Sum-Enchanted Evenings

The Fun and Joy of Mathematics

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LECTURE 9

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School of Mathematics & Statistics
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Outline

Introduction

History of Astronomy II

Applications of Maths

Symmetry and Group Theory

Topology II

Distraction 4: A4 Paper Sheets

Outline

Meaning and Content of Mathematics

The word Mathematics comes from Greek $\mu\alpha\theta\eta\mu\alpha$ (máthéma), meaning "knowledge" or "study" or "learning".

It is the study of topics such as

- Quantity (numbers)
- Structure (patterns)
- Space (geometry)
- Change (analysis).

Reminder: Going upstairs in 1's and 2's.

Outline

The Scientific Revolution

INTRODUCTION

This week, we will look at developments in the sixteenth and seventeenth centuries.

Nicolaus Copernicus
1473 – 1543

Tycho Brahe
1546 – 1601

Johannes Kepler
1571 – 1630

Galileo Galilei
1564 – 1642

Figure from mathigon.org

The Heliocentric Model

In 1543, *Nicolaus Copernicus* (1473–1543) published *“On the Revolutions of the Celestial Spheres”*.

He explained that the Sun is at the centre of the universe and that the Earth and planets move around it in circular orbits.

Danish astronomer *Tycho Brahe* (1546–1601) made very accurate observations of the movements of the planets, and developed his own model of the solar system.

Johannes Kepler (1571–1630)

Johannes Kepler (1571–1630) succeeded Brahe as imperial mathematician.

After many years of struggling, Kepler had formulated his three Laws of Planetary Motion.

Kepler's Laws describe the solar system as we know it to be true today.

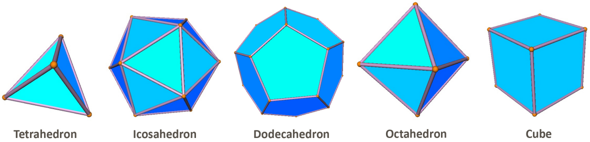
Kepler's Laws

- **The planets move on elliptical orbits, with the Sun at one of the two foci. This explains why the Sun appears larger at some times of the year and smaller at others.**
- **A line joining the planet and the Sun sweeps out equal areas in equal times. This means that a planet moves faster when close to the Sun, and slower when further away.**
- **The square of the orbital period is proportional to the cube of the mean radius of the orbit. This law allows us to find the orbital time of a planet if we know the size of the orbit.**

The *Mysterium Cosmographicum*

There were six known planets in Kepler's time: Mercury, Venus, Earth, Mars, Jupiter, Saturn.

There are precisely five platonic solids:



This gave Kepler an extraordinary idea!

`https://thatsmaths.com/2016/10/13/`

`\keplers-magnificent-mysterium-cosmographicum/`

Galileo Galelii (1564–1630)

Galileo introduced the *telescope* to astronomy, and made some dramatic discoveries.

He observed the four large moons of Jupiter *revolving around that planet.*

He established the laws of inertia that underlie Newton's dynamical laws.

Four Remarkable Scientists

Figure from mathigon.org

Isaac Newton (1642–1727)

In 1687, Isaac Newton published the Principia Mathematica. He established the mathematical foundations of dynamics.

Between any two masses there is a force:

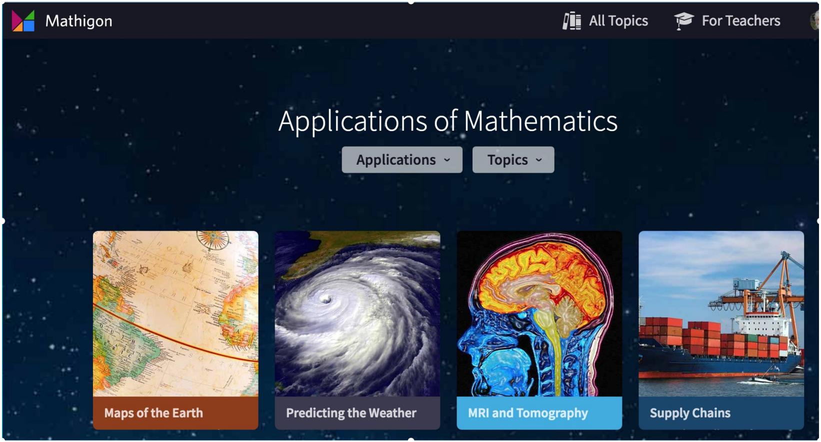
$$F = \frac{GMm}{r^2}$$

This is the force of gravity and gravity is what makes the planets move around the Sun.

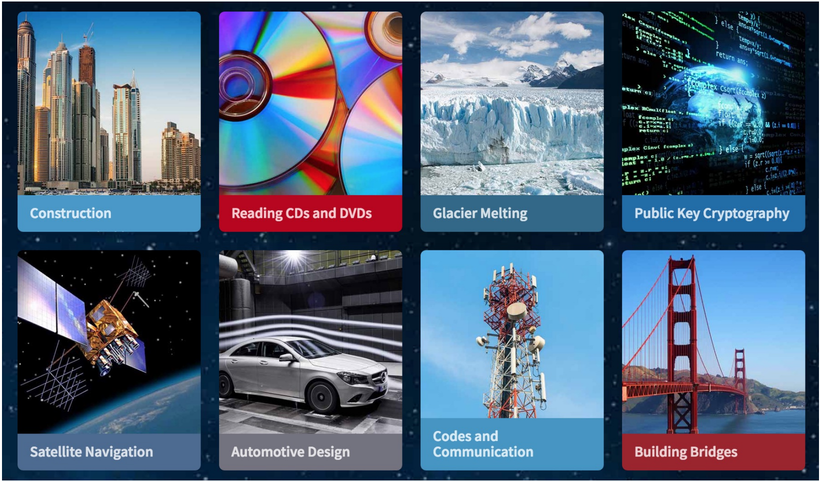
Newton's Laws imply and explain Kepler's laws.

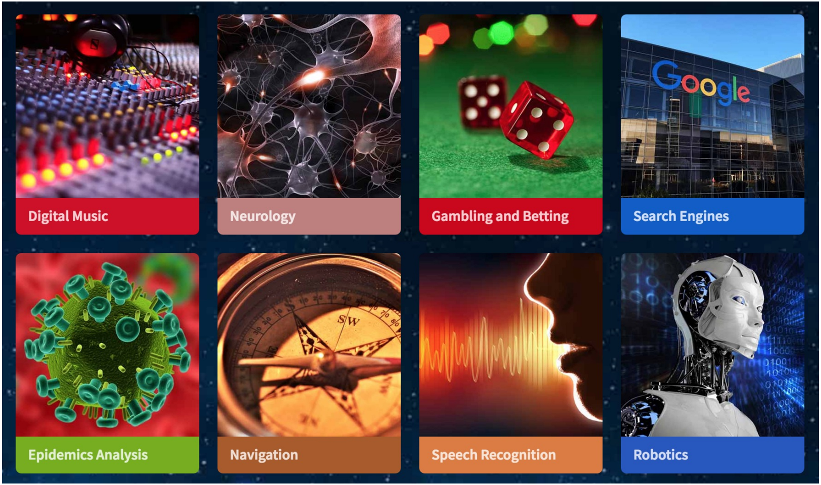
Outline

Applications on mathigon.org



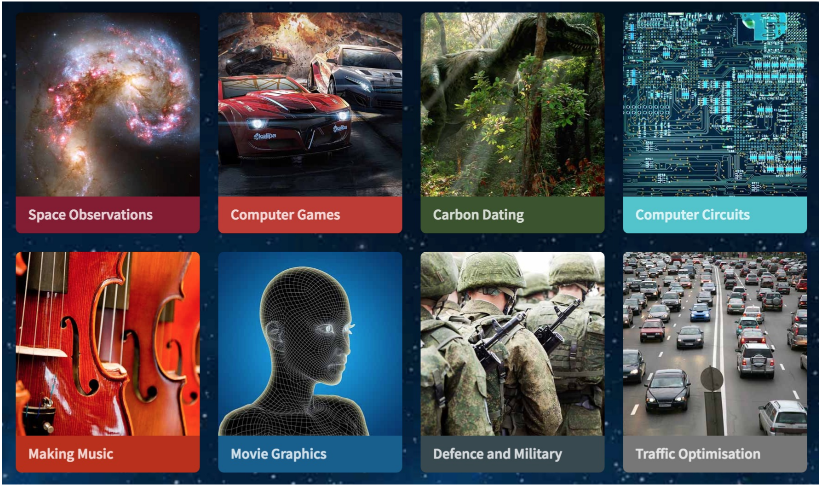

Maps of the Earth
Predicting the Weather
MRI and Tomography
Supply Chains
Finance and Banking
Internet and Phones
Cosmology
Computers

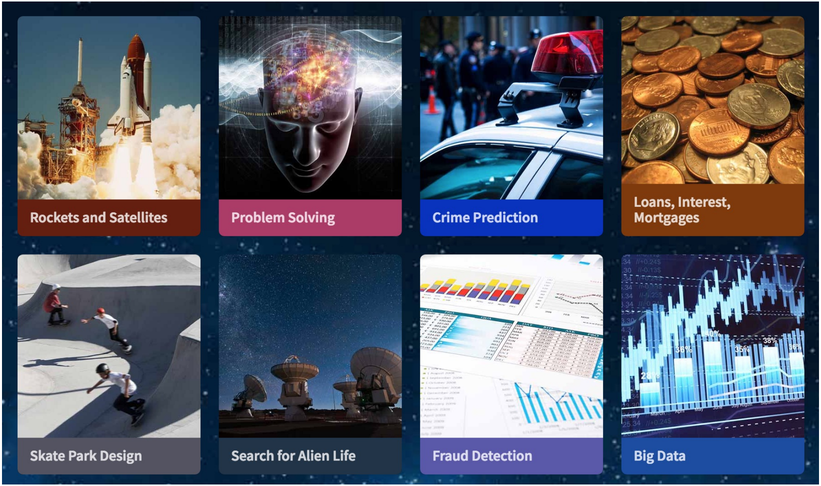

Construction
Reading CDs and DVDs
Glacier Melting
Public Key Cryptography
Satellite Navigation
Automotive Design
Codes and Communication
Building Bridges

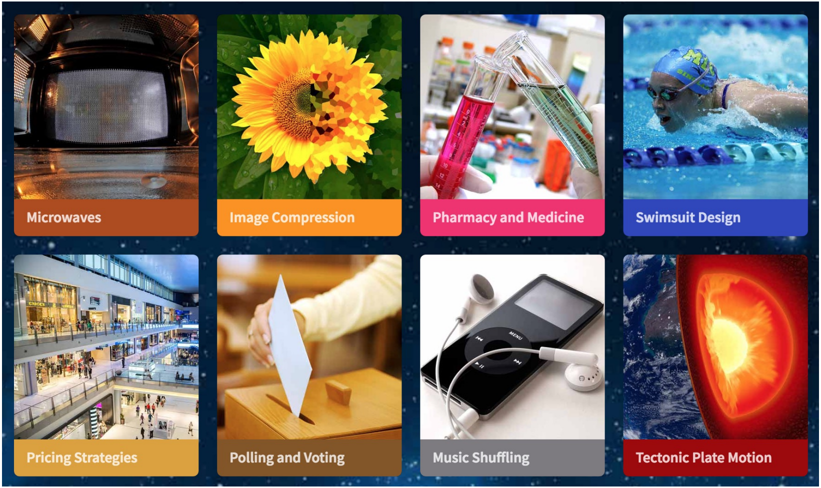


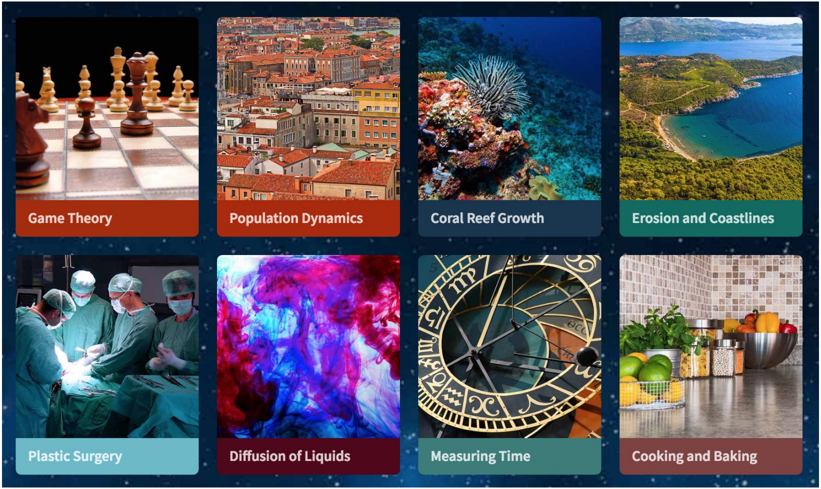
Digital Music
Neurology
Gambling and Betting
Google
Search Engines
Epidemics Analysis
Navigation
Speech Recognition
Robotics

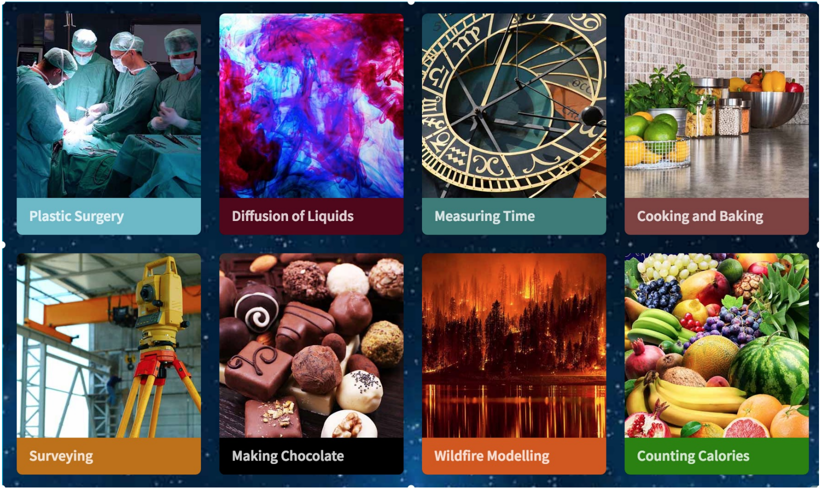

Football Scoring
Volcano Monitoring
Lottery
Roller Coaster Design
Breaking the Enigma
Public Transportation
Crowd Control
Insurance


Space Observations
Computer Games
Carbon Dating
Computer Circuits
Making Music
Movie Graphics
Defence and Military
Traffic Optimisation


Rockets and Satellites
Problem Solving
Crime Prediction
Loans, Interest, Mortgages
Skate Park Design
Search for Alien Life
Fraud Detection
Big Data


Microwaves
Image Compression
Pharmacy and Medicine
Swimsuit Design
Pricing Strategies
Polling and Voting
Music Shuffling
Tectonic Plate Motion


Game Theory
Population Dynamics
Coral Reef Growth
Erosion and Coastlines
Plastic Surgery
Diffusion of Liquids
Measuring Time
Cooking and Baking


Plastic Surgery
Diffusion of Liquids
Measuring Time
Cooking and Baking
Surveying
Making Chocolate
Wildfire Modelling
Counting Calories

Outline

Symmetry and Group Theory

Symmetry is an essentially *geometric* concept.

The mathematical theory of symmetry is *algebraic.*

The key concept is that of a group.

A group is a *set of elements* such that any two elements can be combined to produce another.

Instead of giving the mathematical definition, I will give an example to make things clear.

The *Janus Group*: D_1

The group of symmetries of the human face and of all biological forms with bilateral symmetry.

I : The Identity transformation

R : Reflection about central line

Table : First Dihedral Group D_1.

	I	R
I	I	R
R	R	I

This is how we combine, or *multiply* transformations.

Formal Definition of a Group

Let G be a set, for example

$$G = \{g_1, g_2, g_3, g_4\}$$

Suppose we can *combine* two elements g_1 and g_2 of G to get another element of G, say $g_1 \times g_2$.

For $(G, \times)$ to be a group, we must have:

1. **Closure:** $g_1 \in G \wedge g_2 \in G \implies g_1 \times g_2 \in G$.
2. **Associativity:** $g_1 \times (g_2 \times g_3) = (g_1 \times g_2) \times g_3$.
3. **Identity:** $\exists e \in G : \forall g \in G \;\; e \times g = g \times e = g$.
4. **Inverse:** $\forall g \in G \; \exists h \in G : g \times h = h \times g = e$.

The Book Group (Klein 4-Group)

I : **Identity transformation**
R : **Rotation about central point**
H : **Rotation about horizontal line**
V : **Rotation about vertical line.**

Table : Second Dihedral Group D_2.

	I	R	H	V
I	I	R	H	V
R	R	I	V	H
H	H	V	I	R
V	V	H	R	I

Elementary Examples of Symmetry

1. Line segment [a,b]. Two symmetries:
   Leave alone or flip around.
2. Full line. Double infinity:
   Translate or flip and translate ($\approx \mathbb{R}$).
3. An equilateral triangle.
4. Polygon Rotations and Flips.
5. Cyclic Groups.
6. Symmetries of the Circle (an infinite Group).
7. Symmetries of Rectangle: Klein Group, Book Group. Dihedral Group D_2.
8. Clock Group: $\mathbb{Z}/12\mathbb{Z} \approx \mathbb{Z}_{12}$. Integers modulo 12.
9. Integers $\mathbb{Z}$

Group Isomorphisms

Two groups that are essentially the same, except for the names of the elements, are said to be isomorphic.

Isomorphic groups are like identical twins.

The Janus Group (symmetries of the face) and the symmetric group S_2 (permutations of two symbols) are isometric.

Symmetric Group or Permutation Group

The symmetric group S_n on a set of n symbols is the group of all the permutation operations of the n distinct symbols.

The group operation is *composition* of permutations. There are $n!$ permutations, so the order of S_n is $n!$.

Symmetric Group of order 3 (S_3) is isomorphic to the Dihedral group of order 3 (D_3).

The smallest non-abelian group has 6 elements. It is the dihedral group D_3.

See Wikipedia: *Dihedral group of order 6".*

Symmetries of the Tetrahedron

A regular tetrahedron has 12 rotational symmetries, and a symmetry order of 24 including transformations that combine a reflection and a rotation.

The group of all symmetries is isomorphic to the group S_4, the symmetric group of permutations of four objects.

There is exactly one symmetry for each permutation of the vertices of the tetrahedron.

The set of orientation-preserving symmetries forms a group referred to as the alternating subgroup A_4 of S_4.

Outline

Topology: a Major Branch of Mathematics

Topology is all about continuity and connectivity.

Here are some of the topics in Topology:

- **The Bridges of Königsberg**
- **Doughnuts and Coffee-cups**
- **Knots and Links**
- **Nodes and Edges: Graphs**
- **The Möbius Band**

In this lecture, we look at *Knots and Links.*

Pretzel Puzzle

Look at the two "pretzels" here:

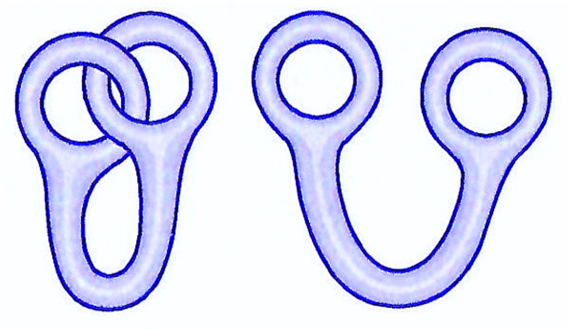


Figure : Two "Pretzels". Are they equivalent?

Knot Theory

A knot is an embedding of the unit circle S^1 into three-dimensional space R^3.

Two knots are equivalent if one can be distorted into the other without breaking it.

A knot is a mapping of the unit circle into three-space.

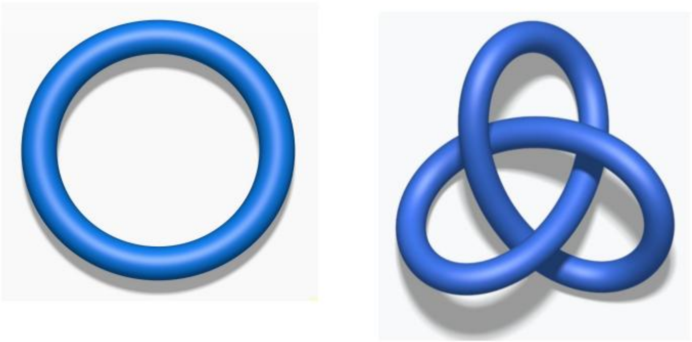


Figure : Left: Unknot. Right: Trefoil.

These two knots aren't equivalent: we can't distort the circle into the trefoil without breaking it.

Knots that are Mirror Images

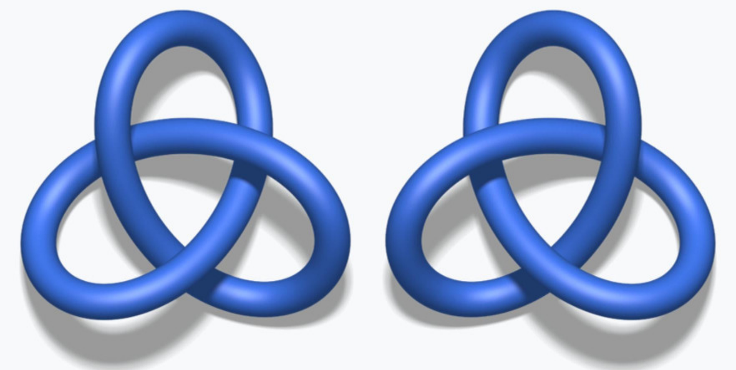


Figure : Levo and Dextro Trefoils.

These knots are mirror images but are not equivalent. We cannot change one into the other without breaking it.

The Simplest Knots and Links

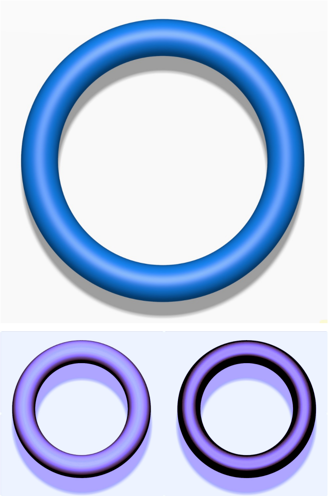


Figure : Top: The Unknot. Bottom: The Unlink.

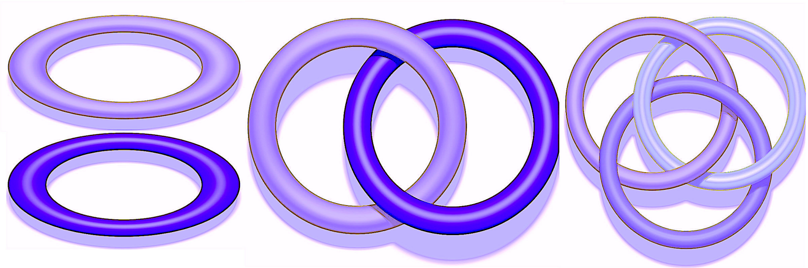


Figure : Unlink, Hopf Link and Borromean Rings.

The Hopf Link

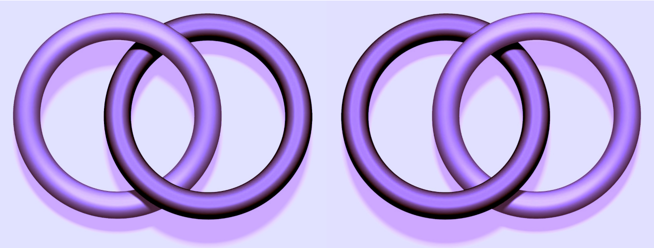


Figure : The Hopf Link and its mirror image. Equivalent?

Rings of Borromeo

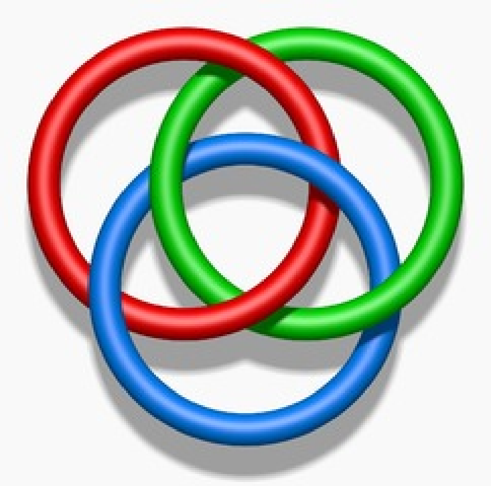


Figure : No two rings are linked! Are the three?

Genus of a Surface

The genus of a topological surface is,
in simple terms, the number of holes in it.

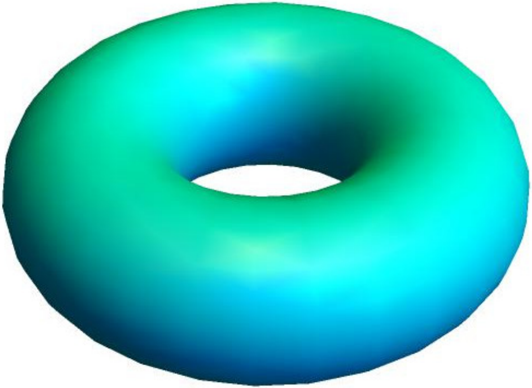
A sphere has no holes, so has genus 0.

A donut has one hole, so has genus 1.

Surfaces can have any number of holes; any genus.

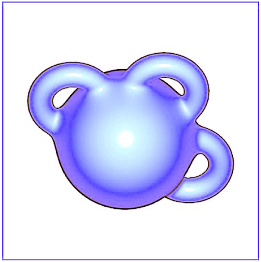
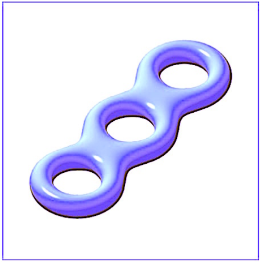
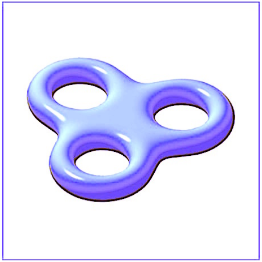
The Sphere, of Genus 0

The Torus, of Genus 1



The Double Torus, of Genus 2

Some Surfaces of Genus 3

Topologists have classified all surfaces in 3-space.

Link between Number Theory and Physics

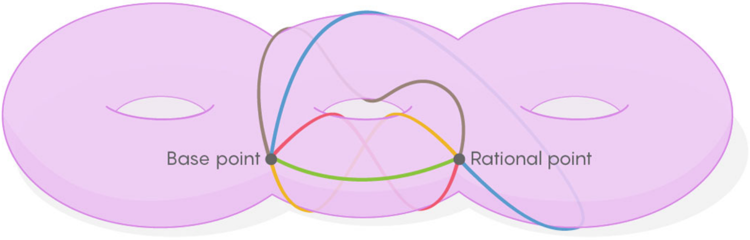
Forty years ago, physics and and topology had little or nothing to do with one another.

In the 1980s, mathematicians and physicists found ways to use physics to study the properties of shapes.

The field has never looked back.

`http://www.quantamagazine.org/secret-link-uncovered-between-pure-math-and-physics-20171201/`

Triple Torus



THREE-HOLED TORUS: Paths connect the base point with a rational point.

Figure : Rational solutions of $x^4 + y^4 = 1$ are on this surface

Pretzel Puzzle

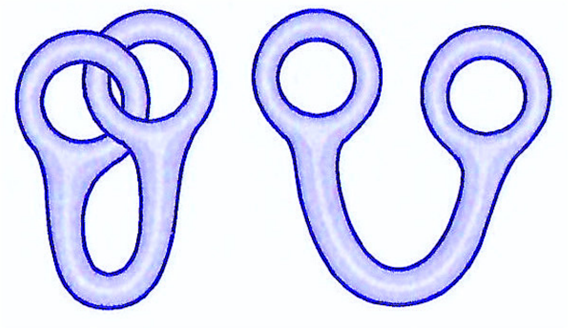


Figure : Two “Pretzels”. Are they equivalent?

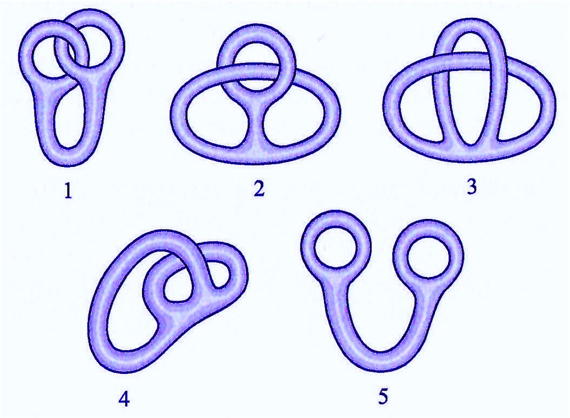


Figure : Equivalence!

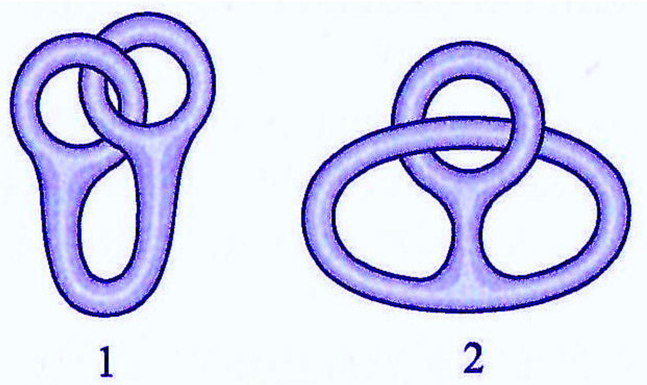


Figure : Make the left-hand loop bigger.

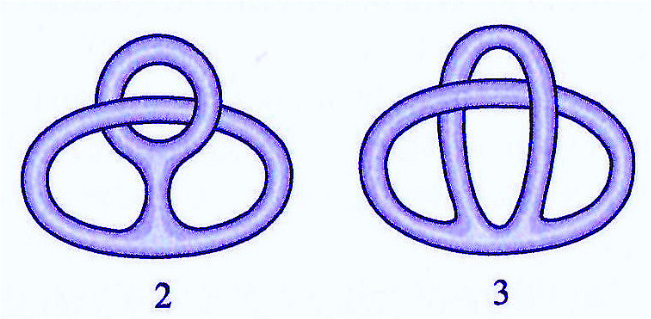


Figure : Make the other loop bigger.

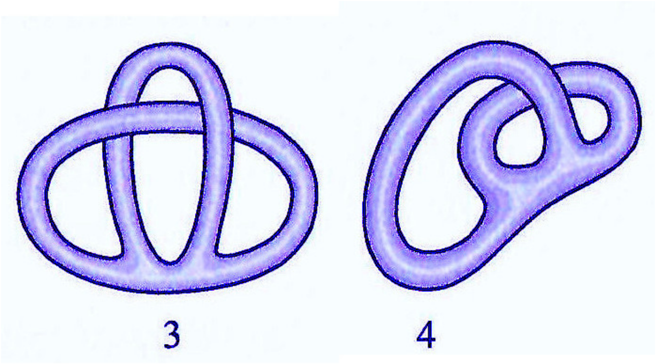


Figure : Pull the top loop away to the side.

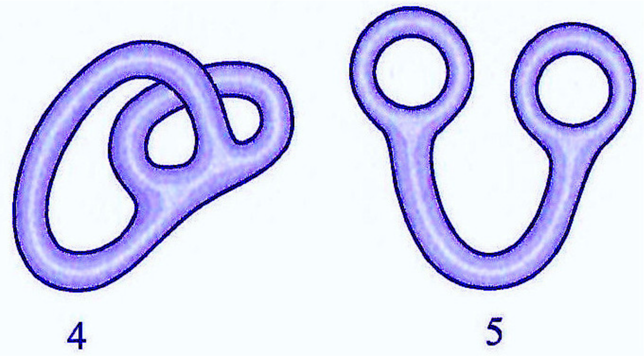


Figure : Smoothly distort to the final form.

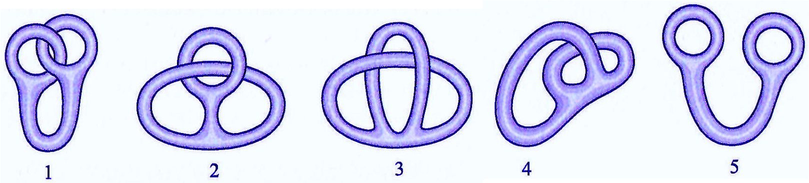


Figure : Combining all the distortions. Equivalence!

Another Surprising Result

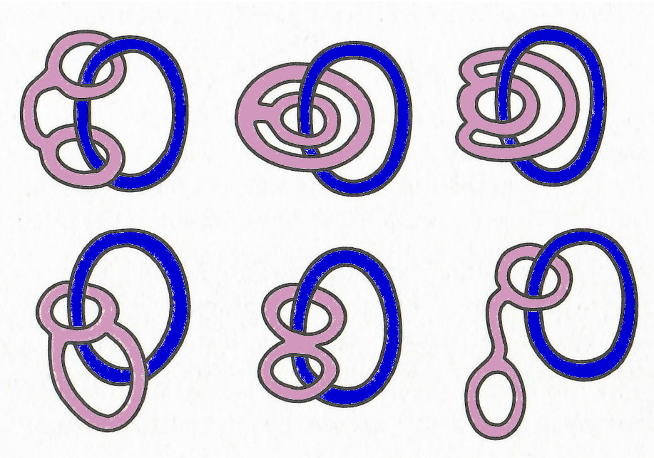


Figure : We can unlink one of the hand-cuffs.

Outline

Introduction

History of Astronomy II

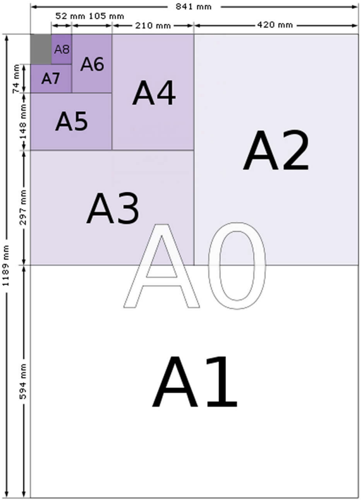
Applications of Maths

Symmetry and Group Theory

Topology II

Distraction 4: A4 Paper Sheets

Standard Paper Sizes



Standard sizes of A-series paper.

The ratio of heights to widths is always $\sqrt{2}$.

Making a Square

The standard sizes of paper are designed so that each has the same shape (or aspect ratio), and the largest, A0, has an area of one square metre.

PUZZLE:
Is it possible to form a square out of sheets of A4 sized paper (without them overlapping)?

Thank you