Sum-Enchanted Evenings

The Fun and Joy of Mathematics

•

LECTURE 7

Peter Lynch
School of Mathematics & Statistics
University College Dublin

Evening Course, UCD, Autumn 2017

Outline

Outline

Introduction

Irrational Numbers

Distraction 6: Slicing a Pizza

Pascal's Triangle

Distraction 7: Plus Magazine

Music: Harmonics

Symmetry I

Meaning and Content of Mathematics

The word **Mathematics** comes from Greek $\mu\alpha\theta\eta\mu\alpha$ (máthéma), meaning "knowledge" or "study" or "learning".

It is the study of topics such as

- Quantity (numbers)
- Structure (patterns)
- Space (geometry)
- Change (analysis).

Outline of Lecture 2

Reminder: QI Video on Factorial 52

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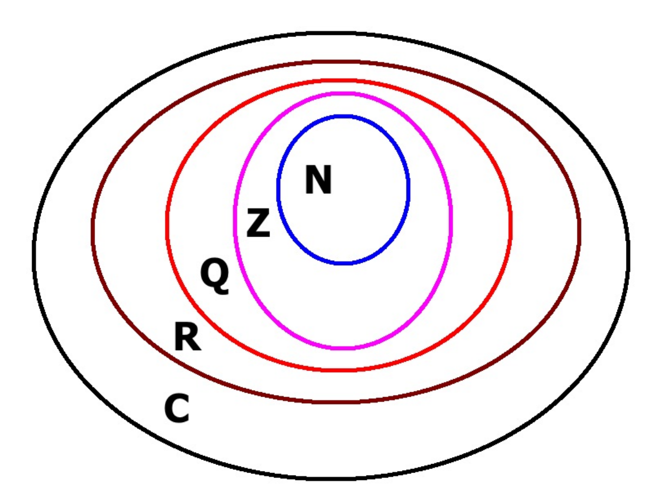
Symmetry I

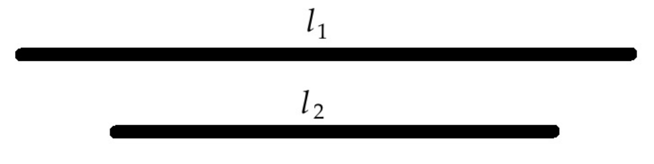
The Hierarchy of Numbers



$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$$

Incommensurability

Suppose we have two line segments



Can we find a unit of measurement such that both lines are a whole number of units?

Can they be co-measured? Are they commensurable?

Are ℓ_1 and ℓ_2 commensurable?

If so, let the unit of measurement be λ.

Then

$$\begin{aligned} \ell_1 &= m\lambda\,, \quad m \in \mathbb{N} \\ \ell_2 &= n\lambda\,, \quad n \in \mathbb{N} \end{aligned}$$

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Therefore

$$\frac{\ell_1}{\ell_2} = \frac{m\lambda}{n\lambda} = \frac{m}{n}$$

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Therefore

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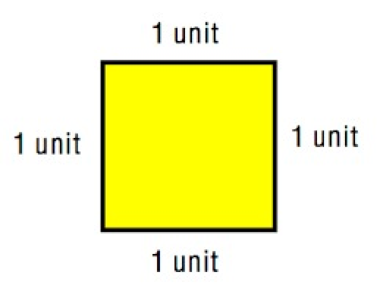
If not, then ℓ_1 and ℓ_2 are incommensurable.

Irrational Numbers

If the side of a square is of length 1, then the diagonal has length $\sqrt{2}$ (by the Theorem of Pythagoras).



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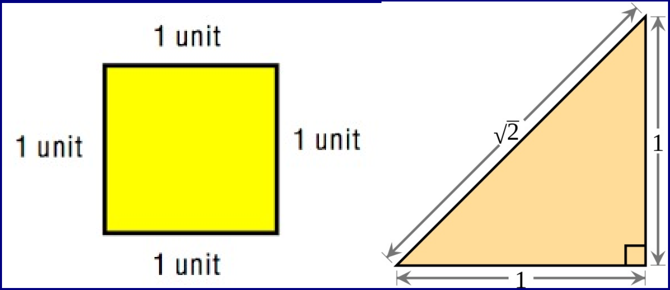

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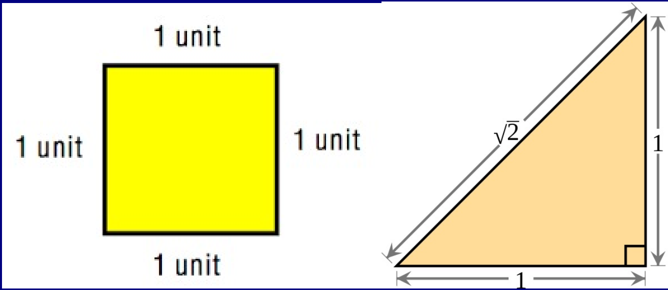

The ratio between the diagonal and the side is:

$$\frac{\textbf{Diagonal}}{\textbf{Side Length}} = \sqrt{2}$$

Irrationality of $\sqrt{2}$

For the Pythagoreans, numbers were of two types:

1. Whole numbers
2. Ratios of whole numbers

There were no other numbers.

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For example, suppose $p = 42$ and $q = 30$. Then

$$\frac{p}{q} = \frac{42}{30} = \frac{7 \times 6}{5 \times 6} = \frac{7}{5}$$

Remark on *Reductio ad Absurdum.*

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"How often have I said to you that when you have eliminated the impossible, whatever remains, however improbable, must be the truth?"

The Sign of the Four (1890)

We say that p and q are relatively prime:
They have no common factors.

In particular, p and q cannot both be even numbers.

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Now square both sides of the equation $\sqrt{2} = p/q$:

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This means that p^2 is even. Therefore p is even.

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$$p^2 = (2r)^2 = 4r^2 = 2q^2 \qquad \text{or} \qquad 2r^2 = q^2$$

But this means that q^2 is even. So, q is even.

Both p and q are even. This is a contradiction.

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This assumption has led to a contradiction.

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It is not a ratio of whole numbers.

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$\kappa\rho\iota\sigma\eta$ $\kappa\alpha\tau\alpha\sigma\tau\rho o\phi\eta$!

$\sqrt{2}$ and the Development of Mathematics

The discovery of irrational quantities had a dramatic effect on the development of mathematics.

Legend has it that the discoveror of this fact was thrown from a ship and drowned.

The result was that focus now fell on geometry, and arithmetic or number theory was neglected.

The problems were not resolved for many centuries.

Outline

Distraction 6: Slicing a Pizza

Cut the pizza using three straight cuts.

There should be exactly one piece of pepperoni on each slice of pizza.

Outline

Pascal’s Triangle

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

Pascal's Triangle

$$\binom{0}{0}$$

$$\binom{1}{0} \quad \binom{1}{1}$$

$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$

$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$

$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$

Pascal's Triangle

Pascal's triangle is a triangular array of the binomial coefficients.

It is named after French mathematician Blaise Pascal.

It was studied centuries before him in:

- India (Pingala, C2BC)
- Persia (Omar Khayyam, C11AD)
- China (Yang Hui, C13AD).

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Draw Pascal's triangle on the board.

Pascal's Triangle

The rows of Pascal's triangle are numbered starting with row n = 0 at the top (0-th row).

The entries in each row are numbered from the left beginning with k = 0.

The triangle is easily constructed:

- **A single entry 1 in row 0.**
- **Add numbers above for each new row.**

The entry in the nth row and k-th column of Pascal's triangle is denoted $\binom{n}{k}$.

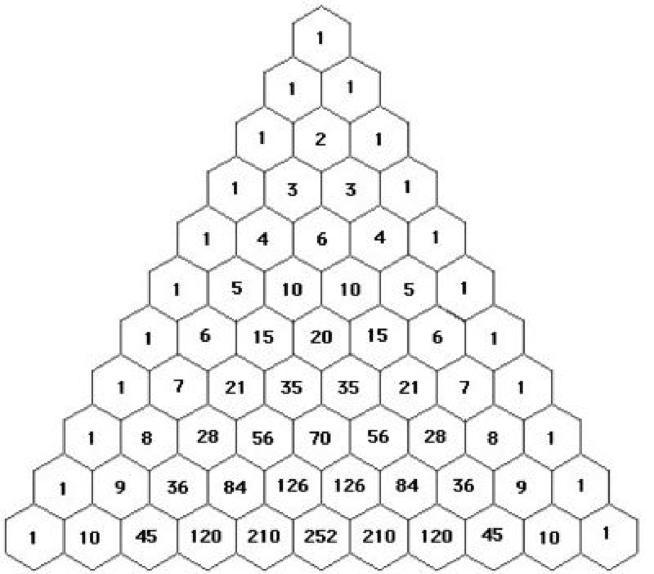
The entry in the topmost row is $\binom{0}{0} = 1$.

Pascal's Identity

The construction of the triangle may be written:

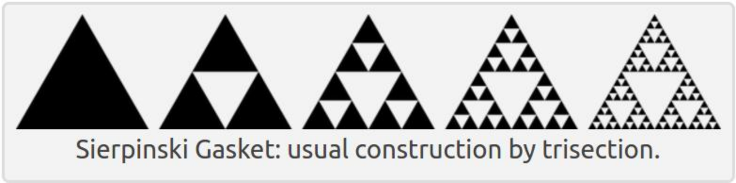
$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

This relationship is known as Pascal's Identity.


1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1

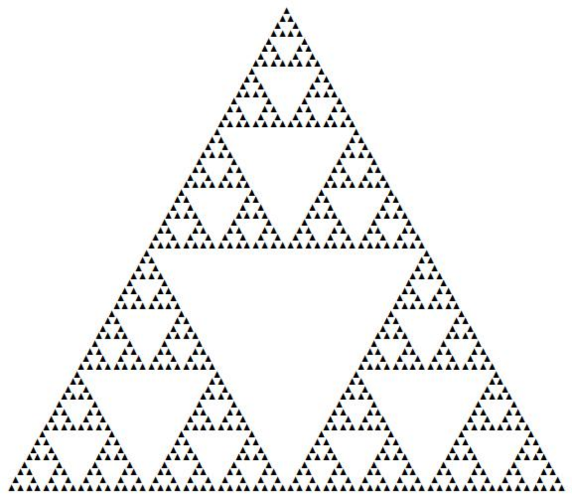
1							
1	1						
1	2	1					
1	3	3	1				
1	4	6	4	1			
1	5	10	10	5	1		
1	6	15	20	15	6	1	
1	7	21	35	35	21	7	1

Sierpinski's Gasket



Sierpinski's Gasket is constructed by starting with an equilateral triangle, and successively removing the central triangle at each scale.

Sierpinski's Gasket



Sierpinski's Gasket in Pascal's Triangle

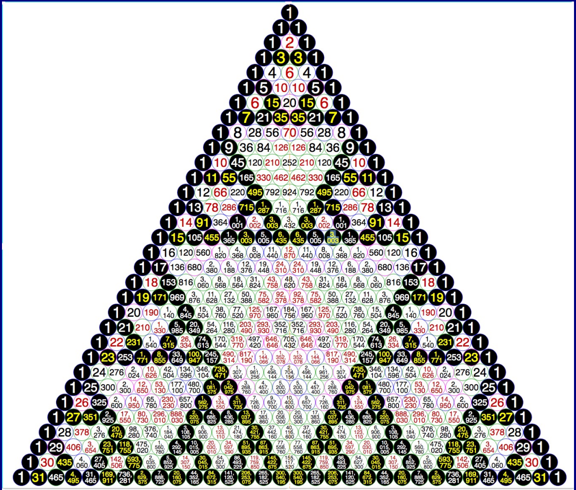


Figure : Odd numbers are in black

Remember Walking in Manhattan?

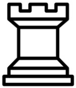
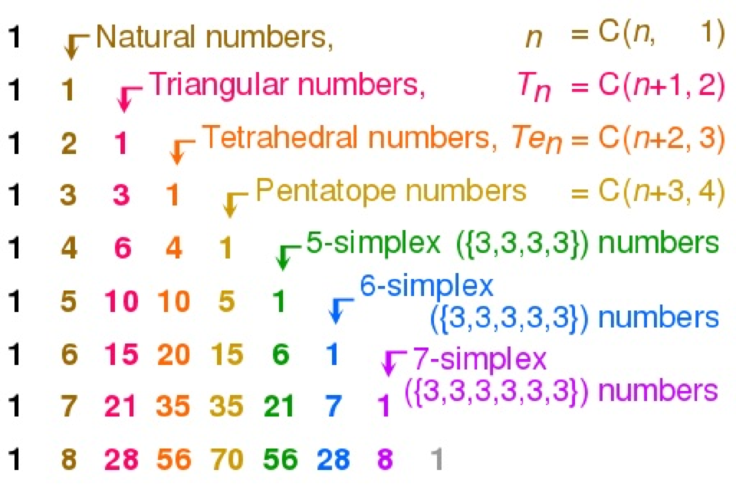
	1	1	1
1	2	3	4
1	3	6	10
1	4	10	20

Figure : Number of routes for a rook in chess.

Geometric Numbers in Pascal's Triangle



Outline

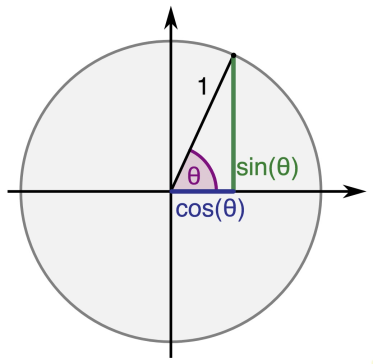
Distraction 7: Plus Magazine

PLUS: The Mathematics e-zine

https://plus.maths.org/

Outline

Definition of Circular Functions



We define the functions

$$y = \sin\theta$$

$$x = \cos\theta$$

using this diagram.

Reference

`https://en.wikipedia.org/wiki/Sine`

Sine Waves and Harmonics

A **pure tone** is represented by a **sine wave**

$$y = \sin \omega t$$

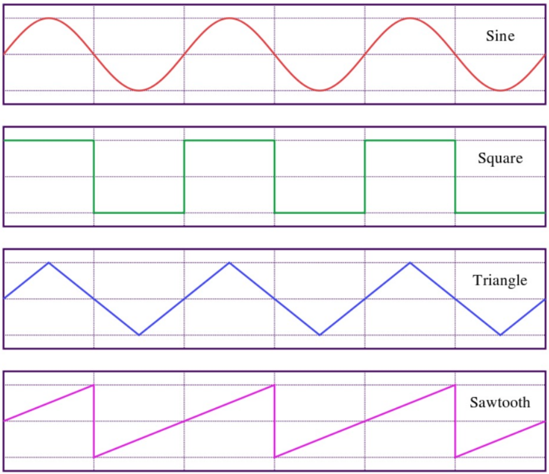
Its ***n*-th harmonic** is represented by

$$y = \sin n\omega t$$

To hear these, go to

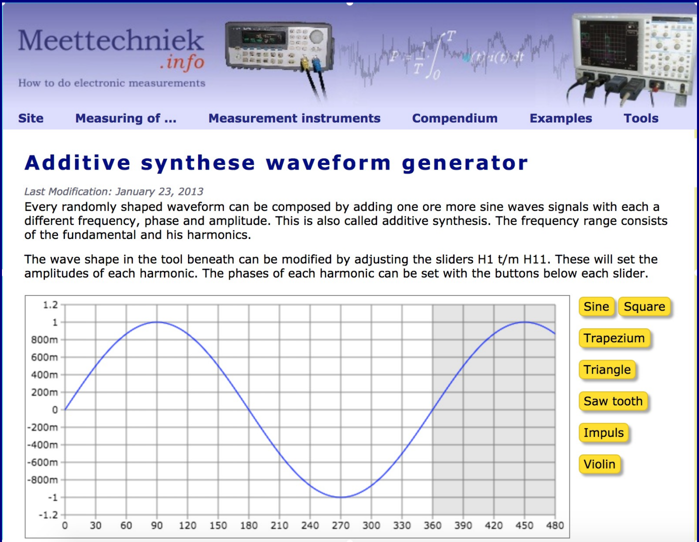
`https://meettechniek.info/additional/additive-synthesis.html`

Various Wave Forms

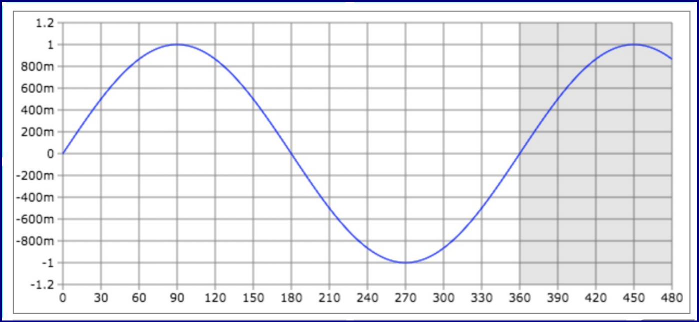


Sine, square, triangle, and sawtooth waveforms

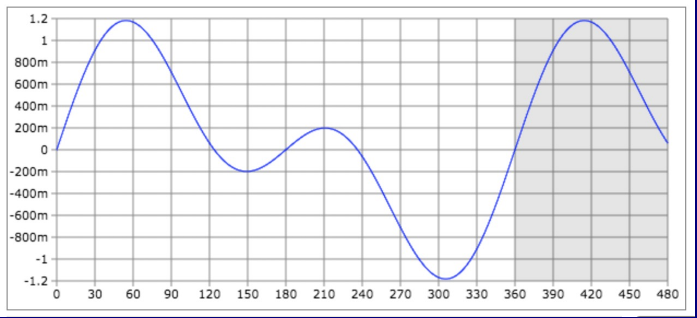
Online Waveform Generator



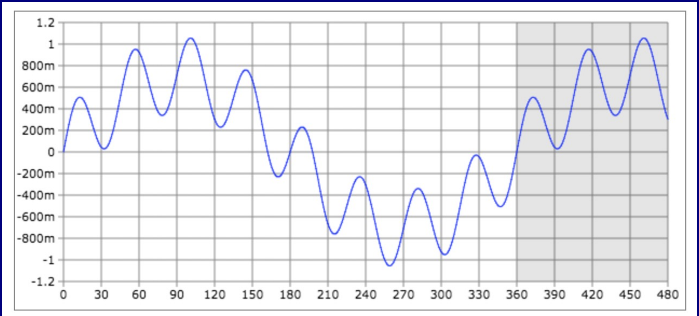
Pure Sine Wave



Sine Wave and First Harmonic



Sine Wave and Eighth Harmonic



Outline

Ubiquity and Beauty of Symmetry

Symmetry is all around us.

- Many buildings are symmetric.
- Our bodies have bilateral symmetry.
- Crystals have great symmetry.
- Viruses can display stunning symmetries.
- At the sub-atomic scale, symmetry reigns.
- Galaxies have many symmetries.

The Taj Mahal

A Face with Symmetry: Halle Berry

Halle Berry | Berry Halle

An Asymmetric Face: You know Who!

Symmetry and Group Theory

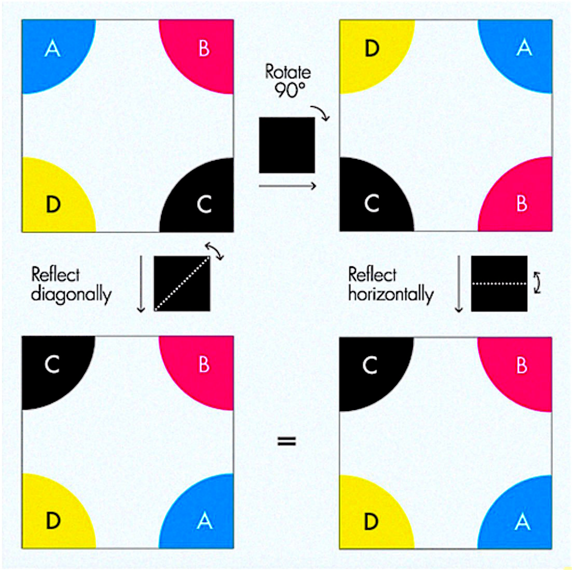
Symmetry is an essentially geometric concept.

The mathematical theory of symmetry is algebraic.
The key concept is that of a group.

A group is a set of elements such that any two elements can be combined to produce another.

Instead of giving the mathematical definition,
I give an example to make things clear.

Non-Commutative Operations



The Klein 4-Group

Take a book, place it on the table and draw a rectangle around it. In how many ways can the book fit into the rectangle?

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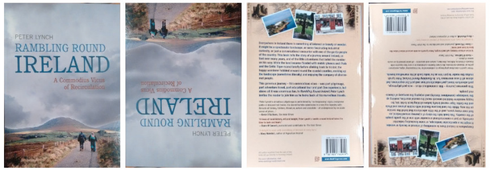
Take a book, place it on the table and draw a rectangle around it. In how many ways can the book fit into the rectangle?

Once a single corner of the book is put at the top left corner, there is no further lee-way.

There are four ways to fit the book in the rectangle.



The four orientations of the book can be described in terms of four simple rotations:

- **I:** Place book upright with front cover upright
- **X:** Rotate 180° about horizontal through centre
- **Y:** Rotate 180° about vertical through centre
- **Z:** Rotate 180° about perp. through centre

Multiplication Table

*	**I**	**X**	**Y**	**Z**
I	I	X	Y	Z
X	X	I	Z	Y
Y	Y	Z	I	X
Z	Z	Y	X	I

There are several sub-groups:

$$\{I, X, Y, Z\} \quad \{I, X\} \quad \{I, Y\} \quad \{I, Z\} \quad \{I\}$$

Twelve-tone Music

Table : Klein 4-Group.

	P	R	I	RI
P	P	R	I	RI
R	R	P	RI	I
I	I	RI	P	R
RI	RI	I	R	P

The Klein 4-group is the basic group of transformations in twelve tone music.

The operations are retrogression (R), inversion (I) and the composion (RI), which is also a rotation operation.

Numbers of Low-Order Groups

Order n	# Groups[6]	Abelian	Non-Abelian
0	0	0	0
1	1	1	0
2	1	1	0
3	1	1	0
4	2	2	0
5	1	1	0
6	2	1	1
7	1	1	0
8	5	3	2
9	2	2	0
10	2	1	1
11	1	1	0
12	5	2	3
13	1	1	0
14	2	1	1
15	1	1	0
16	14	5	9

Table of number of groups of orders up to sixteen.

Commutative groups are called **Abelian** groups.

Groups that do not commute are **Non-Abelian.**

The smallest non-Abelian group is of order 6.

From 2 to 3 Dimensional Symmetry

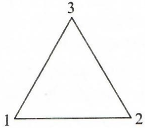


Fig. 4.6

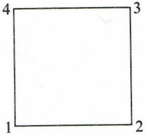
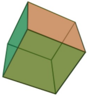
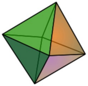
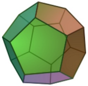
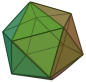
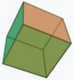
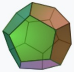
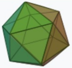


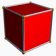
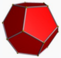
Fig. 4.7

Tetrahedron	Cube	Octahedron	Dodecahedron	Icosahedron
Four faces	Six faces	Eight faces	Twelve faces	Twenty faces
 (Animation)	 (Animation)	 (Animation)	 (Animation)	 (Animation)

The Five Platonic Solids

Polyhedron		Vertices	Edges	Faces
tetrahedron		4	6	4
cube		8	12	6
octahedron		6	12	8
dodecahedron		20	30	12
icosahedron		12	30	20

Platonic Solids: Euler's Gem

Name	Image	Vertices V	Edges E	Faces F	Euler characteristic: $V - E + F$
Tetrahedron		4	6	4	**2**
Hexahedron or cube		8	12	6	**2**
Octahedron		6	12	8	**2**
Dodecahedron		20	30	12	**2**
Icosahedron		12	30	20	**2**

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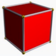
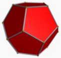
Hexahedron or cube		8	12	6	**2**
Octahedron		6	12	8	**2**
Dodecahedron		20	30	12	**2**
Icosahedron		12	30	20	**2**

Mnemonic: Very Easy Formula 2 remember!

Dual Polyhedra

Every polyhedron is associated with a dual.

The vertices of the polyhedron correspond to the faces of its dual. The faces of the polyhedron correspond to the vertices of its dual.

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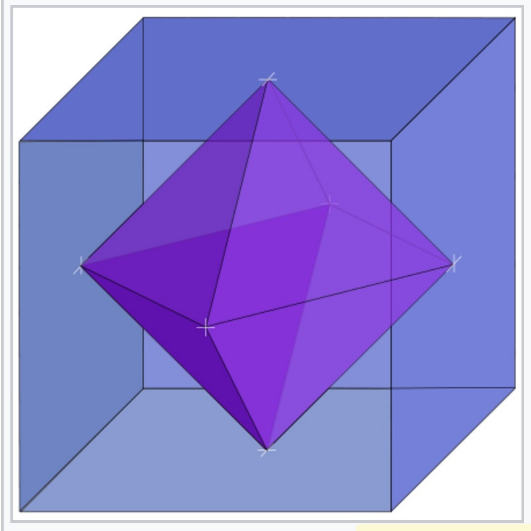
The dual of the dual is the original!

Duality preserves the symmetry of the polyhedron.

Cube and Octahedron are Dual



Cube and Octahedron are Dual

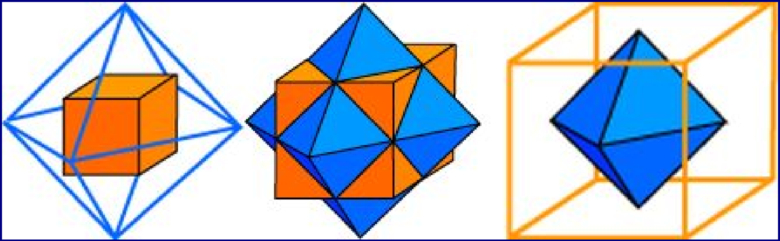


Figure : Tetrahedron and dual.

Dodecahedron and Icosahedron are Dual

Figure : Tetrahedron and dual.

Tetrhedron is its own Dual

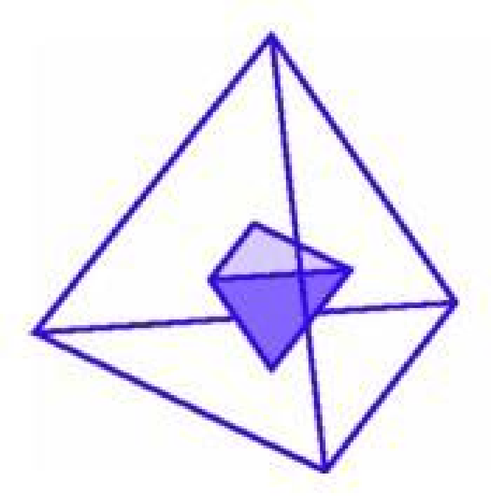


Figure : Tetrahedron and dual.

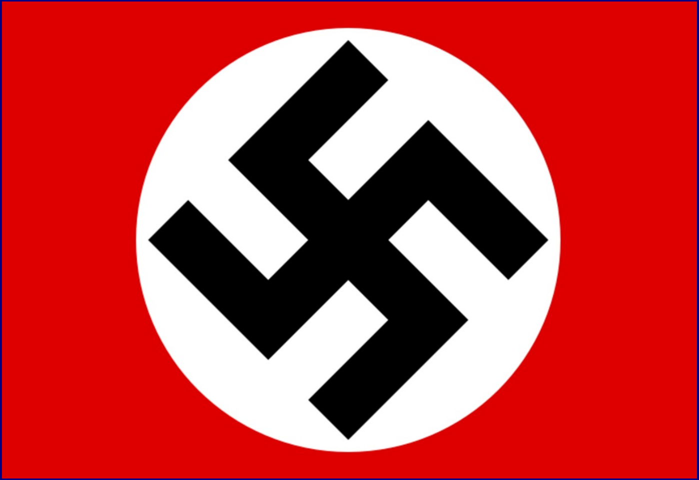
Threefold Symmetry: Z_3

Threefold Symmetry: Z_3

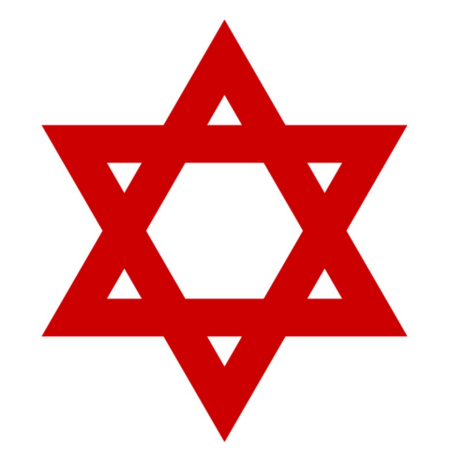
Threefold Symmetry: Z_3

Z_3 Symmetry

Z_4 Symmetry

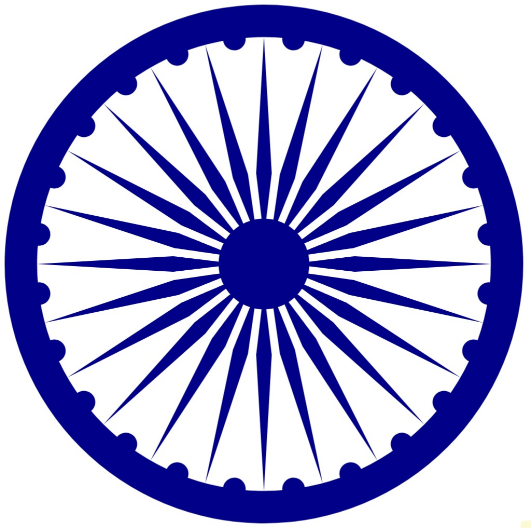


Star of David (D_6 Symmetry)



Flag of India (D_1)

Ashoka Chakra (D_{24})



Thank you