo Galilei
1564 – 1642

Figure from mathigon.org



Isaac Newton (1642–1727)

In 1687, Isaac Newton published the **Principia Mathematica**. He established the mathematical foundations of dynamics.

Between any two masses there is a force:

$$F = \frac{GMm}{r^2}$$

This is the **force of gravity** and gravity is what makes the planets move around the Sun.

Newton's Laws imply and explain Kepler's laws.



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Applications on **mathigon.org**

Mathigon

All Topics For Teachers

Applications of Mathematics

Applications ▾ Topics ▾

- 
Maps of the Earth
- 
Predicting the Weather
- 
MRI and Tomography
- 
Supply Chains





Maps of the Earth



Predicting the Weather



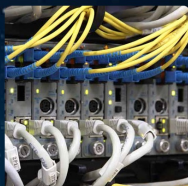
MRI and Tomography



Supply Chains



Finance and Banking



Internet and Phones



Cosmology



Computers





Construction



Reading CDs and DVDs



Glacier Melting



Public Key Cryptography



Satellite Navigation



Automotive Design



Codes and Communication

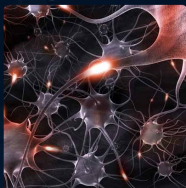


Building Bridges

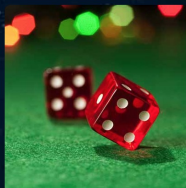




Digital Music



Neurology



Gambling and Betting



Search Engines



Epidemics Analysis



Navigation



Speech Recognition



Robotics





Football Scoring



Volcano Monitoring



Lottery



Roller Coaster Design



Breaking the Enigma



Public Transportation



Crowd Control



Insurance





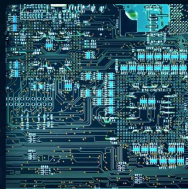
Space Observations



Computer Games



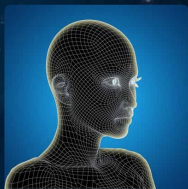
Carbon Dating



Computer Circuits



Making Music



Movie Graphics



Defence and Military



Traffic Optimisation





Rockets and Satellites



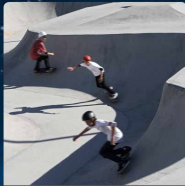
Problem Solving



Crime Prediction



Loans, Interest, Mortgages



Skate Park Design



Search for Alien Life

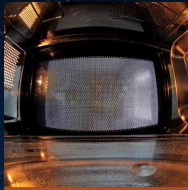


Fraud Detection



Big Data





Microwaves



Image Compression



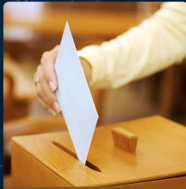
Pharmacy and Medicine



Swimsuit Design



Pricing Strategies



Polling and Voting



Music Shuffling



Tectonic Plate Motion





Game Theory



Population Dynamics



Coral Reef Growth



Erosion and Coastlines



Plastic Surgery



Diffusion of Liquids



Measuring Time



Cooking and Baking





Plastic Surgery



Diffusion of Liquids



Measuring Time



Cooking and Baking



Surveying



Making Chocolate



Wildfire Modelling



Counting Calories



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Symmetry and Group Theory

Symmetry is an essentially **geometric** concept.

The mathematical theory of symmetry is **algebraic**.

The key concept is that of a **group**.



Symmetry and Group Theory

Symmetry is an essentially **geometric** concept.

The mathematical theory of symmetry is **algebraic**.

The key concept is that of a **group**.

A group is a **set of elements** such that any two elements can be combined to produce another.

Instead of giving the mathematical **definition**,
I will give an **example** to make things clear.



The *Janus Group*: D_1

The group of symmetries of the human face and of all biological forms with **bilateral symmetry**.

I : The Identity transformation

R : Reflection about central line

Table : First Dihedral Group D_1 .

	I	R
I	I	R
R	R	I

This is how we combine, or **multiply** transformations.



Formal Definition of a Group

Let G be a set, for example

$$G = \{g_1, g_2, g_3, g_4\}$$

Suppose we can **combine** two elements g_1 and g_2 of G to get another element of G , say $g_1 \times g_2$.



Formal Definition of a Group

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For (G, \times) to be a group, we must have:

1. **Closure:** $g_1 \in G \wedge g_2 \in G \implies g_1 \times g_2 \in G$.
2. **Associativity:** $g_1 \times (g_2 \times g_3) = (g_1 \times g_2) \times g_3$.
3. **Identity:** $\exists e \in G : \forall g \in G \ e \times g = g \times e = g$.
4. **Inverse:** $\forall g \in G \exists h \in G : g \times h = h \times g = e$.



The Book Group (Klein 4-Group)

- I** : Identity transformation
- R** : Rotation about central point
- H** : Rotation about horizontal line
- V** : Rotation about vertical line.

Table : Second Dihedral Group D_2 .

	I	R	H	V
I	I	R	H	V
R	R	I	V	H
H	H	V	I	R
V	V	H	R	I



Elementary Examples of Symmetry

1. **Line segment $[a,b]$. Two symmetries:**
Leave alone or flip around.
2. **Full line. Double infinity:**
Translate or flip and translate ($\approx \mathbb{R}$).
3. **An equilateral triangle.**
4. **Polygon Rotations and Flips.**
5. **Cyclic Groups.**
6. **Symmetries of the Circle (an infinite Group).**
7. **Symmetries of Rectangle: Klein Group, Book Group. Dihedral Group D_2 .**
8. **Clock Group: $\mathbb{Z}/12\mathbb{Z} \approx \mathbb{Z}_{12}$. Integers modulo 12.**
9. **Integers \mathbb{Z}**



Group Isomorphisms

Two groups that are essentially the same, except for the names of the elements, are said to be **isomorphic**.

Isomorphic groups are like identical twins.

The Janus Group (symmetries of the face) and the symmetric group S_2 (permutations of two symbols) are isometric.



Symmetric Group or Permutation Group

The symmetric group S_n on a set of n symbols is the group of all the permutation operations of the n distinct symbols.

The group operation is *composition* of permutations. There are $n!$ permutations, so the order of S_n is $n!$.

Symmetric Group of order 3 (S_3) is isomorphic to the Dihedral group of order 3 (D_3).

The smallest non-abelian group has 6 elements. It is the dihedral group D_3 .

See Wikipedia: [Dihedral group of order 6](#)".



Symmetries of the Tetrahedron

A regular tetrahedron has 12 rotational symmetries, and a symmetry order of 24 including transformations that combine a reflection and a rotation.

The group of all symmetries is isomorphic to the group S_4 , the symmetric group of permutations of four objects.

There is exactly one symmetry for each permutation of the vertices of the tetrahedron.

The set of orientation-preserving symmetries forms a group referred to as the alternating subgroup A_4 of S_4 .



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Topology: a Major Branch of Mathematics

Topology is all about **continuity** and **connectivity**.

Here are some of the topics in Topology:

- ▶ The Bridges of Königsberg
- ▶ Doughnuts and Coffee-cups
- ▶ Knots and Links
- ▶ Nodes and Edges: Graphs
- ▶ The Möbius Band

In this lecture, we look at **Knots and Links**.



Pretzel Puzzle

Look at the two “pretzels” here:

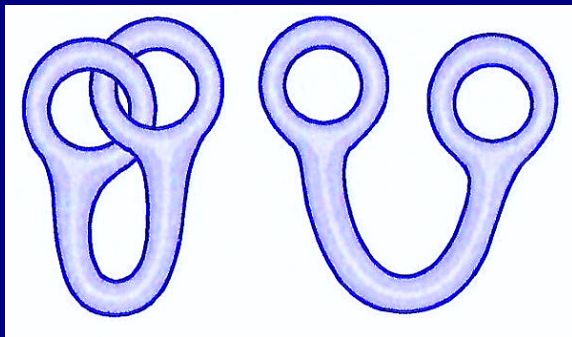


Figure : Two “Pretzels”. Are they equivalent?



Knot Theory

A **knot** is an embedding of the unit circle S^1 into three-dimensional space \mathbb{R}^3 .

Two knots are equivalent if one can be distorted into the other without breaking it.



A knot is a mapping of the unit circle into three-space.

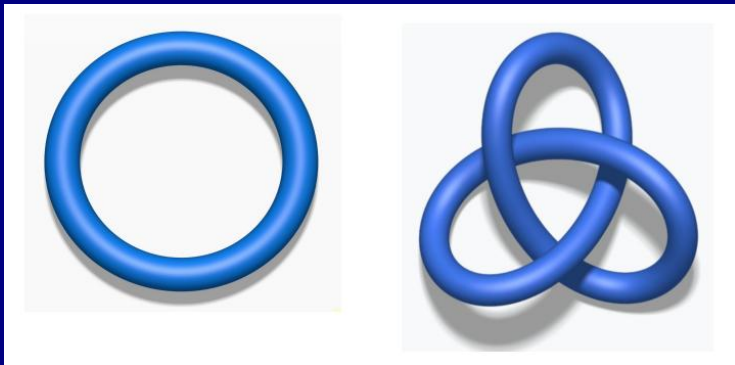


Figure : Left: Unknot. Right: Trefoil.

These two knots aren't equivalent: we can't distort the circle into the trefoil without breaking it.



Knots that are Mirror Images

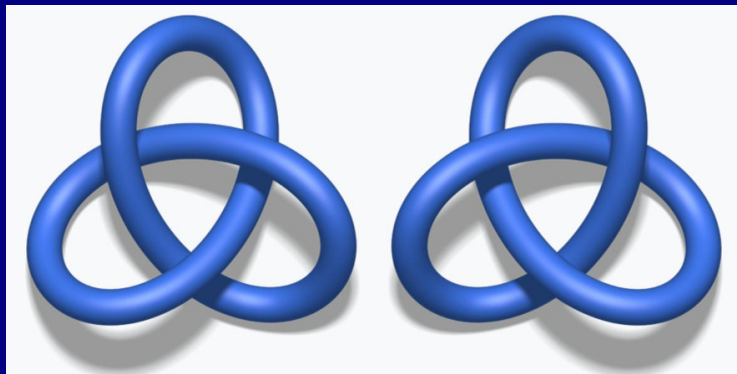


Figure : Levo and Dextro Trefoils.

These knots are mirror images but are not equivalent. We cannot change one into the other without breaking it.



The Simplest Knots and Links

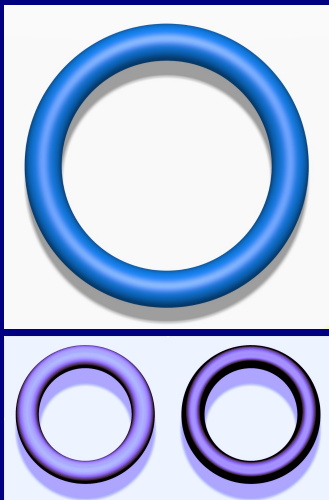


Figure : Top: The Unknot. Bottom: The Unlink.



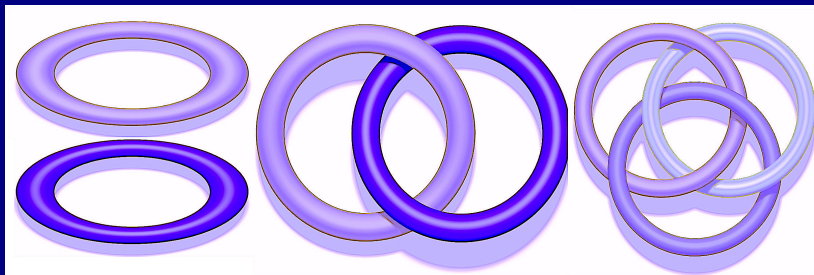


Figure : Unlink, Hopf Link and Borromean Rings.

The Hopf Link

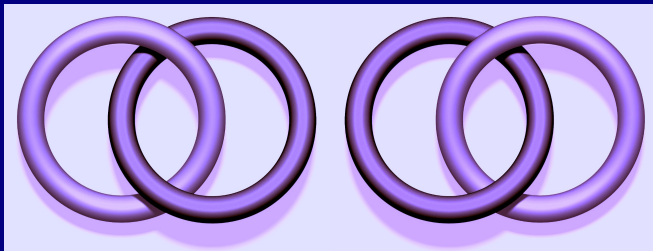


Figure : The Hopf Link and its mirror image. Equivalent?

Rings of Borromeo

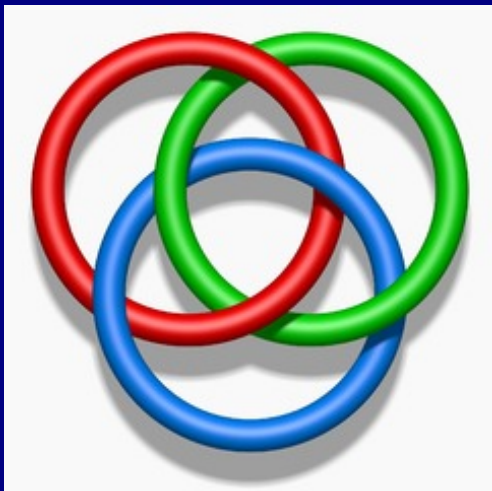


Figure : No two rings are linked! Are the three?



Genus of a Surface

The **genus** of a topological surface is, in simple terms, the **number of holes in it**.

A sphere has no holes, so has genus 0.

A donut has one hole, so has genus 1.

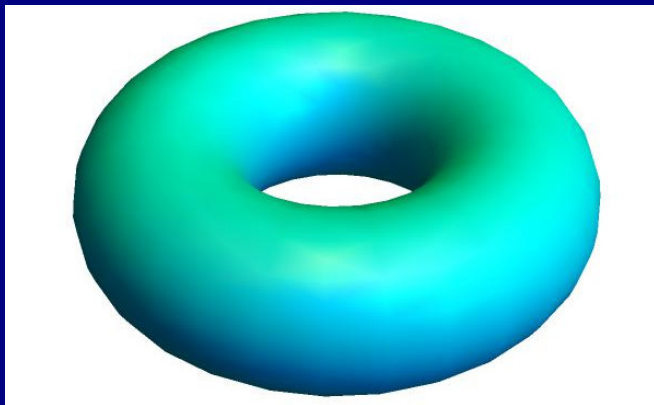
Surfaces can have any number of holes; any genus.



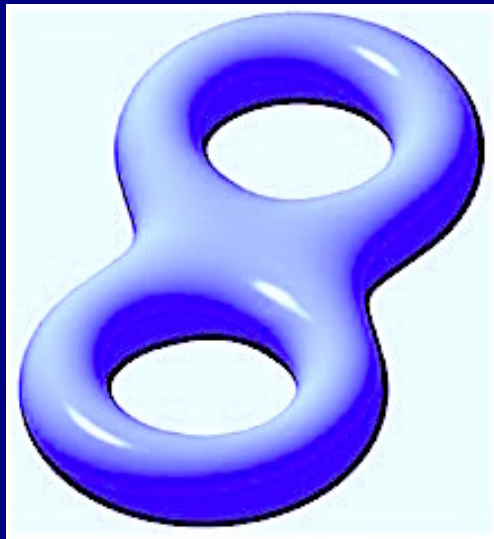
The Sphere, of Genus 0



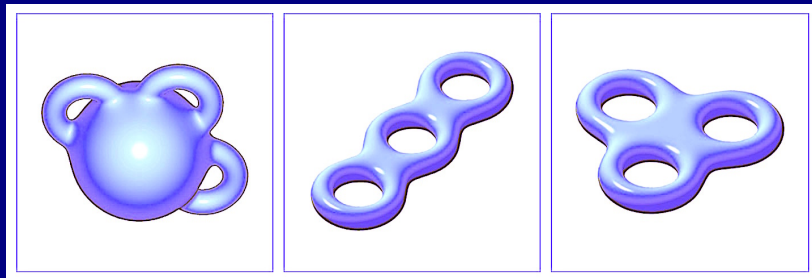
The Torus, of Genus 1



The Double Torus, of Genus 2



Some Surfaces of Genus 3



Topologists have classified all surfaces in 3-space.



Link between Number Theory and Physics

Forty years ago, physics and topology had little or nothing to do with one another.

In the 1980s, mathematicians and physicists found ways to use physics to study the properties of shapes.

The field has never looked back.

<http://www.quantamagazine.org/secret-link-uncovered-between-pure-math-and-physics-20171201/>



Triple Torus

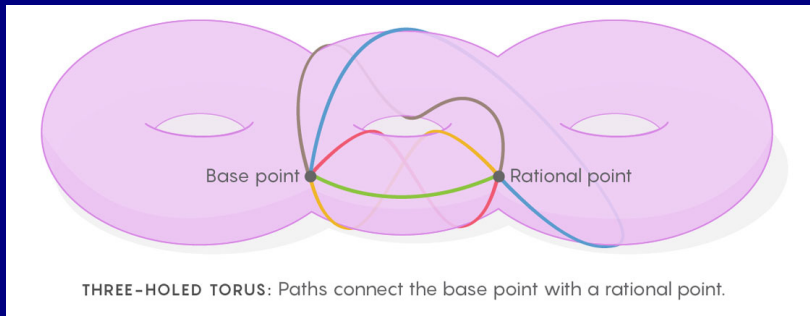


Figure : Rational solutions of $x^4 + y^4 = 1$ are on this surface



Pretzel Puzzle

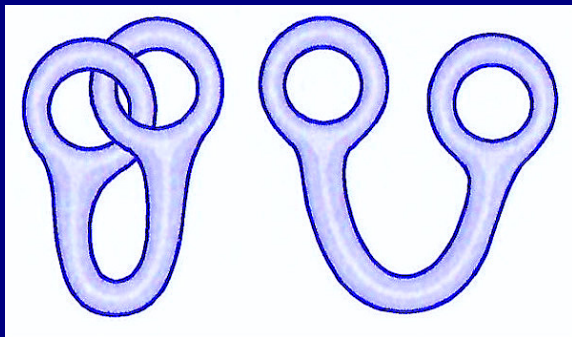


Figure : Two “Pretzels”. Are they equivalent?

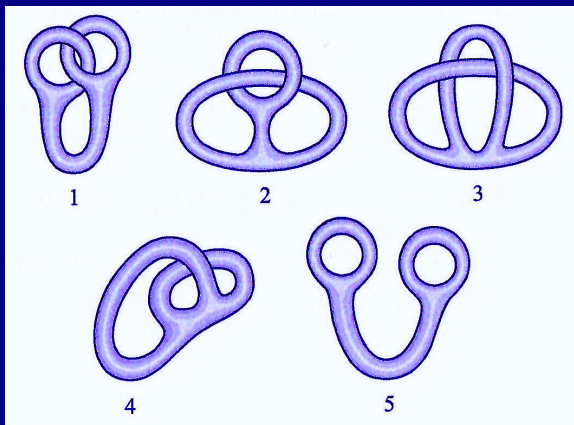


Figure : Equivalence!

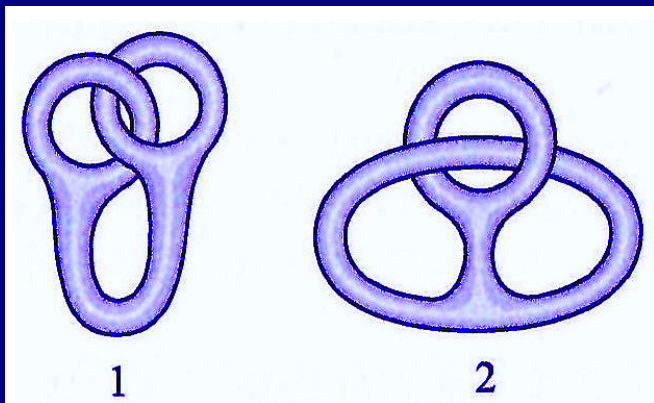


Figure : Make the left-hand loop bigger.

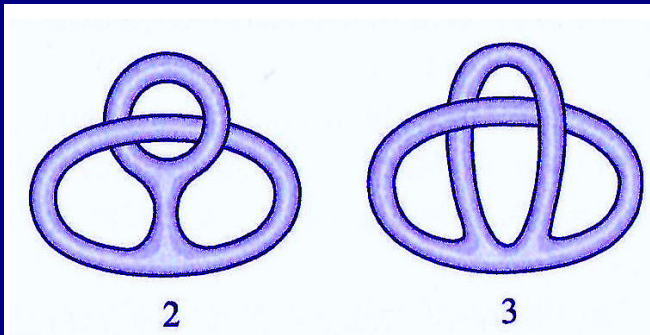


Figure : Make the other loop bigger.

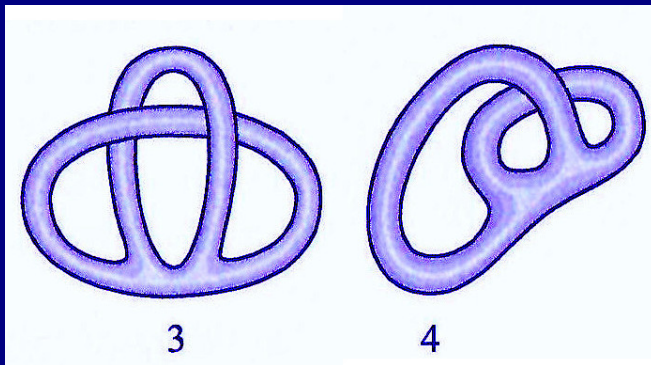


Figure : Pull the top loop away to the side.

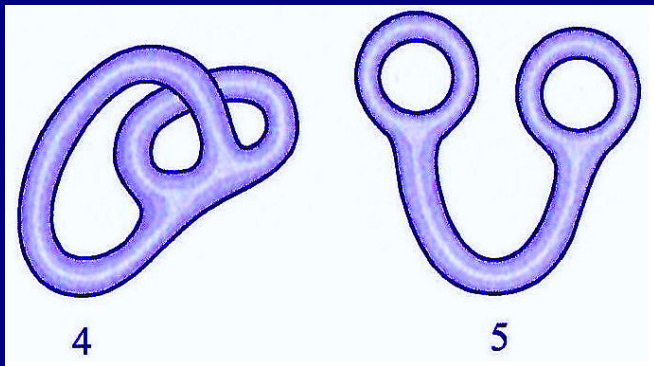


Figure : Smoothly distort to the final form.

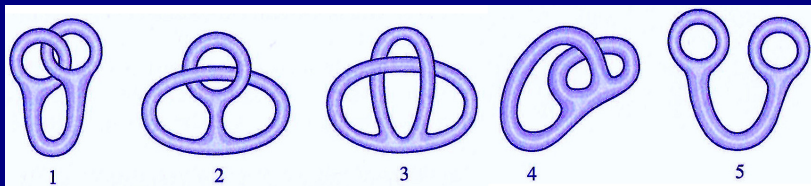


Figure : Combining all the distortions. Equivalence!

Another Surprising Result

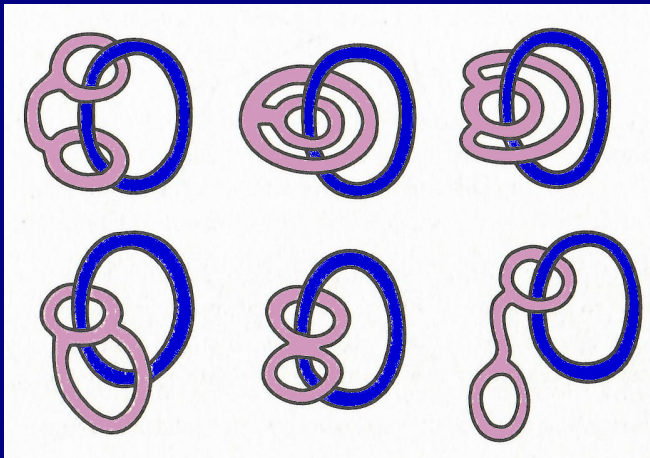


Figure : We can unlink one of the hand-cuffs.

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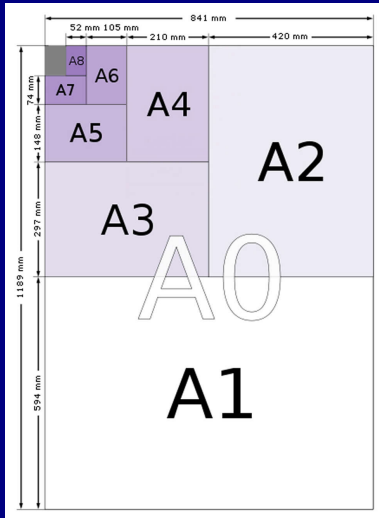
Symmetry and Group Theory

Topology II

Distraction 4: A4 Paper Sheets



Standard Paper Sizes



Standard sizes of
A-series paper.

The ratio of heights to
widths is always $\sqrt{2}$.



Making a Square

The standard sizes of paper are designed so that each has the same shape (or aspect ratio), and the largest, A0, has an area of one square metre.

PUZZLE:

Is it possible to form a square out of sheets of A4 sized paper (without them overlapping)?



Thank you

