Sum-Enchanted Evenings

The Fun and Joy of Mathematics

•

LECTURE 8

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School of Mathematics & Statistics
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Evening Course, UCD, Autumn 2017

Outline

Outline

Introduction

Euler's Gem

History of Astronomy I

Distraction 8: Sum by Inspection

The Real Number Line

Symmetries of Triangle and Square

Meaning and Content of Mathematics

The word Mathematics comes from Greek $\mu\alpha\theta\eta\mu\alpha$ (máthéma), meaning "knowledge" or "study" or "learning".

It is the study of topics such as

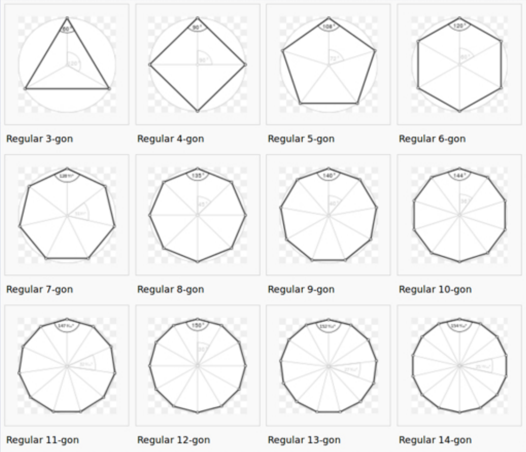
- Quantity (numbers)
- Structure (patterns)
- Space (geometry)
- Change (analysis).

Outline

Euler's polyhedron formula.

Carving up the globe.

Regular Polygons



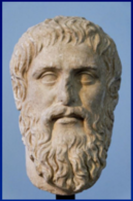
The Platonic Solids
(polyhedra)

Tetrahedron (four faces)	Cube or hexahedron (six faces)	Octahedron (eight faces)	Dodecahedron (twelve faces)	Icosahedron (twenty faces)
				

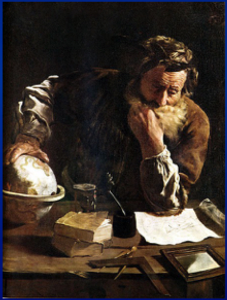
These five regular polyhedra were discovered in ancient Greece, perhaps by Pythagoras.

Plato used them as models of the universe.

They are analysed in Book XIII of Euclid's Elements.

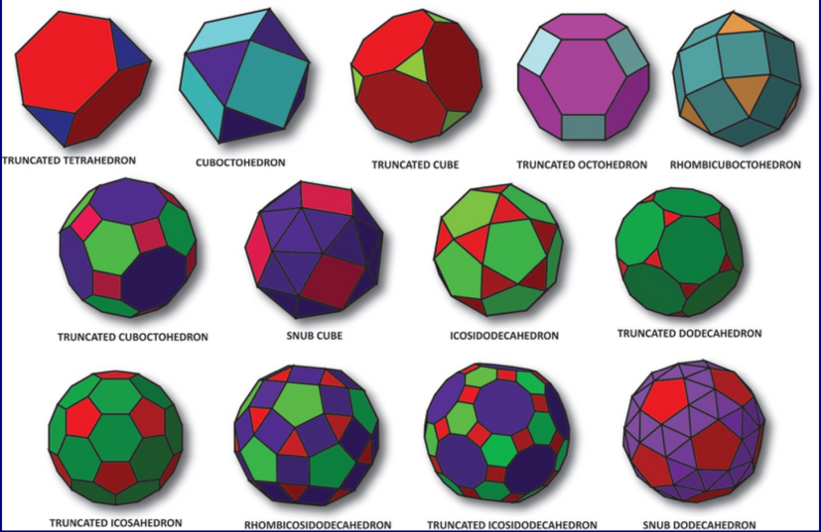


There are only five Platonic solids.



But Archimedes found, using different types of polygons, that he could construct 13 new solids.

The Thirteen Archimedean Solids



Euler's Polyhedron Formula

The great Swiss mathematician, Leonard Euler, noticed that, for all (convex) polyhedra,

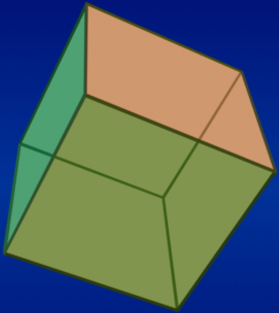
$$V - E + F = 2$$

where

- V = Number of vertices
- E = Number of edges
- F = Number of faces

Mnemonic: *Very Easy Formula*

For example, a Cube

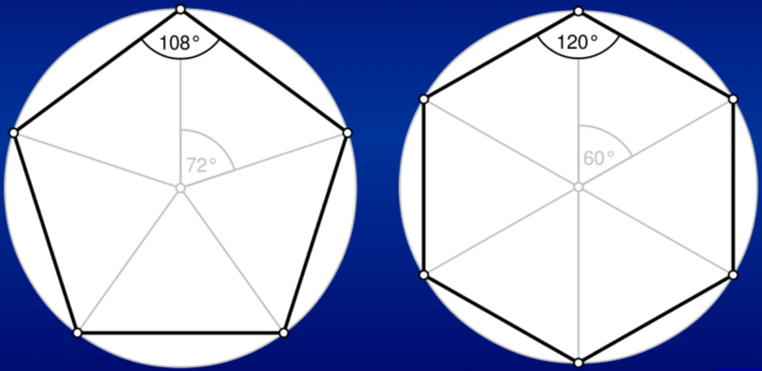


Number of vertices: V = 8
Number of edges: E = 12
Number of faces: F = 6

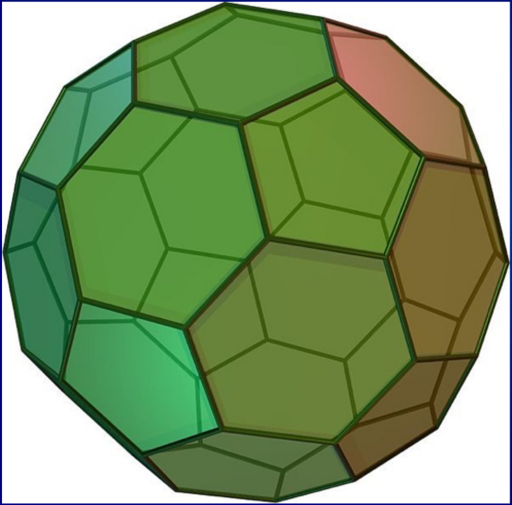
$$(V - E + F) = (8 - 12 + 6) = 2$$

Mnemonic: *Very Easy Formula*

Pentagons and Hexagons



The Truncated Icosahedron

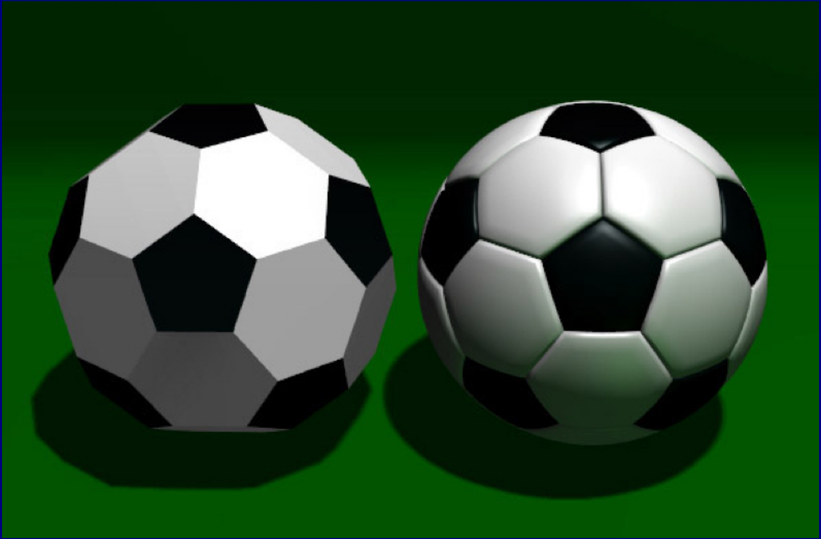


An Archimedean solid
with
pentagonal and
hexagonal faces.

The Truncated Icosahedron

Whare have you seen this before?

The Truncated Icosahedron



The "**Buckyball**", introduced at the
1970 World Cup Finals in Mexico.

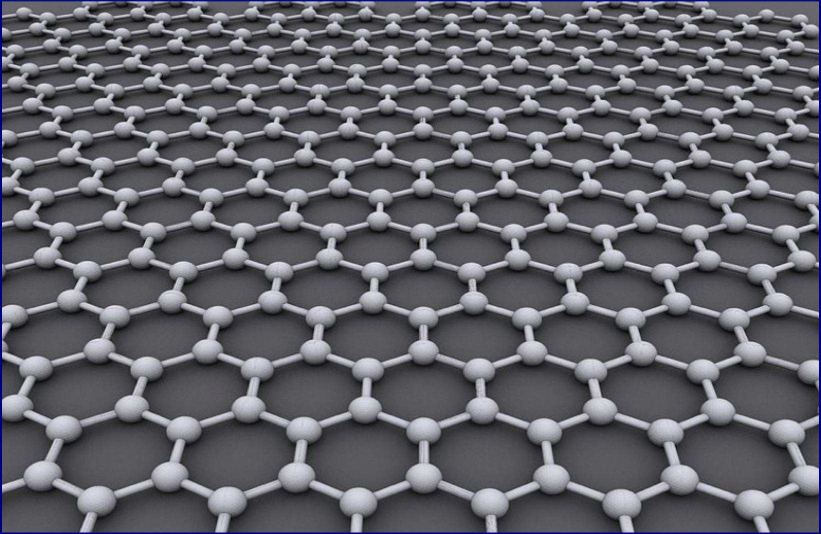
It has 32 panels: 20 hexagons and 12 pentagons.


A Geodesic Dome designed by the American architect
Richard Buckminster "Bucky" Fuller.

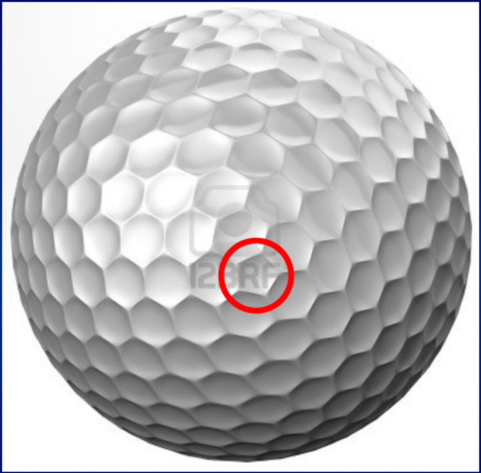
Buckminsterfullerene is a molecule with formula C_{60}

It was first synthesized in 1985.

Graphene
A hexagonal pattern of carbon one atom thick



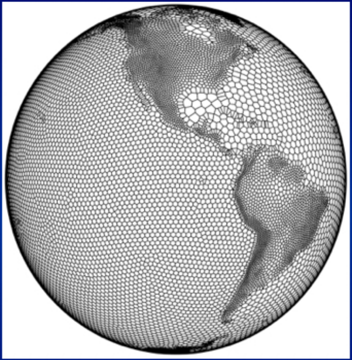

123RF



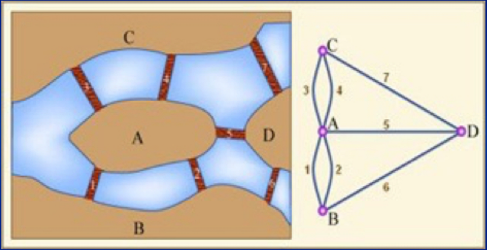
Euler's Polyhedron Formula

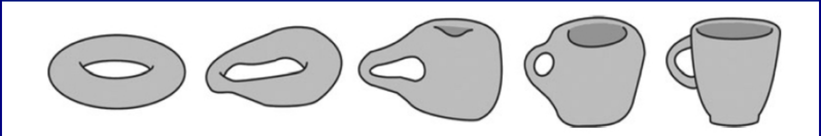
$$V - E + F = 2$$

still holds.

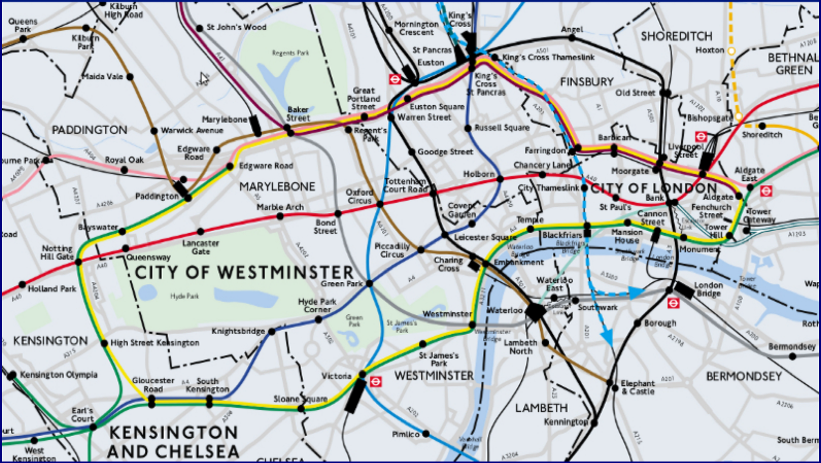


Topology
is often called
Rubber Sheet Geometry



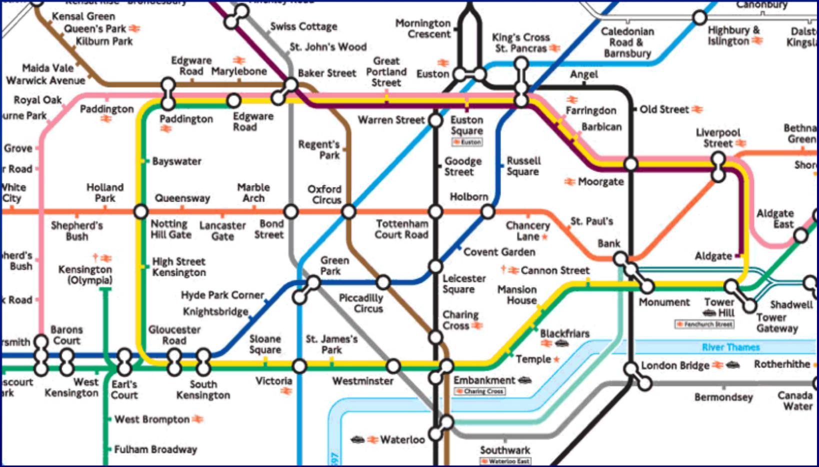


Topology and the London Underground
Topographical Map



Topology and the London Underground

Topological Map



Outline

The Ancient Greeks

Mathematics and Astronomy are intimately linked.

Two of the strands of the Quadrivium were Geometry (static) and Cosmology (dynamic space).

Greek astronomers like Claudius Ptolemy (c.90–168AD) believed that the Earth is at the centre of the universe.

The Sun and planets move around the Earth in orbits that are of the most perfect of all shapes, circles.

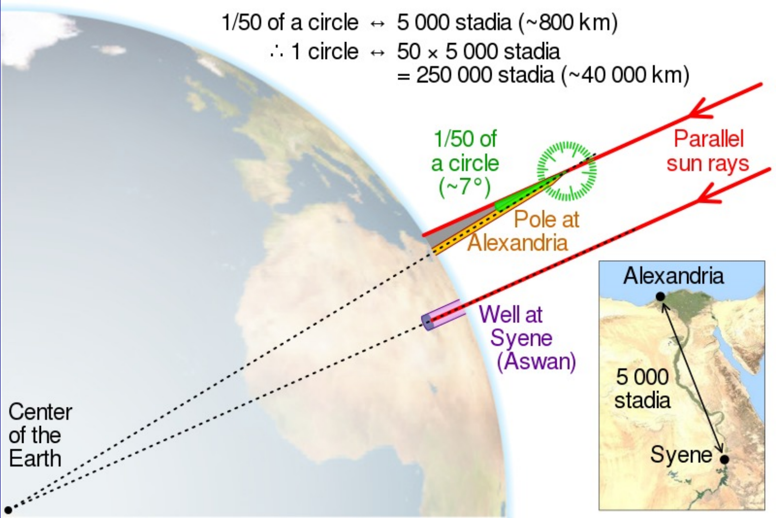
Aristarchus of Samos (c.310–230 BC)

Aristarchus of Samos ('A$\rho\iota\sigma\tau\alpha\rho\chi o\varsigma$), astronomer and mathematician, presented the first model that placed the Sun at the center of the universe, with the Earth revolving around it.

The original writing of Aristarchus is lost, but Archimedes wrote in his Sand Reckoner:

"His hypotheses are that the fixed stars and the Sun remain unmoved, that the Earth revolves about the Sun on the circumference of a circle, the Sun lying in the middle of the orbit, . . . "

Eratosthenes (c.276–194 BC)



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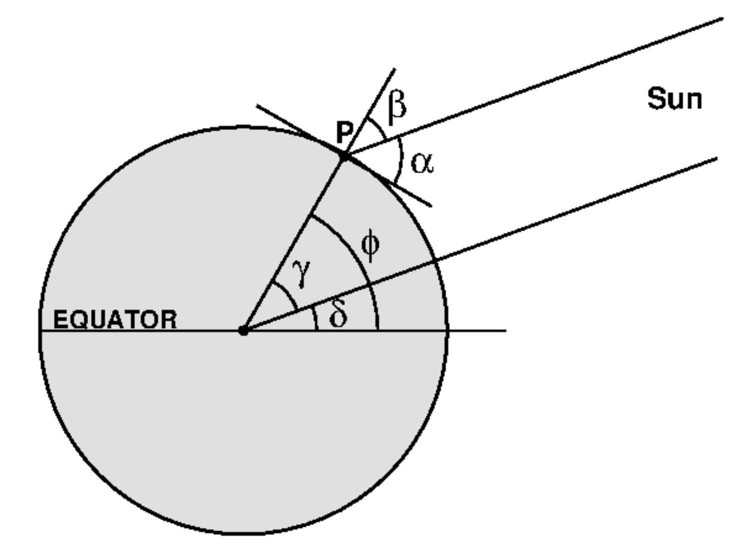


Hipparchus (c.190–120 BC)

Hipparchus of Nicaea ($'I\pi\pi\alpha\rho\chi o\varsigma$) was a Greek astronomer, geographer, and mathematician.

Regarded as the greatest astronomer of antiquity.

Often considered to be the founder of trigonometry.

He is famous for his discovery of the precession of the equinoxes, compilation of the first comprehensive star catalog of the western world, and invention of the astrolabe and (perhaps) the armillary sphere.

Claudius Ptolemy (c.AD 100–170)

Claudius Ptolemy was a Greco-Roman astronomer, mathematician, geographer and astrologer.

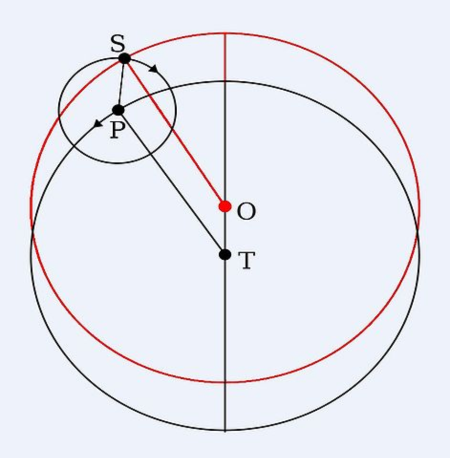
He lived in the city of Alexandria in the Roman province of Egypt and held Roman citizenship. Ptolemy wrote several scientific treatises:

- An astronomical treatise (the Almagest) originally called Mathematical Treatise (Mathematike Syntaxis).
- A book on geography.
- An astrological treatise.

Ptolemy's Almagest is the only surviving comprehensive ancient treatise on astronomy.

Ptolemy's Model

Ptolemy's model was geocentric and was universally accepted until the appearance of simpler heliocentric models during the scientific revolution.



Epicycles Rule

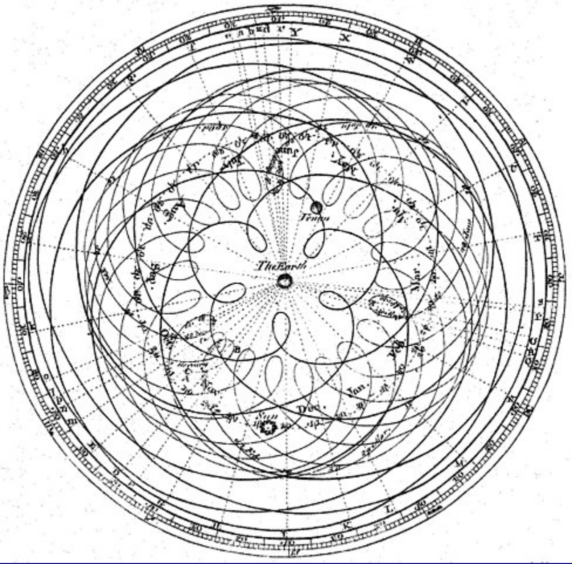
According to Norwood Russell Hanson (science historian):

There is no bilaterally symmetrical, nor eccentrically periodic curve used in any branch of astrophysics or observational astronomy which could not be smoothly plotted as the resultant motion of a point turning within a constellation of epicycles, finite in number, revolving around a fixed deferent.

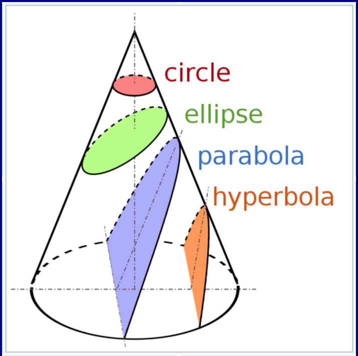
"The Mathematical Power of Epicyclical Astronomy", 1960

Any path — periodic or not, closed or open — can be represented by an infinite number of epicycles.

Ptolemaic Epicycles



Conic Sections



Circles are special cases of curves called conic sections.

They are formed by a plane cutting a cone at various angles.

Conics were studied by Apollonius of Perga (late 3rd – early 2nd centuries BC).

https://en.wikipedia.org/wiki/Conic_section

The Scientific Revolution

TRAILER

Next week, we will look at developments in the sixteenth and seventeenth centuries.

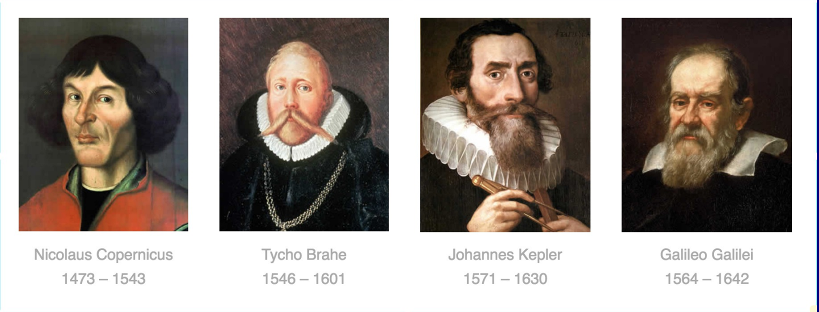


Figure from mathigon.org

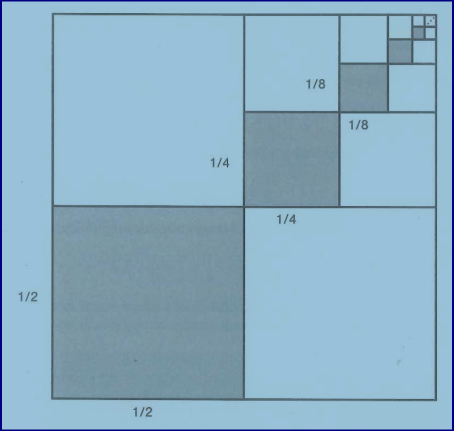
Outline

Distraction 8: Sum by Inspection

Can you guess the sum of this series:

$$\left(\frac{1}{2}\right)^2 + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{8}\right)^2 + \left(\frac{1}{16}\right)^2 + \cdots$$

Distraction 8: Sum by Inspection



We will find the shaded area without calculation

Proof by Inspection

Look at the figure in two different ways

At each scale, we have three squares the same size, and we keep one of them (black) and omit the others.

So, the area of the shaded squares is $\frac{1}{3}$.

Proof by Inspection

Look at the figure in two different ways

At each scale, we have three squares the same size, and we keep one of them (black) and omit the others.

So, the area of the shaded squares is $\frac{1}{3}$.

However, it is also given by the series

$$\left(\frac{1}{2}\right)^2 + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{8}\right)^2 + \left(\frac{1}{16}\right)^2 + \cdots$$

Therefore we can sum the series:

$$\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \cdots = \frac{1}{3}$$

Outline

The Real Numbers

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We need to be able to assign a number to a line of any length.

The Pythagoreans found that no number known to them gave the diagonal of a unit square.

It is as if there are gaps in the number system.

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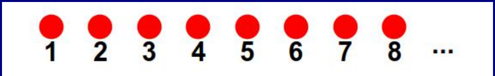
It is as if there are gaps in the number system.

We look at the rational numbers and show how to complete them: how to fill in the gaps.

The set $\mathbb{N}$ is infinite, but each element is isolated.



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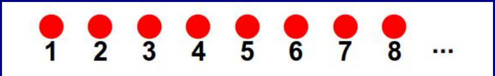

The set $\mathbb{Q}$ is infinite and also dense:
between any two rationals there is another rational.

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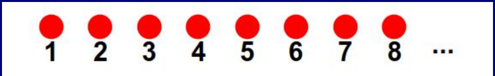

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PROOF: Let $r_1 = p_1/q_1$ and $r_2 = p_2/q_2$ be rationals.

$$\bar{r} = \tfrac{1}{2}(r_1 + r_2) = \frac{1}{2}\left(\frac{p_1}{q_1} + \frac{p_2}{q_2}\right) = \frac{p_1 q_2 + q_1 p_2}{2 q_1 q_2}$$

is another rational between them: $r_1 < \bar{r} < r_2$.

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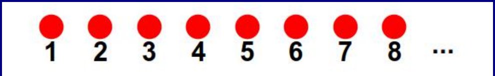

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Although $\mathbb{Q}$ is dense, there are gaps.
The line of rationals is discontinuous.

We complete it—filling in the gaps—by defining the limit of any sequence of rationals as a real number.

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WARNING:
We are glossing over a number of fundamental ideas of mathematical analysis:

- What is an infinite sequence?
- What is the limit of a sequence?

We will return later to these ideas.

To give a particular example, we know that

$$\sqrt{2} = 1.41421356\ldots$$

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$$\{1, 1.4, 1.41, 1.414, 1.4142, 1.41421, 1.414213, \ldots\}$$

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In terms of fractions, this is the sequence

$$\left\{1, \frac{14}{10}, \frac{141}{100}, \frac{1414}{1000}, \frac{14142}{10000}, \frac{141421}{100000}, \frac{1414213}{1000000}, \ldots\right\}$$

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EXERCISE:
Construct a sequence in $\mathbb{Q}$ that tends to π.

The Real Number Line

The set of **Real Numbers**, $\mathbb{R}$, contains all the rational numbers in $\mathbb{Q}$ and also all the limits of sequences of rationals [technically, all ‘Cauchy sequences’].

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We may assume that

- Every point on the number line corresponds to a real number.
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- Every real number corresponds to a point on the number line.

PHYSICS: There are unknown aspects of the microscopic structure of spacetime! These go beyond our ‘Universe of Discourse’.

Now we have the chain of sets:

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$$

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- Algebraic numbers like $\sqrt{2}$.
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We denote the algebraic numbers by $\mathbb{A}$.

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We will soon talk about prime numbers $\mathbb{P}$.
They are subset of the natural numbers, so

$$\mathbb{P} \subset \mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{A} \subset \mathbb{R}$$

Outline

Symmetries of the Triangle and Square: The Dihedral Groups D_3 and D_4

Let's look at symmetries of the triangle and square.

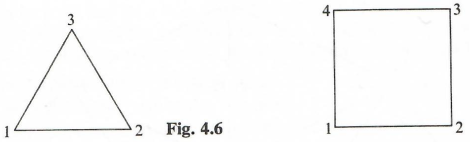
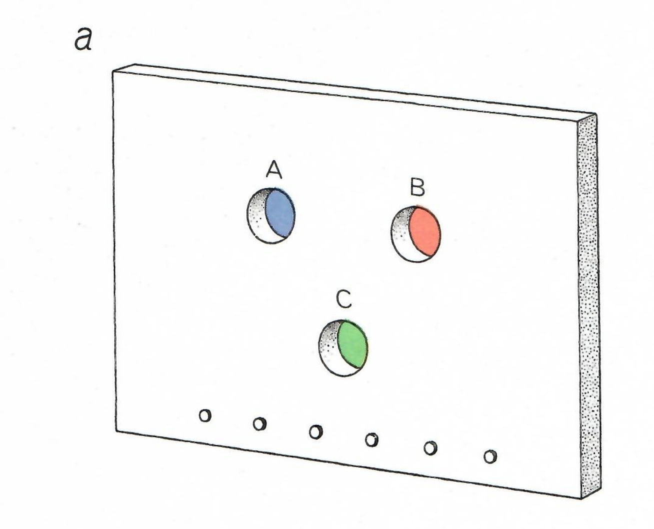
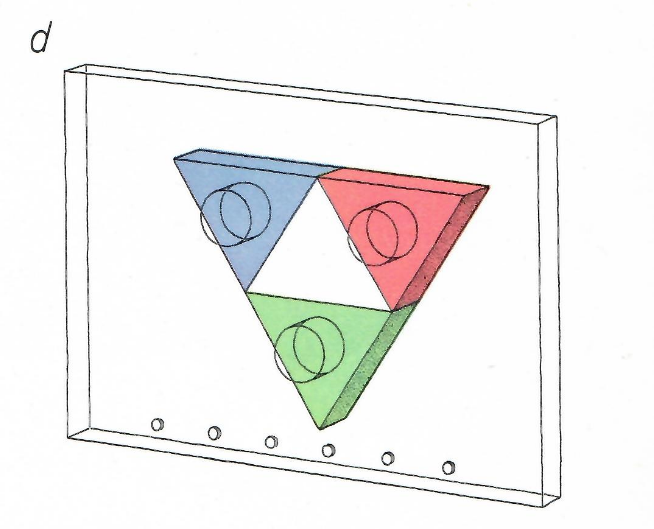


Fig. 4.6 Fig. 4.7

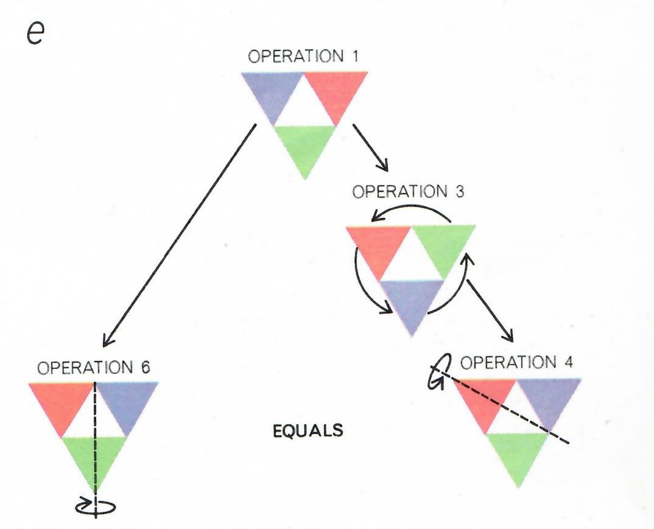
They correspond to the dihedral groups D_3 and D_4.

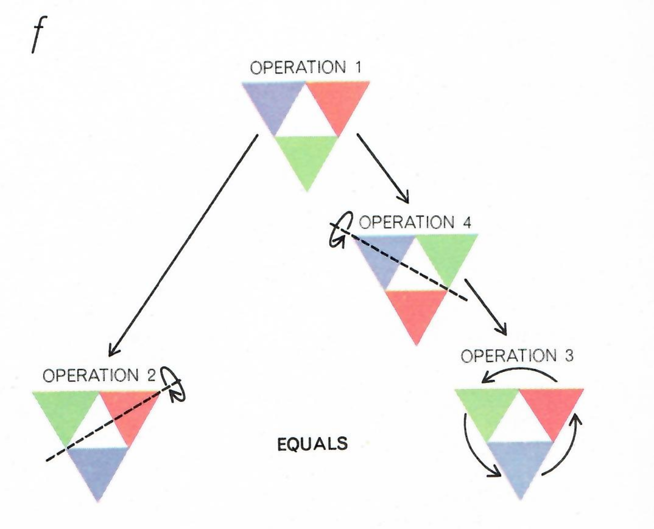

a
A
B
C

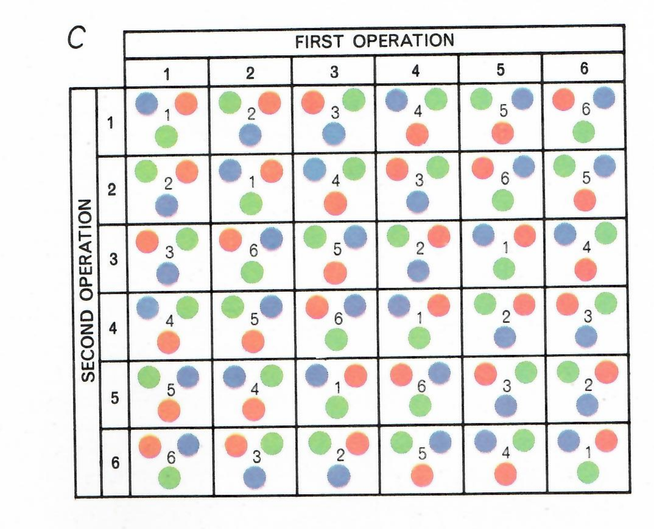

d

b

OPERATION	RESULT
1. NO CHANGE:	
2. SWITCH A AND C:	
3. REPLACE A BY B, B BY C, C BY A:	
4. SWITCH C AND B:	
5. REPLACE A BY C, B BY A, C BY B:	
6. SWITCH A AND B:	


e
OPERATION 1
OPERATION 3
OPERATION 6
EQUALS
OPERATION 4


f
OPERATION 1
OPERATION 4
OPERATION 3
OPERATION 2
EQUALS


C
FIRST OPERATION
1 2 3 4 5 6
SECOND OPERATION
1 | 1 2 3 4 5 6
2 | 2 1 4 3 6 5
3 | 3 6 5 2 1 4
4 | 4 5 6 1 2 3
5 | 5 4 1 6 3 2
6 | 6 3 2 5 4 1

Symbols for Transformations of Triangle

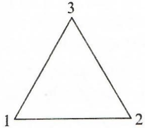


Fig. 4.6

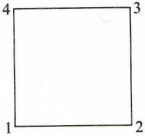


Fig. 4.7

$$\rho_0 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \quad \mu_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix},$$

$$\rho_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \quad \mu_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix},$$

$$\rho_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, \quad \mu_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}.$$

The Third Dihedral Group D_3

	ρ_0	ρ_1	ρ_2	μ_1	μ_2	μ_3
ρ_0	ρ_0	ρ_1	ρ_2	μ_1	μ_2	μ_3
ρ_1	ρ_1	ρ_2	ρ_0	μ_2	μ_3	μ_1
ρ_2	ρ_2	ρ_0	ρ_1	μ_3	μ_1	μ_2
μ_1	μ_1	μ_3	μ_2	ρ_0	ρ_2	ρ_1
μ_2	μ_2	μ_1	μ_3	ρ_1	ρ_0	ρ_2
μ_3	μ_3	μ_2	μ_1	ρ_2	ρ_1	ρ_0

Fig. 4.5

Subgroup Z_3 of Third Dihedral Group D_3

	ρ_0	ρ_1	ρ_2
ρ_0	ρ_0	ρ_1	ρ_2
ρ_1	ρ_1	ρ_2	ρ_0
ρ_2	ρ_2	ρ_0	ρ_1

Fig. 4.5

The Third Dihedral Group D_3

	ρ_0	ρ_1	ρ_2	μ_1	μ_2	μ_3
ρ_0	ρ_0	ρ_1	ρ_2	μ_1	μ_2	μ_3
ρ_1	ρ_1	ρ_2	ρ_0	μ_2	μ_3	μ_1
ρ_2	ρ_2	ρ_0	ρ_1	μ_3	μ_1	μ_2
μ_1	μ_1	μ_3	μ_2	ρ_0	ρ_2	ρ_1
μ_2	μ_2	μ_1	μ_3	ρ_1	ρ_0	ρ_2
μ_3	μ_3	μ_2	μ_1	ρ_2	ρ_1	ρ_0

Fig. 4.5

Subgroup Z_2 of Third Dihedral Group D_3

	ρ_0
ρ_0	ρ_0

μ_1
μ_1

μ_1	μ_1

ρ_0

Fig. 4.5

Symbols for Transformations of Square

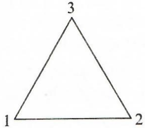


Fig. 4.6

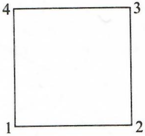


Fig. 4.7

$$\rho_0 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}, \qquad \mu_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix},$$

$$\rho_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}, \qquad \mu_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix},$$

$$\rho_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}, \qquad \delta_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix},$$

$$\rho_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}, \qquad \delta_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix}.$$

The Fourth Dihedral Group D_4

	ρ_0	ρ_1	ρ_2	ρ_3	μ_1	μ_2	δ_1	δ_2
ρ_0	ρ_0	ρ_1	ρ_2	ρ_3	μ_1	μ_2	δ_1	δ_2
ρ_1	ρ_1	ρ_2	ρ_3	ρ_0	δ_2	δ_1	μ_1	μ_2
ρ_2	ρ_2	ρ_3	ρ_0	ρ_1	μ_2	μ_1	δ_2	δ_1
ρ_3	ρ_3	ρ_0	ρ_1	ρ_2	δ_1	δ_2	μ_2	μ_1
μ_1	μ_1	δ_1	μ_2	δ_2	ρ_0	ρ_2	ρ_1	ρ_3
μ_2	μ_2	δ_2	μ_1	δ_1	ρ_2	ρ_0	ρ_3	ρ_1
δ_1	δ_1	μ_2	δ_2	μ_1	ρ_3	ρ_1	ρ_0	ρ_2
δ_2	δ_2	μ_1	δ_1	μ_2	ρ_1	ρ_3	ρ_2	ρ_0

Fig. 4.8

Thank you