Sum-Enchanted Evenings

The Fun and Joy of Mathematics

•

LECTURE 6

Peter Lynch
School of Mathematics & Statistics
University College Dublin

Evening Course, UCD, Autumn 2017

Outline

Outline

Introduction

Combinatorics

Distraction 10

Music and Mathematics II

Meaning and Content of Mathematics

The word Mathematics comes from Greek $\mu\alpha\theta\eta\mu\alpha$ (máthéma), meaning "knowledge" or "study" or "learning".

It is the study of topics such as

- Quantity (numbers)
- Structure (patterns)
- Space (geometry)
- Change (analysis).

Outline

Combinatorics

Combinatorics is a branch of mathematics primarily concerned with counting.

Combinatorics arises throughout pure mathematics: algebra, probability theory, topology and geometry.

It is intimately connected with graph theory.

In addition, there are many practical applications.

Combinatorics is important in computer science, e.g., for analysis of algorithms. "Big Data".

A Simple Example

We have a combination lock with four barrels.

Is this secure?

A Simple Example

We have a combination lock with four barrels.

Is this secure?

How many combinations are there?
How long would it take to break the lock?

A Simple Example

The first barrel may be set in any of 10 ways.
So that makes $N = 10$ settings.

For each of these, the second barrel has 10 settings.
So that makes $N = 10 \times 10$ variations.

A Simple Example

The first barrel may be set in any of 10 ways.
So that makes $N = 10$ settings.

For each of these, the second barrel has 10 settings.
So that makes $N = 10 \times 10$ variations.

Likewise for the third and fourth barrels.

So the total number of settings is

$$N = (10 \times 10 \times 10 \times 10) = 10,000 = 10^4 \, .$$

A Simple Example

The total number of settings is

$$N = 10^4 \, .$$

Suppose a 'baddie' could try one code every second.

How long would it take him to break the lock?

A Simple Example

The total number of settings is

$$N = 10^4 \, .$$

Suppose a ‘baddie’ could try one code every second.

How long would it take him to break the lock?

On the order of 10^4 seconds, or about 3 hours.

A Simple Example

Now suppose each digit of the code to open the lock is used only once.

A Simple Example

Now suppose each digit of the code to open the lock is used only once.

The first barrel may be set in any of 10 ways. So that makes $N = 10$ settings.

The second barrel now has 9 settings. So that makes $N = 10 \times 9$ variations.

Likewise, the third and fourth barrels have 8 and 7 possible values.

A Simple Example

Now suppose each digit of the code to open the lock is used only once.

The first barrel may be set in any of 10 ways. So that makes $N = 10$ settings.

The second barrel now has 9 settings. So that makes $N = 10 \times 9$ variations.

Likewise, the third and fourth barrels have 8 and 7 possible values.

So the total number of settings is

$$N = 10 \times 9 \times 8 \times 7 = 5,040 .$$

A Simple Example

The number of lock codes with 4 distinct digits is

$$N = 10 \times 9 \times 8 \times 7 = 5,040\,.$$

This is an example of a **permutation.**

A Simple Example

The number of lock codes with 4 distinct digits is

$$N = 10 \times 9 \times 8 \times 7 = 5,040\,.$$

This is an example of a permutation.

It is the answer to the question:

How many four-digit numbers are there with no digit repeated.

The permutations respect the ordering of the digits.

Factorial n or $n!$

The product of the first n numbers is called factorial n, written $n!$

$$n! = 1 \times 2 \times 3 \times \cdots \times (n-1) \times n\,.$$

Factorial n or $n!$

The product of the first n numbers is called factorial n, written $n!$

$$n! = 1 \times 2 \times 3 \times \cdots \times (n-1) \times n\,.$$

For small n, it is easy to calculate $n!$.

$1! = 1 \qquad 2! = 2 \qquad 3! = 6 \qquad 4! = 24 \qquad 5! = 120$

Factorial n or $n!$

The product of the first n numbers is called factorial n, written $n!$

$$n! = 1 \times 2 \times 3 \times \cdots \times (n-1) \times n\,.$$

For small n, it is easy to calculate $n!$.

$$1! = 1 \qquad 2! = 2 \qquad 3! = 6 \qquad 4! = 24 \qquad 5! = 120$$

As n gets larger, factorial n grows very rapidly.

$$10! = 3,628,800 \qquad 20! \approx 2.433 \times 10^{18} \qquad 30! \approx 2.653 \times 10^{32}$$

Use of Factorials: A Simple Example

Back to the lock codes with 4 distinct digits:

$$N = 10 \times 9 \times 8 \times 7 = 5,040\,.$$

We can write this as

$$N = \frac{10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{6 \times 5 \times 4 \times 3 \times 2 \times 1}$$

Use of Factorials: A Simple Example

Back to the lock codes with 4 distinct digits:

$$N = 10 \times 9 \times 8 \times 7 = 5,040\,.$$

We can write this as

$$N = \frac{10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{6 \times 5 \times 4 \times 3 \times 2 \times 1}$$

This can be written using factorials:

$$N = \frac{10!}{6!}$$

Permutations and Combinations

Depending on the particular problem, the order may or may not be important.

For a lock combination, the order of the barrels is obviously important: getting the right digits in the wrong order will not open the lock!

For the lock problem we need Permutations.

Permutations and Combinations

Depending on the particular problem, the order may or may not be important.

For a lock combination, the order of the barrels is obviously important: getting the right digits in the wrong order will not open the lock!

For the lock problem we need Permutations.

For a hand of five cards in poker, the order in which the cards are dealt doesn't matter: the same hand results from any of [HOW MANY] dealing orders?

For the Poker hand we need Combinations.

Permutations and Combinations

The number of permutations of 2 numbers from 5:

$$^5P_2 = 5 \times 4 = 20$$

As we have seen, this can be written

$$^5P_2 = \frac{5!}{3!} = \frac{5!}{(5-2)!}$$

Permutations and Combinations

The number of **permutations** of 2 numbers from 5:

$$^5P_2 = 5 \times 4 = 20$$

As we have seen, this can be written

$$^5P_2 = \frac{5!}{3!} = \frac{5!}{(5-2)!}$$

The number of **combinations** of 2 numbers from 5:

$$^5C_2 = \binom{5}{2} = (5 \times 4)/2 = 10$$

More generally, we have

$$^nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

The National Lottery: 49/6

How to win? Just pick the right six numbers!

Choose 6 numbers from 49.

The National Lottery: 49/6

How to win? Just pick the right six numbers!

Choose 6 numbers from 49.

How many choices are there?

Do we need permutations or combinations?

The National Lottery: 49/6

- The first number can be any of 49.
- The second number can be any of 48.

$$\vdots$$

- The sixth number can be any of 44.

The National Lottery: 49/6

- The first number can be any of 49.
- The second number can be any of 48.

$$\vdots$$

- The sixth number can be any of 44.

So, the total number of choices seems to be

$$(49 \times 48 \times 47 \times 46 \times 45 \times 44) = \frac{49 \cdot 48 \cdot 47 \cdots 3 \cdot 2 \cdot 1}{43 \cdot 42 \cdot 41 \cdots 3 \cdot 2 \cdot 1}$$

The National Lottery: 49/6

- The first number can be any of 49.
- The second number can be any of 48.

$$\vdots$$

- The sixth number can be any of 44.

So, the total number of choices seems to be

$$(49 \times 48 \times 47 \times 46 \times 45 \times 44) = \frac{49 \cdot 48 \cdot 47 \cdots 3 \cdot 2 \cdot 1}{43 \cdot 42 \cdot 41 \cdots 3 \cdot 2 \cdot 1}$$

The total number of possible choices is

$$\frac{49!}{43!} = \frac{49!}{(49-6)!} = {}^{49}P_6$$

The National Lottery: 49/6

But many choices result in the same six numbers.

The number of possible orderings of 6 numbers is

$$6 \times 5 \times 4 \times 3 \times 2 \times 1 = 6!$$

The National Lottery: 49/6

But many choices result in the same six numbers.

The number of possible orderings of 6 numbers is

$$6 \times 5 \times 4 \times 3 \times 2 \times 1 = 6!$$

We must divide by this number.

The National Lottery: 49/6

But many choices result in the same six numbers.

The number of possible orderings of 6 numbers is

$$6 \times 5 \times 4 \times 3 \times 2 \times 1 = 6!$$

We must divide by this number.

Therefore, the total number of combinations is

$$\binom{49}{6} = \frac{49 \cdot 48 \cdot 47 \cdot 46 \cdot 45 \cdot 44}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \frac{49!}{(49-6)!6!}$$

We read this as "49 choose 6".

The National Lottery: 49/6

How big is this?

$$\binom{49}{6} = 13,983,816 \approx 14 \text{ million}$$

The National Lottery: 49/6

How big is this?

$$\binom{49}{6} = 13,983,816 \approx 14 \text{ million}$$

- How long before you win the lottery?
- How much will you spend?
- How much will you win?
- Is it a good investment?

Lotto “Fix”, Saturday 7 October 2017

QI Feature

How many different ways are there of ordering a deck of cards?

QI Feature

How many different ways are there of ordering a deck of cards?

Google for “QI Factorial 52”

`https://youtu.be/SLIvwtIuC3Y`

80,658,175,170,943,878,571,660,636,856,403,766, 975,289,505,440,883,277,824,000,000,000,000.

$$8 \times 10^{67}$$

Pascal's Identity

We will show how $\binom{n}{r}$ can be built up inductively.

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$$

Pascal's Identity

We will show how $\binom{n}{r}$ can be built up inductively.

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$$

$$\begin{aligned}
\binom{n-1}{r-1} + \binom{n-1}{r} &= \frac{(n-1)!}{(r-1)!(n-r)!} + \frac{(n-1)!}{r!(n-1-r)!} \\
&= \frac{r(n-1)!}{r!(n-r)!} + \frac{(n-r)(n-1)!}{r!(n-r)!} \\
&= \frac{(r+n-r)(n-r)(n-1)!}{r!(n-r)!} \\
&= \frac{n!}{r!(n-r)!} = \binom{n}{r}.
\end{aligned}$$

Pascal's Identity

We will show how $\binom{n}{r}$ can be built up inductively.

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$$

Pascal's Identity

We will show how $\binom{n}{r}$ can be built up inductively.

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$$

Split the n objects into $n-1$ and 1.

Pascal's Identity

We will show how $\binom{n}{r}$ can be built up inductively.

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$$

Split the n objects into $n-1$ and 1.

Now split all the choices of r objects into two:

- **Those choices that contain the n-th object.**
- **Those choices that do not contain it.**

Pascal's Identity

We will show how $\binom{n}{r}$ can be built up inductively.

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$$

Split the n objects into $n-1$ and 1.

Now split all the choices of r objects into two:

- Those choices that contain the n-th object.
- Those choices that do not contain it.

The first group has $\binom{n-1}{r-1}$ choices.

The second group has $\binom{n-1}{r}$ choices. QED.

A Walk on the Wild Side

Suppose you are in a city with a grid of streets.

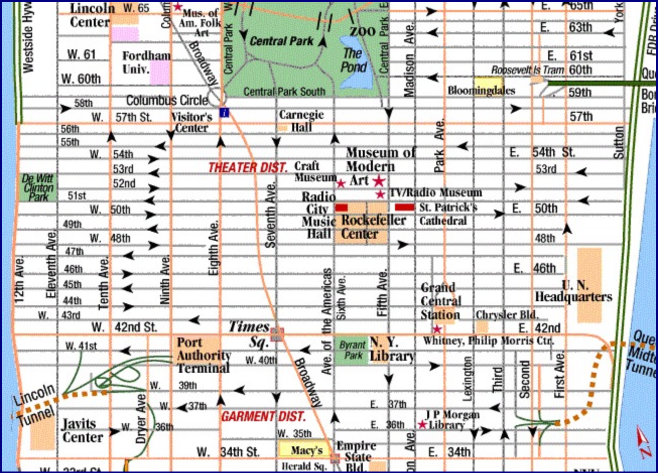
Suppose each block is 100m$\times$100m.

Suppose you have to travel m blocks East
and n blocks South.

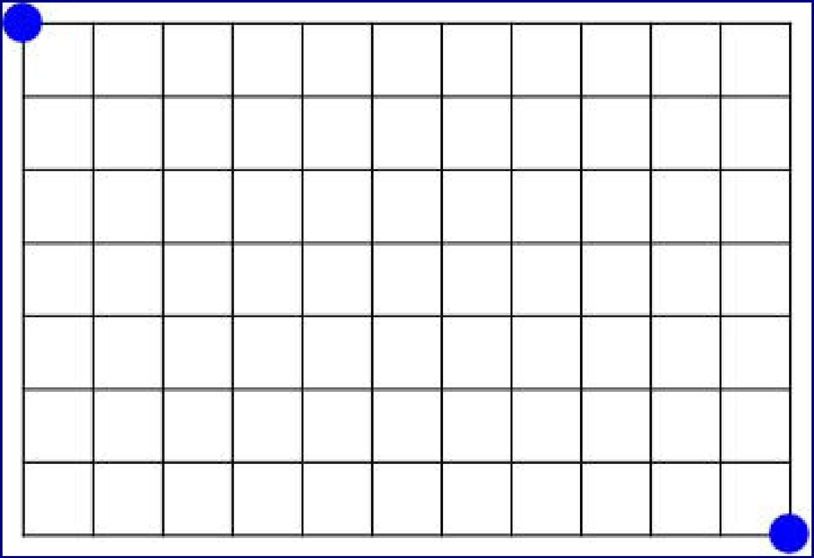
The shortest route is $(m + n)\times$100m.

How many ways of making the journey are there?

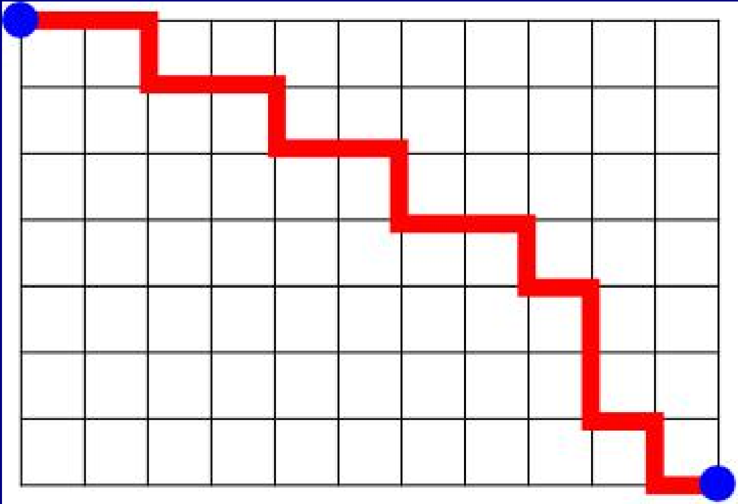
Midtown Manhattan Street Grid



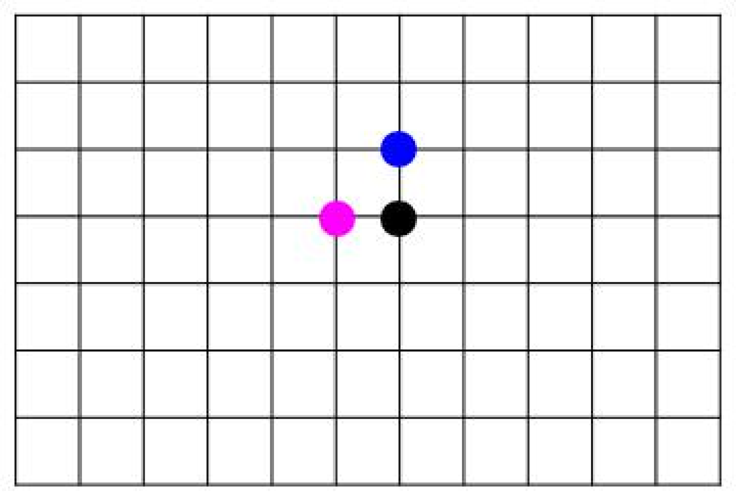
Manhattan Grid



Manhattan Grid

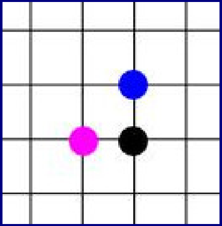


Manhattan Grid



Manhattan Grid

We consider **progressive paths** (going East or South). No backtracking West or North is permitted.



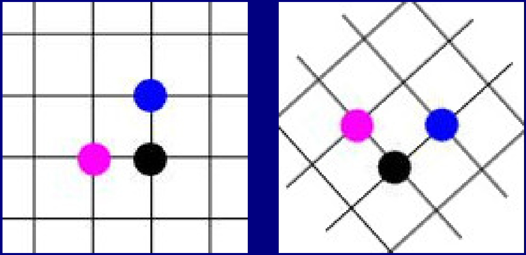
Every path through the Black Point must pass through either the Blue or Magenta Point.

No path passes through both Blue and Magenta Points.

Therefore: The number of paths to the Black Point is the sum of the paths to the Blue and Magenta Points.

Manhattan Grid

The Path-number for any point is the number of (progressive) paths from the origin to that point.



In the right-hand figure above, the Path-number for any point is the sum of the Path-numbers for the two points (diagonally) above it.

Pascal's Triangle

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

Pascal's Triangle

$$\binom{0}{0}$$

$$\binom{1}{0} \quad \binom{1}{1}$$

$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$

$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$

$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$

Outline

Distraction 10: Mathematics Everywhere

Look around the room.
What mathematical forms do you see?

I asked that question while waiting
in Blackrock Station for a train.

Some results of that quest are shown in
`https://thatsmaths.com/2016/05/26/mathematics-everywhere-in-blackrock-station/`

Just go to `thatsmaths.com`
and search for "Blackrock".

Outline

PItch and Frequency

Every pure musical pitch has a frequency.

Doubling the frequency corresponds
to moving up a full octave.

A musical note consists of a base frequency or pitch,
called the fundamental, and a series of harmonics.

PItch and Frequency

Every pure musical pitch has a frequency.

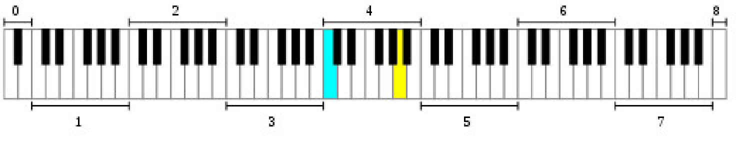
Doubling the frequency corresponds
to moving up a full octave.

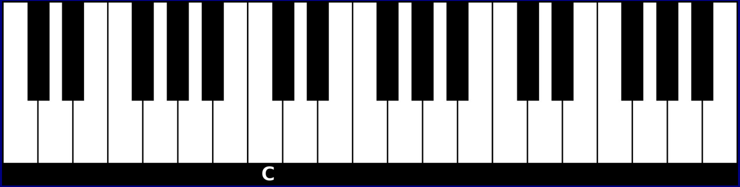
A musical note consists of a base frequency or pitch, called the fundamental, and a series of harmonics.

Harmonics are oscillations whose frequencies are whole-number multiples of the base frequency.

The note A sounds quite distinct when played on an oboe and a clarinet: the fundamentals are the same, but the overtones or harmonics are not.

The Piano Keyboard





Middle C

C is the first note of the C major scale.
Middle C is the ‘central note on the piano.

It is commonly pitched at 261.63 Hz.

The standard frequency of the note A4 is 440 Hz.

$$261.63 \times 2^{9/12} = 440$$

Middle C

C is the first note of the C major scale.
Middle C is the ‘central note on the piano.

It is commonly pitched at 261.63 Hz.

The standard frequency of the note A4 is 440 Hz.

$$261.63 \times 2^{9/12} = 440$$

Where does the peculiar factor $2^{9/12}$ come from?

We will look at **well-tempered scales** later.

Pythagorean Tuning

Pythagoras discovered that a perfect fifth — with frequency ratio 3:2 — is especially harmonious.

The entire musical scale can be constructed using only the ratios 2:1 (octaves) and 3:2 (fifths).

In the **tonic sol-fa** scale the eight notes of the major scale are **Do, Re, Mi, Fa, So, La, Ti, Do.**

Pythagorean Tuning

Starting with Do, the ratio 3:2 brings us to So.

Moving up another fifth, we have the ratio 9:4.

Reducing this by 2 to remain within the octave, we get 9:8, the note Re.

Pythagorean Tuning

Starting with Do, the ratio 3:2 brings us to So.

Moving up another fifth, we have the ratio 9:4.

Reducing this by 2 to remain within the octave, we get 9:8, the note Re.

Moving up another fifth gives a ratio 27:16 and brings us to La.

Continuing thus, we get all the “white notes” in the major scale. This is called Pythagorean tuning.

The Piano Keyboard

Do	Re	Mi	Fa	So	La	Ti	Do
1:1	9:8	81:64	4:3	3:2	27:16	243:128	2:1

The Pythagorean Comma

The Pythagoreans noticed that $2^{19} \approx 3^{12}$.

Going up twelve fifths, with ratio $(3/2)^{12}$ and down seven octaves $(1/2)^7$ gets us back (almost) to our starting point.

The number $3^{12}/2^{19} \approx 1.01364$ is called the Pythagorean comma.

The Pythagorean Comma

The Pythagoreans noticed that $2^{19} \approx 3^{12}$.

Going up twelve fifths, with ratio $(3/2)^{12}$ and down seven octaves $(1/2)^7$ gets us back (almost) to our starting point.

The number $3^{12}/2^{19} \approx 1.01364$ is called the Pythagorean comma.

As a result, the 13th note we get is not quite the same as the starting note. The two notes are enharmonics.

Triads and Just Intonation

The triad — three notes separated by 4 and 3 semitones, such as C–E–G — is of central importance in western music.

In the tuning scheme of Pythagoras, the third (C–E) has a frequency ratio of 81:64.

Triads and Just Intonation

The triad — three notes separated by 4 and 3 semitones, such as C–E–G — is of central importance in western music.

In the tuning scheme of Pythagoras, the third (C–E) has a frequency ratio of 81:64.

Generally, ratios with smaller numbers result in more pleasant sensations of sound.

Replacing 81:64 by 80:64 = 5:4 the three notes of the triad C–E–G are in the ratio 4:5:6.

Triads and Just Intonation

Likewise, changing the sixth note (A or "La") from 27:16 to 25:15 = 5:3 makes F–A–C a perfect triad with the frequency ratios 4:5:6.

Triads and Just Intonation

Likewise, changing the sixth note (A or "La") from 27:16 to 25:15 = 5:3 makes F–A–C a perfect triad with the frequency ratios 4:5:6.

Finally, if the monstrous 243:128 is replaced by 240:128 = 15:8, we get a scheme of tuning called just intonation.

Do	Re	Mi	Fa	So	La	Ti	Do
1:1	9:8	5:4	4:3	3:2	5:3	15:8	2:1

Pythagorean and Just Intonation

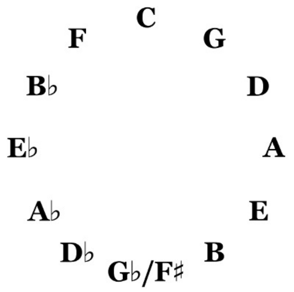
Do	Re	Mi	Fa	So	La	Ti	Do
1:1	9:8	81:64	4:3	3:2	27:16	243:128	2:1

Pythagorean intonation.

Do	Re	Mi	Fa	So	La	Ti	Do
1:1	9:8	5:4	4:3	3:2	5:3	15:8	2:1

Just intonation.

Organizing Scheme: the Circle of Fifths

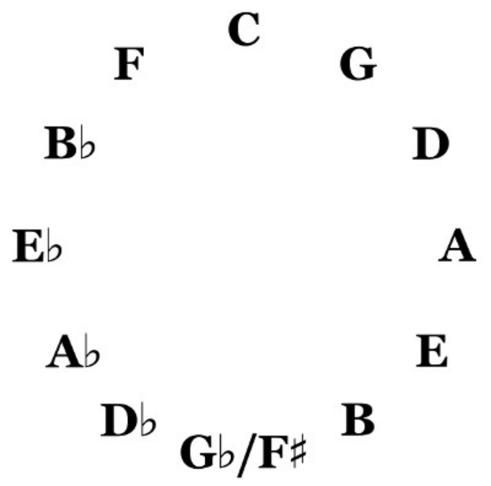


The **Circle of Fifths** represents the relationship between musical pitch and key signature.

It shows the twelve tones of the chromatic scale.

The Circle is useful in harmonising melodies and building chords.

Organizing Scheme: the Circle of Fifths



Tempered Scales

We will look at this topic next week!

Thank you