Sum-Enchanted Evenings

The Fun and Joy of Mathematics

•

LECTURE 5

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Outline

Outline

Meaning and Content of Mathematics

The word Mathematics comes from Greek $\mu\alpha\theta\eta\mu\alpha$ (máthéma), meaning "knowledge" or "study" or "learning".

It is the study of topics such as

- **Quantity (numbers)**
- **Structure (patterns)**
- **Space (geometry)**
- **Change (analysis).**

Outline

What are Axioms?

How can we prove a theorem, if we have nothing to start from?

We cannot prove something using nothing.

We need some starting point.

The basic building blocks are called Axioms.

Axioms are not proved, but are assumed true.

What are Axioms?

Axioms are important because the entire body of mathematics rests upon them.

If there are too few axioms, we can prove very little of interest from them.

If there are too many axioms, we can prove almost any result from them.

Consistency:
We must not have axioms that contradict each other.

What are Axioms?

Mathematicians assume axioms are true without being able to prove them.

This is not problematic, because axioms *are normally intuitively obvious.*

There are usually only a few axioms.

For example, we may assume that

$$a \times b = b \times a$$

for any two numbers a and b.

But Hamilton found that for two *quaternions,*

$$A \times B \neq B \times A \, .$$

Different sets of axioms lead to different kinds of mathematics.

Every area of mathematics has its own set of basic axioms.

When mathematicians have proven a theorem, they publish it for other mathematicians to check.

Sometimes a mistake in the proof is found.

Sometimes an error is not found for many years (e.g., an early "proof" of the *Four Colour Theorem.*)

In principle, it is possible to break a proof into steps starting from the basic axioms.

Euclid’s Axioms of Geomery

Euclid based his “Elements of Geometry” on a set of five postulates or axioms:

"Let the following be postulated":

1. "To draw a straight line from any point to any point."
2. "To produce [extend] a finite straight line continuously in a straight line."
3. "To describe a circle with any centre and distance [radius]."
4. "That all right angles are equal to one another."
5. *The parallel postulate*: "That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles."

The fifth postulate, the parallel postulate, has been a great source of controversy and confusion. This has led to *completely new areas of mathematics.*

Peano's Axioms of Arithmetic

Giuseppi Peano constructed five axioms to build up the set $\mathbb{N}$ of natural numbers:

$$\exists 0 : 0 \in \mathbb{N}$$

$$\forall n \in \mathbb{N} : \exists n' \in \mathbb{N}$$

$$\neg(\exists n \in \mathbb{N} : n' = 0)$$

$$\forall m, n \in \mathbb{N} : m' = n' \Rightarrow m = n$$

$$\forall A \subseteq \mathbb{N} : (0 \in A \wedge (n \in A \Rightarrow n' \in A)) \Rightarrow A = \mathbb{N}$$

The natural numbers may then be extended to the integers, rational numbers and real numbers.

Axioms of Set Theory

Set theory is the basic language of mathematics.

Many mathematical problems can be formulated in the language of set theory.

To prove them we need the *Set Theory Axioms.*

The most widely accepted axioms are the set of nine Zermelo-Fraenkel (ZF) axioms.

A tenth axiom, may also be assumed, the *Axiom of Choice.*

Zermelo-Fraenkel axioms

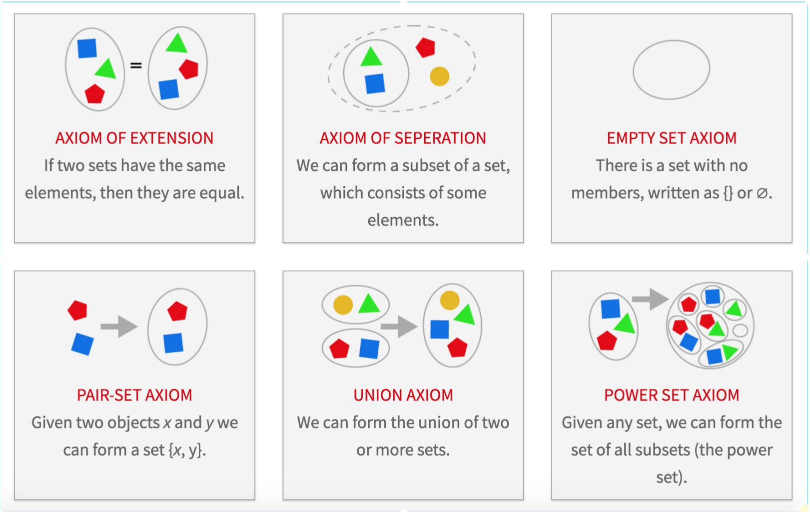


Image from Mathigon.org

Zermelo-Fraenkel axioms

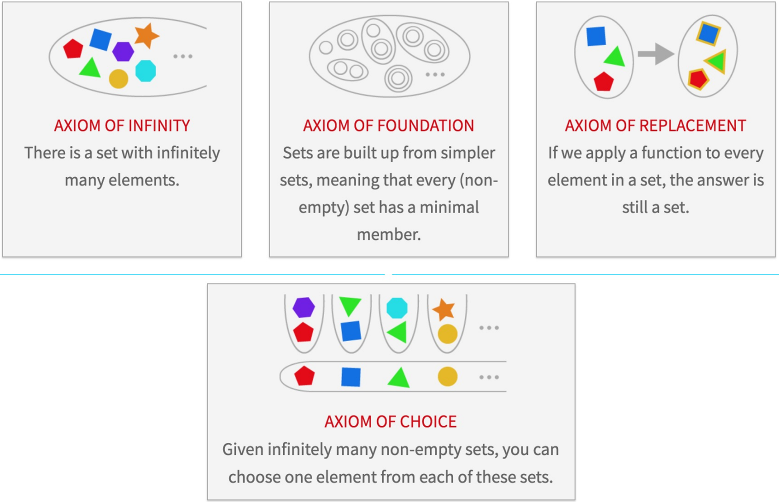


Image from Mathigon.org

Axiom of Choice

The Axiom of Choice (AC) looks just as innocuous as the other nine axioms.
However it has unexpected consequences.

We can use AC to prove that it is possible to cut a sphere into five pieces and reassemble them into two spheres, each identical to the initial sphere.

This result is called the Banach-Tarski Theorem.

The five pieces have fractal boundaries:
they can't actually be made in practice.

Also, they are not *measurable:*
they have no definite volume.

The Current Status

There is an active debate among logicians about whether to accept the Axiom of Choice or not.

Every collection of axioms forms a different "mathematical world".

Different theorems may be true in different worlds.

The question is: are we happy to live in a world where we can make two spheres from one.

See Wikipedia article: Axiom of Choice

Outline

The Connection

The Pythagoreans' quest was to find *the eternal laws of the Universe,* and they organized their studies into the scheme later known as the Quadrivium.

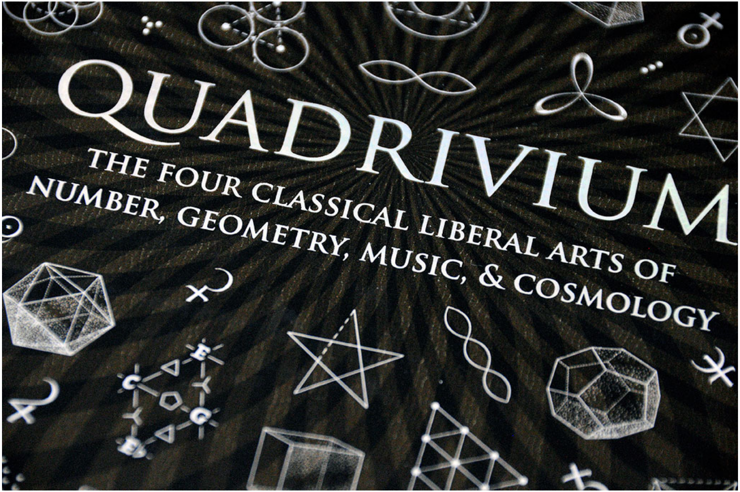
It comprised four disciplines:

- **Arithmetic**
- **Geometry**
- **Music**
- **Astronomy**

Pythagorean Quotation:

- **"There is geometry in the humming of the strings, There is music in the spacing of the spheres."**

The Quadrivium



Static/Dynamic. Pure/Applied

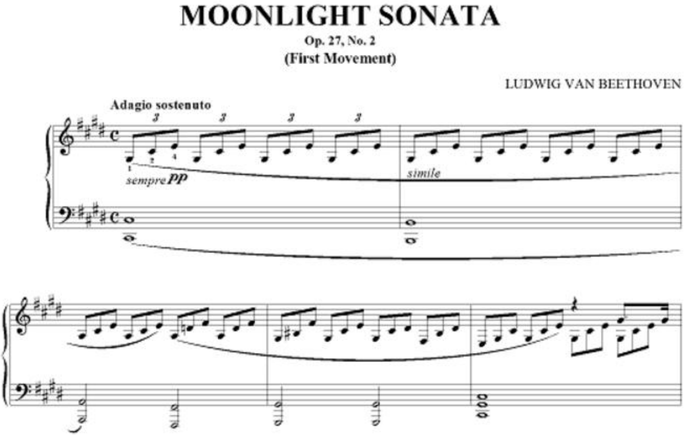
- **Arithmetic: Static number**
- **Music: Dynamic number**

Arithmetic represents numbers at rest.

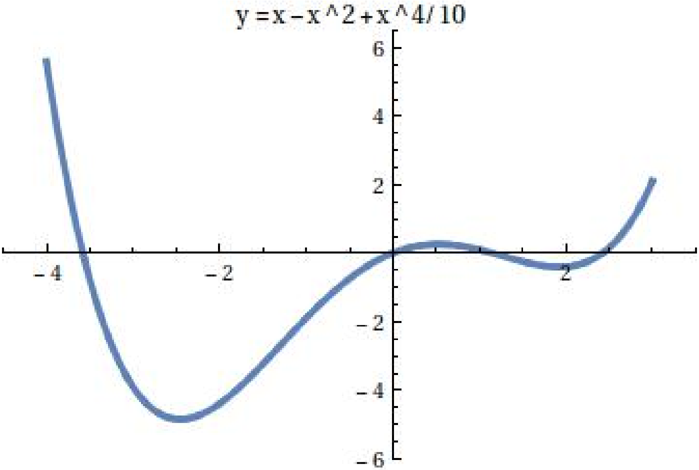
Music is numbers in motion.

Arithmetic is *pure or abstract* in nature.

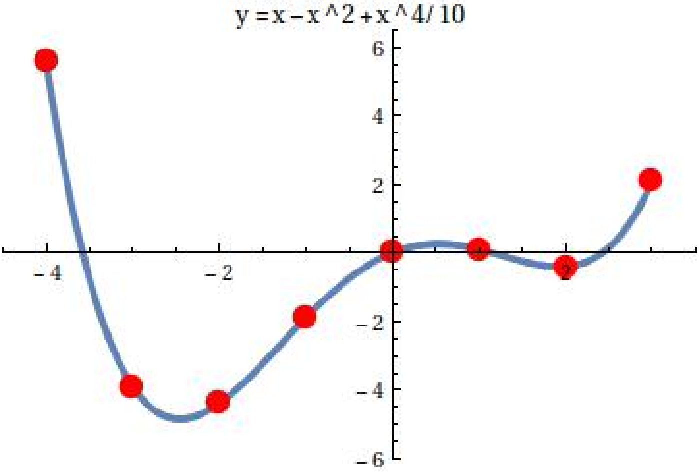
Music is *applied or concrete* in nature.

Music is Written using a Special Notation

The opening bars of Beethoven's *Moonlight Sonata*

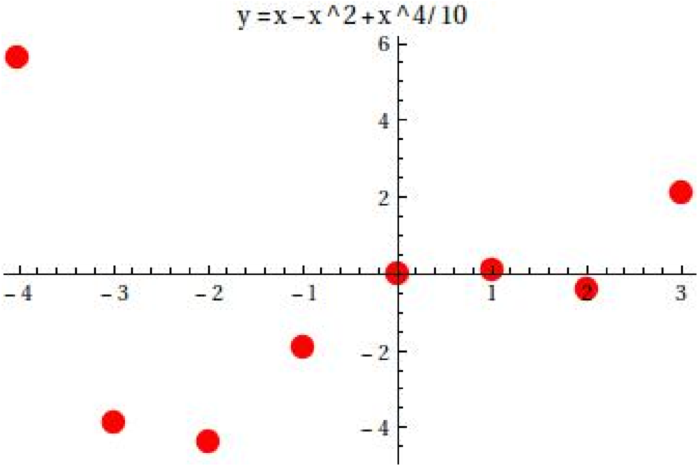
A Mathematical Graph: Continuous



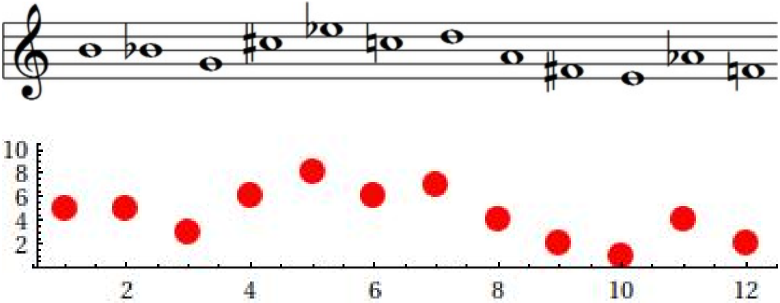
A Graph: Discrete and Continuous



A Mathematical Graph: Discrete



A Musical Score: One Voice



A musical score is just a graph of *pitch versus time.*

Outline

The Greek Alphabet, Part 4

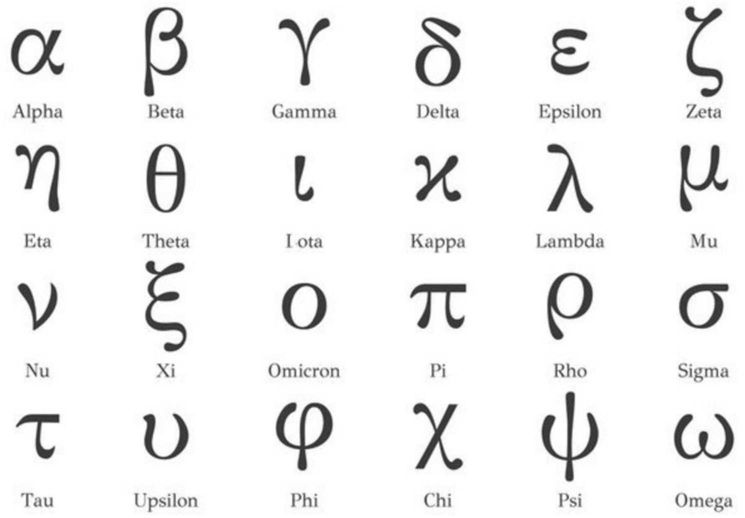


Figure : 24 beautiful letters

The Last Six Letters

We will consider the final group of six letters.

$$\tau \quad \upsilon \quad \phi \quad \chi \quad \psi \quad \omega$$

$$\mathrm{T} \quad \Upsilon \quad \Phi \quad \mathrm{X} \quad \Psi \quad \Omega$$

Let us focus first on the *small letters*
and come back to the big ones later.

$$\tau \quad \upsilon \quad \phi \quad \chi \quad \psi \quad \omega$$

Tau: You have certainly heard of a Tau-cross: τ.

Upsilon (υ) or u-psilon means 'bare u'.
It is often transliterated as 'y'.

Phi (ϕ) is ' f ', often used for latitude
(as λ is often used for longitude).

Chi (χ) has a 'ch' or 'k' sound.

Psi (ψ) is very common: psychology, etc.

Omega (ω) is the end: Alpha and Omega $\left(\frac{\mathrm{A}}{\Omega}\right)$.

Now you know 24 letters. You should get a diploma.

A Few Greek Words (for practice)

$\kappa\omega\mu\alpha$
$\psi\upsilon\kappa\eta$
$\kappa\rho\iota\sigma\iota\varsigma$

$\alpha\nu\alpha\theta\epsilon\mu\alpha$
$\alpha\mu\beta\rho o\sigma\iota\alpha$
$\kappa\alpha\tau\alpha\sigma\tau\rho o\phi\eta$

Coma: $\kappa\omega\mu\alpha$
Psyche: $\psi\upsilon\kappa\eta$
Crisis: $\kappa\rho\iota\sigma\iota\varsigma$

Anathema: $\alpha\nu\alpha\theta\epsilon\mu\alpha$
Ambrosia: $\alpha\mu\beta\rho o\sigma\iota\alpha$
Catastrophe: $\kappa\alpha\tau\alpha\sigma\tau\rho o\phi\eta$


Αεροδρόμιο
Airport
Ξενοφώντος
Ομόνοια
Μητροπόλεως
Ερμού
Πειραιώς
Β. Σοφίας


Ο Θεός

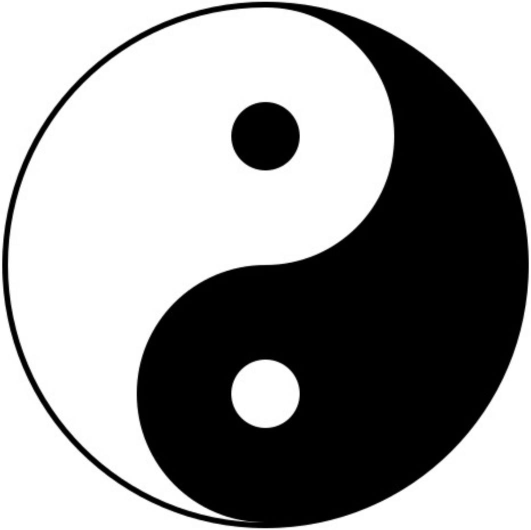

Εκκλησία


Ο Θεός
God
Εκκλησία
Church

End of Greek 104

Outline

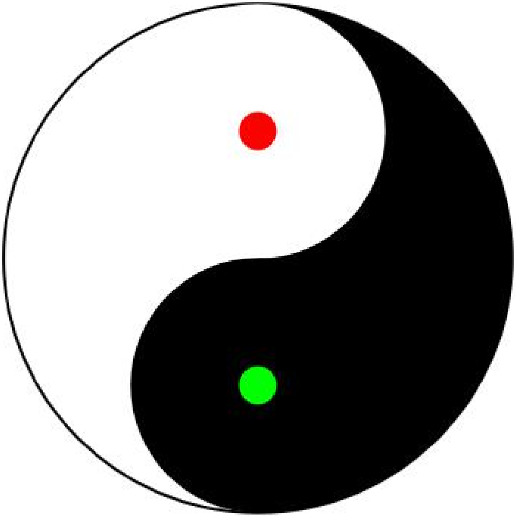
Distraction 9: The Yin Yang Symbol



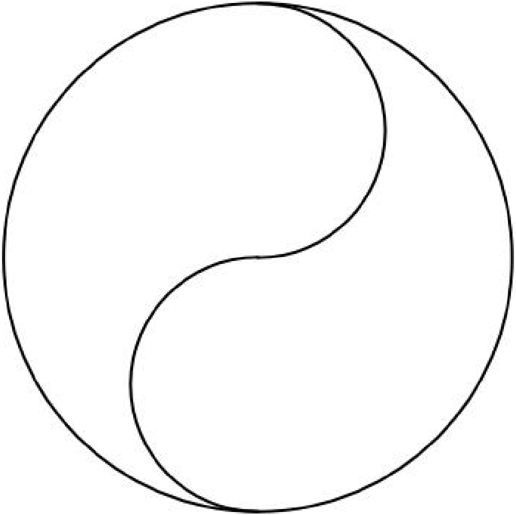
Distraction 9: A Simple Puzzle

Divide this area into two *identical parts* (*congruent* parts) by drawing a *single curve*.

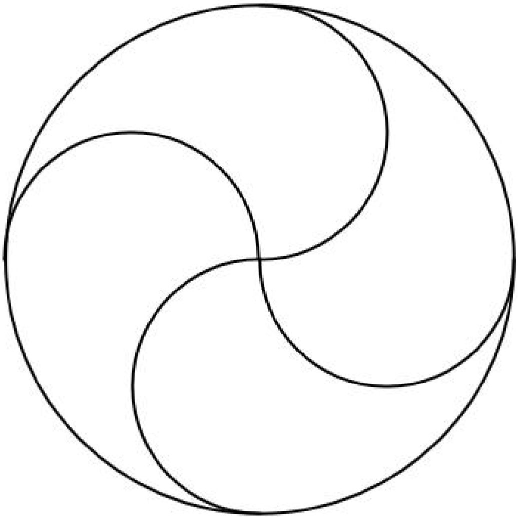
Distraction 9: A Simple Puzzle



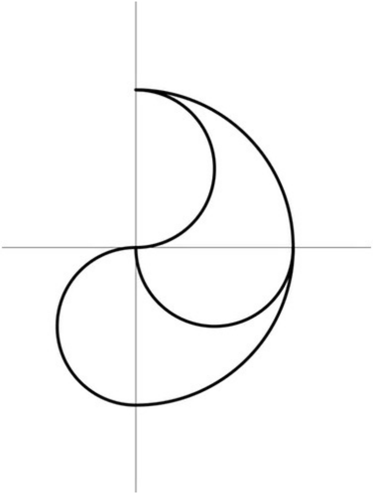
Distraction 9: A Simple Puzzle



Distraction 9: A Simple Puzzle

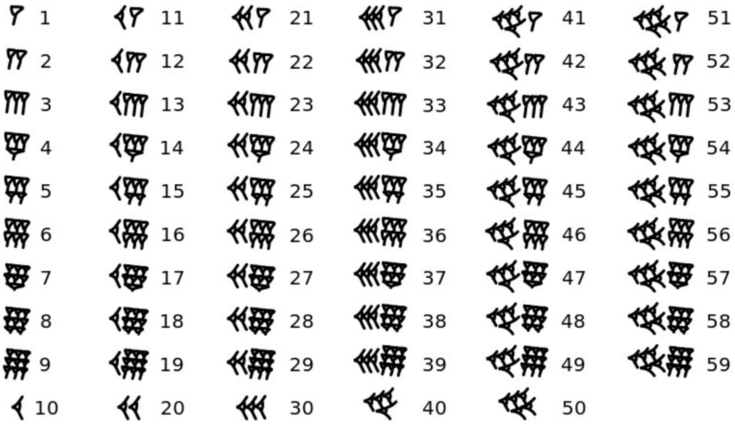


Distraction 9: A Simple Puzzle

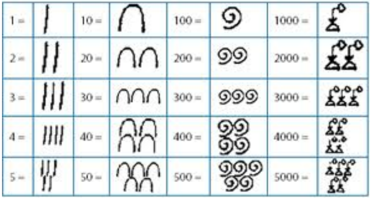


Outline

Babylonian Numerals



Ancient Egyptian Numerals



Ancient Hebrew and Greek Numerals

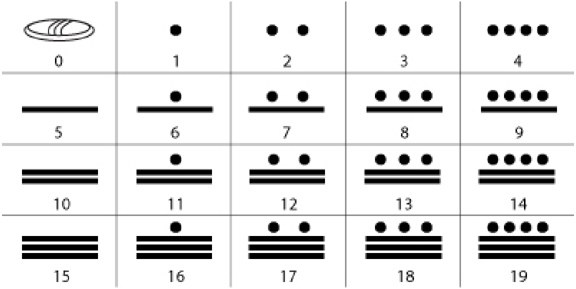
8	7	6	5	4	3	2	1
ח	ז	ו	ה	ד	ג	ב	א
Chet	Zayin	Vav	Hey	Dalet	Gimmel	Bet	Aleph

70	60	50	40	30	20	10	9
ע	ס	נ	מ	ל	כ	י	ט
Ayin	Samekh	Nun	Mem	Lamed	Kaf	Yod	Tet

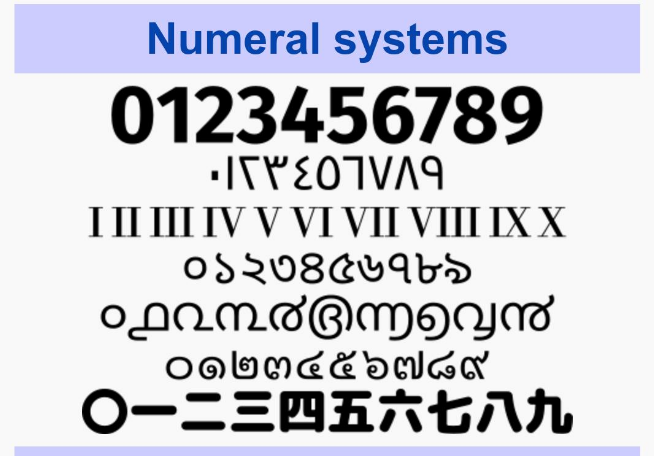
1	α	alpha	10	ι	iota	100	ρ	rho
2	β	beta	20	κ	kappa	200	σ	sigma
3	γ	gamma	30	λ	lambda	300	τ	tau
4	δ	delta	40	μ	mu	400	υ	upsilon
5	ϵ	epsilon	50	ν	nu	500	ϕ	phi
6	ϛ	vau*	60	ξ	xi	600	χ	chi
7	ζ	zeta	70	o	omicron	700	ψ	psi
8	η	eta	80	π	pi	800	ω	omega
9	θ	theta	90	ϙ	koppa*	900	ϡ	sampi

*vau, koppa, and sampi are obsolete characters

Mayan Numerals



Various Numeral Systems



Wikipedia: Hindu-Arabic Numeral System

Roman Numerals

I	1	XXI	21	XLI	41
II	2	XXII	22	XLII	42
III	3	XXIII	23	XLIII	43
IV	4	XXIV	24	XLIV	44
V	5	XXV	25	XLV	45
VI	6	XXVI	26	XLVI	46
VII	7	XXVII	27	XLVII	47
VIII	8	XXVIII	28	XLVIII	48
IX	9	XXIX	29	XLIX	49
X	10	XXX	30	L	50
XI	11	XXXI	31	LI	51
XII	12	XXXII	32	LII	52
XIII	13	XXXIII	33	LIII	53
XIV	14	XXXIV	34	LIV	54
XV	15	XXXV	35	LV	55
XVI	16	XXXVI	36	LVI	56
XVII	17	XXXVII	37	LVII	57
XVIII	18	XXXVIII	38	LVIII	58
XIX	19	XXXIX	39	LIX	59
XX	20	XL	40	LX	60

In order: $M\ D\ C\ L\ X\ V\ I\ = 1666$

How to Multiply Roman Numbers

Table : Multiplication Table for Roman Numbers.

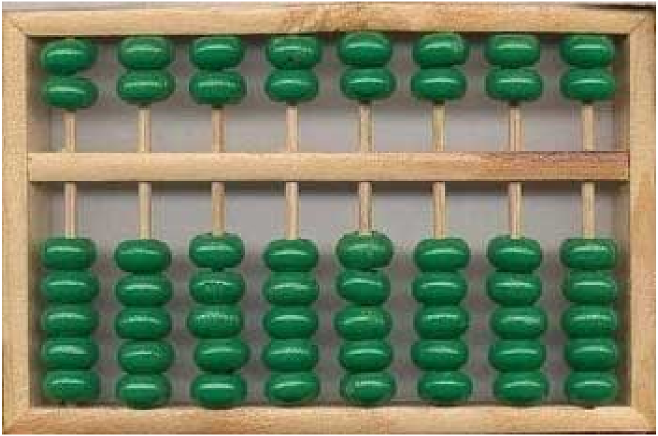
	I	**V**	**X**	**L**	**C**	**D**	**M**
I	*I*	*V*	*X*	*L*	*C*	*D*	*M*
V	*V*	*XXV*	*L*	*CCL*	*D*	*MMD*	$\overline{V}$
X	*X*	*L*	*C*	*D*	*M*	$\overline{V}$	$\overline{X}$
L	*L*	*CCL*	*D*	*MMD*	$\overline{V}$	$\overline{XXV}$	$\overline{L}$
C	*C*	*D*	*M*	$\overline{V}$	$\overline{X}$	$\overline{L}$	$\overline{C}$
D	*D*	*MMD*	$\overline{V}$	$\overline{XXV}$	$\overline{L}$	$\overline{CCL}$	$\overline{D}$
M	*M*	$\overline{V}$	$\overline{X}$	$\overline{L}$	$\overline{C}$	$\overline{D}$	$\overline{M}$

A Roman Abacus

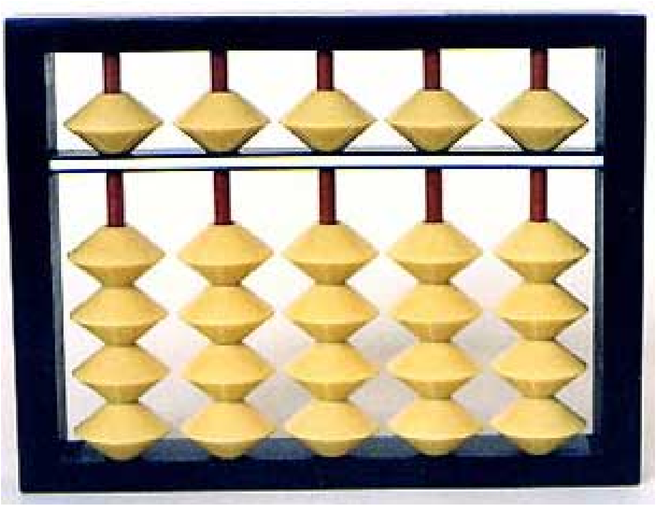
Replica of a Roman abacus from 1st century AD.

Abacus is a Latin word, which comes from the Greek $\alpha\beta\alpha\kappa\alpha\varsigma$ (board or table).

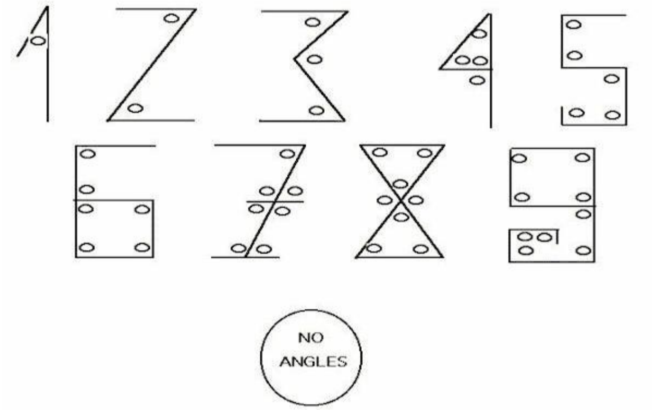
A Chinese Abacus: Suan Pan



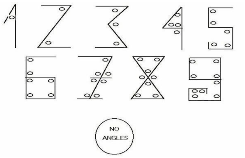
A Japanese Abacus: Soroban



A Different Angle on Numerals



Delightful theory. Almost certainly wrong.



Arguments "for"

1. It is a very simple idea
2. It links symbols to numerical values

Arguments "against"

1. Number forms modified to fit model
2. Complete lack of historical evidence.

The great tragedy of science —

the slaying of a beautiful hypothesis by an ugly fact (T H Huxley)

Outline

A Hierarchy of Numbers

We will introduce a hierarchy of numbers.

Each set is contained in the next one.

They are like a set of nested Russian Dolls:

Matryoshka

The Natural Numbers $\mathbb{N}$

The *counting numbers* were the first to emerge:

$$1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \ \ldots$$

They are also called the *Natural Numbers*.

We can arange the natural numbers in a list.

This list is like a toy computer.

The Natural Numbers $\mathbb{N}$

The set of natural numbers is denoted $\mathbb{N}$.

If n is a natural number, we write $n \in \mathbb{N}$.

Natural numbers can be added: $4 + 2 = 6 \in \mathbb{N}$

But not always subtracted: $4 - 6 = -2 \notin \mathbb{N}$.

To allow for subtraction we have to extend $\mathbb{N}$.

The Integers $\mathbb{Z}$

We extend the counting numbers by adding the negative whole numbers:

$$\ldots \quad -3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad \ldots$$

The whole numbers are also called the *Integers*.

The set of integers is denoted $\mathbb{Z}$.

If k is an integer, we write $k \in \mathbb{Z}$.

Clearly,

$$\mathbb{N} \subset \mathbb{Z}$$

Integers can be added and subtracted.

They can also multiplied:

$$6 \times 4 = 24 \in \mathbb{Z} .$$

However, they cannot usually be divided:

$$\frac{6}{4} = 1\tfrac{1}{2} \notin \mathbb{Z} .$$

To allow for division we have to extend $\mathbb{Z}$.

The Rational Numbers $\mathbb{Q}$

We extend the integers by adding fractions:

$$r = \frac{p}{q} \qquad \text{where } p \text{ and } q \text{ are integers.}$$

These *rational numbers* are ratios of integers.

The set of rational numbers is denoted $\mathbb{Q}$.

If r is a rational number, we write $r \in \mathbb{Q}$.

Clearly,

$$\mathbb{Z} \subset \mathbb{Q}$$

With the Rational Numbers, we can:

Add, Subtract, Multiply and Divide

That is, for any $p \in \mathbb{Q}$ and $q \in \mathbb{Q}$

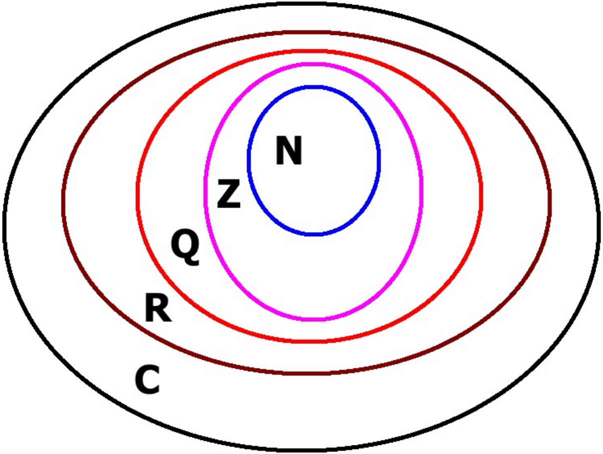
All of $p + q$ $p - q$ $p \times q$ and $p \div q$

are rational numbers.

We say that $\mathbb{Q}$ is *closed under addition, subtraction, multiplication and division.*

But we are not yet finished. $\mathbb{R}$ is yet to come.

The Hierarchy of Numbers



$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$$

The Hierarchy of Numbers

Each set is contained in the next one.

They are like a set of nested Russian Dolls:

Matryoshka

Thank you