Sum-Enchanted Evenings

The Fun and Joy of Mathematics

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LECTURE 3

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Outline

Outline

Meaning and Content of Mathematics

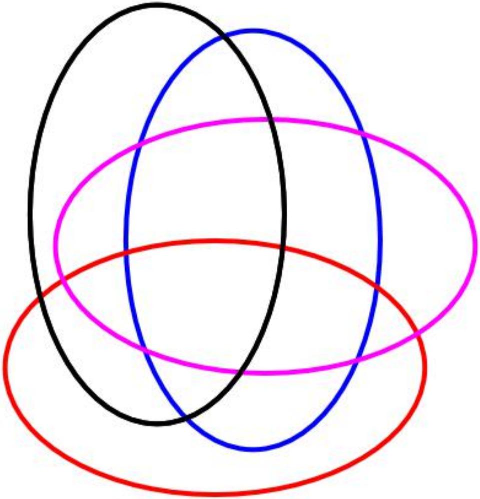
The word Mathematics comes from Greek $\mu\alpha\theta\eta\mu\alpha$ (máthéma), meaning "knowledge" or "study" or "learning".

It is the study of topics such as

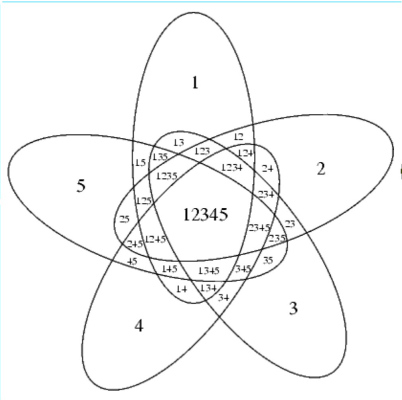
- Quantity (numbers)
- Structure (patterns)
- Space (geometry)
- Change (analysis).

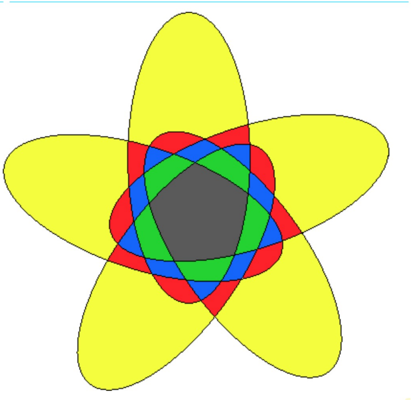
Outline

Venn Diagram for 4 Sets

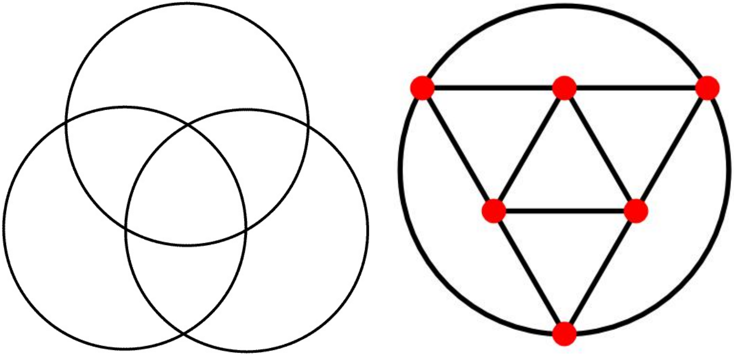


Venn Diagram for 5 Sets



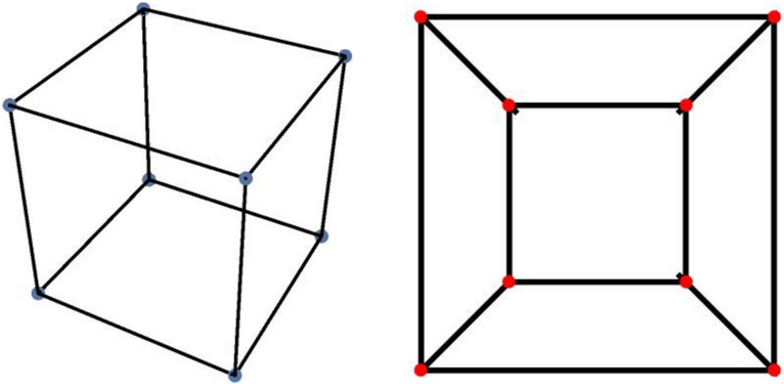


Venn Diagram as a Graph



Graph is equivalent to an octahedron

Venn3 Dual as a Cube



See blog post

Venn Again's Awake

on my mathematical blog `thatsmaths.com`

There is No Largest Number

Children often express bemusement at the idea that there is no largest number.

Given any number, 1 can be added to it to give a larger number.

But the implication that there is no limit to this process is perplexing.

The concept of infinity has exercised the greatest minds throughout the history of human thought.

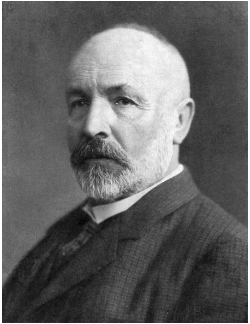
Degrees of Infinity

In the late 19th century, Georg Cantor showed that there are different degrees of infinity.

In fact, there is an infinite hierarchy of infinities.

Cantor brought into prominence several paradoxical results that had a profound impact on the development of logic and of mathematics.

Georg Cantor (1845–1918)



Cantor discovered many remarkable properties of infinite sets.

Cardinality

Finite Sets have a finite number of elements.

Example: The Counties of Ireland form a finite set.

$$\text{Counties} = \{\text{Antrim, Armagh, } \ldots, \text{ Wexford, Wicklow}\}$$

For a finite set A, the *cardinality* of A is:

The number of elements in A

One-to-one Correspondence

A particular number, say 5, is associated with all the sets having five elements.

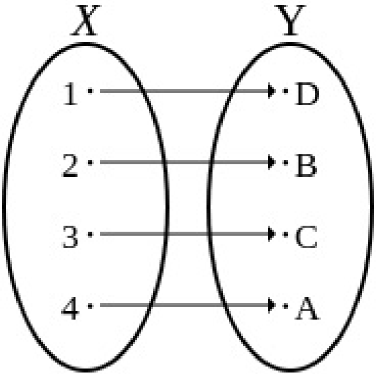
For any two of these sets, we can find a 1-to-1 correspondence between the elements of the two sets.

The number 5 is called the cardinality of these sets.

Generalizing this:

Any two sets are the same size (or cardinality) if there is a 1-to-1 correspondence between them.

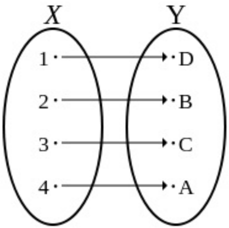
One-to-one Correspondence



Equality of Set Size: 1-1 Correspondence

How do we show that two sets are the same size?

For finite sets, this is straightforward counting.



For infinite sets, we must find a 1-1 correspondence.

Cardinality

The number of elements in a set is called the cardinality of the set.

Cardinality of a set A is written in various ways:

$$|\mathbf{A}| \qquad \|\mathbf{A}\| \qquad \mathrm{card}(\mathbf{A}) \qquad \#(A)$$

For example

$$\#\{\textbf{Irish Counties}\} = 32$$

The Empty Set

We call the set with *no elements* the empty set.

It is denoted by a special symbol

$$\varnothing = \{ \ \}$$

Clearly

$$\#\{ \ \} = 0 \, .$$

We could have a philosophical discussion about the empty set. Is it related to a perfect vacuum?

The Greeks regarded the vacuum as an impossibility.

The Natural Numbers $\mathbb{N}$

The *counting numbers* (positive whole numbers) are

1 2 3 4 5 6 7 8 . . . .

They are also called the *Natural Numbers*.

The set of natural numbers is denoted $\mathbb{N}$.

This is our first infinite set.

We use a special symbol to denote its cardinality:

$$\#(\mathbb{N}) = \aleph_0$$

$$\aleph_0$$

The Power Set

For any set, we can form a new one, the Power Set.

The Power Set is the *set of all subsets of* A.

Suppose the set A has just two elements:

$$\mathbf{A} = \{a, b\}$$

Here are the subsets of A:

$$\{\ \} \qquad \{a\} \qquad \{b\} \qquad \{a, b\}$$

The *power set* is

$$\mathrm{P}[\mathbf{A}] = \Big\{ \{\ \}, \{a\}, \{b\}, \{a, b\} \Big\}$$

Cantor's Theorem

Cantor's theorem states that, for any set A, the power set of A has a strictly greater cardinality than A itself.

$$\#[\mathrm{P}(\mathbf{A})] > \#[A]$$

This holds for both finite and infinite sets.

It means that, for every cardinal number, there is a greater cardinal number.

One-to-one Correspondence

Now we consider sets are infinite:
take all the natural numbers,

$$\mathbb{N} = \{1, 2, 3, ...\}$$

as one set and all the even numbers

$$\mathbb{E} = \{2, 4, 6, ...\}$$

as the other.

By associating each number $n \in \mathbb{N}$ with $2n \in \mathbb{E}$,
we have a perfect 1-to-1 correspondence.

By Cantor's argument, the two sets are the same size:

$$\#[\mathbb{N}] = \#[\mathbb{E}]$$

Again,

$$\#[\mathbb{N}] = \#[\mathbb{E}]$$

But this is *paradoxical:* The set of natural numbers contains all the even numbers

$$\mathbb{E} \subset \mathbb{N}$$

and also all the odd ones.

In an intuitive sense, $\mathbb{N}$ is larger than $\mathbb{E}$.

The same paradoxical result had been deduced by *Galileo* some 250 years earlier.

Cantor carried these ideas much further:

The set of all the real numbers has a degree of infinity, or cardinality, greater than the counting numbers:

$$\#[\mathbb{R}] > \#[\mathbb{N}]$$

Cantor showed this using an ingenious approach called the diagonal argument.

This is a fascinating technique, but we will not give details here.

Review: Infinities Without Limit

For any set A, the power set P(A) is
the collection of all the subsets of A.

Cantor proved P(A) has cardinality greater than A.

For finite sets, this is obvious;
for infinite ones, it was startling.

The result is now known as Cantor's Theorem,
and Cantor used his diagonal argument in proving it.

He thus developed an entire hierarchy
of transfinite cardinal numbers.

Outline

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Enigmas of Infinity

Zeno of Elea devised several paradoxes involving infinity.

He argued that one cannot travel from A to B: to do so, one must first travel half the distance, then half of the remaining half, then half the remainder, and so on.

He concluded that motion is logically impossible.

Zeno was misled by his belief that the sum of finite quantities must grow without limit as more are added.

Enigmas of Infinity

Systematic mathematical study of infinite sets began around 1875 when Georg Cantor developed a theory of transfinite numbers.

He reasoned that the method of comparing the sizes of finite sets could be carried over to infinite ones.

If two finite sets, for example the cards in a deck and the weeks in a year, can be matched up one to one they must have the same number of elements.

Bijections

Mathematicians call a 1:1 correspondence a bijection.

Cantor used this approach to compare infinite sets: if there is a bijection between them, two sets are said to be *the same size.*

Cantor built an entire theory of infinity on this idea.

Hilbert's Hotel

We will look at a fantasy devised by David Hilbert.

We could call it a Gedankenexperiment

It was introduced in 1924 in a lecture
Über das Unendliche.

Hilbert's Hotel

Hilbert's Grand Hotel

Leading German mathematician David Hilbert constructed *an amusing metaphor* to illustrate the surprising and counter-intuitive properties of infinity.

He imagined a hotel with an infinite number of rooms.

Even with the hotel full, there is always room to accommodate an extra guest.

Simply move guest 1 to room 2, guest 2 to room 3 and so on, thereby vacating the first room.

Indeed, an infinite number of new arrivals could be accommodated: for all rooms n, move the guest in room n to room 2n, and magically all the odd-numbered room become vacant.

Indeed, a countably infinite number of busses, each with a countably infinite number of passengers, can be accommodated.

Video:
https://www.youtube.com/watch?v=Uj3_KqkI9Zo&t=191s

http://world.mathigon.org/resources/Infinity/Miss_Marple.mp4

Infinitesimals

Infinite quantities are unboundedly large.

At the opposite extreme, *infinitesimals are infinitely small quantities.*

An infinitesimal is smaller than any finite number, yet greater than zero.

Such quantities were used by Leibniz and Newton in formulating differential and integral calculus.

They were the cause of great controversy that raged for centuries.

Bishop George Berkeley

In his satirical critique on the foundations of mathematics, Irish bishop George Berkeley described infinitesimals as *the ghosts of departed quantities.*

Berkeley's witty polemic was justified: the foundations of mathematical analysis were unsound.

The problems were resolved only by a rigorous theory of limits, devised around 1820 by Augustin-Louis Cauchy and Karl Weierstrass.

Vanishing points used in perspective art correspond to mathematical points at infinity. They allow artists to render forms and distances realistically.

The dutch artist M. C. Escher was a genius at exploiting the concept of infinity and its paradoxes.

Mathematicians deal with infinity on a daily basis. The concept of infinity is essential in analysis (calculus) and set theory.

Zoom in on the arc of a circle. It approximates a line segment, but never quite gets there. Its length would have to be infinitesimal before we could truly call it straight.

Archimedes used this idea in his calculation of π. "A circle is a line under a microscope."

Outline

The Greek Alphabet, Part 2

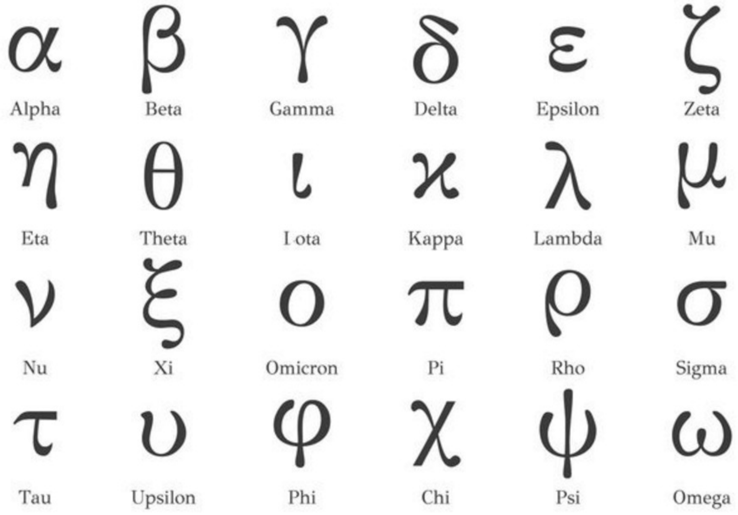


Figure : 24 beautiful letters

The Next Six Letters

We will consider the second group of six letters.

η	θ	ι	κ	λ	μ
H	Θ	I	K	Λ	M

Let us focus first on the *small letters* and come back to the big ones later.

We already met the *Riemann zeta-function;* when the signs alternate, it becomes the *eta-function:*

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} \qquad \eta(z) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^z}$$

Angles are very often denoted θ.

We use the term *iota* for a tiny quantity.
This comes from the Greek letter ι .

The three letters κ, λ, μ are like K, L, M
Also, μ is used for one-millionth: 1μm is a micro-meter.

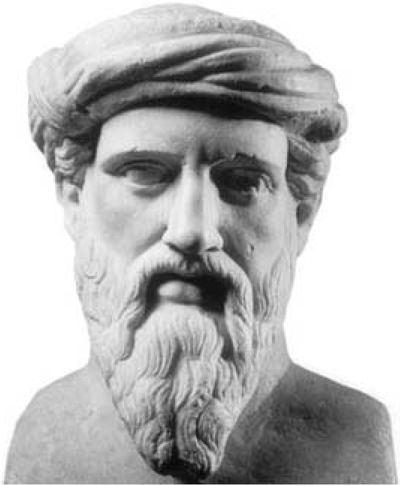
Now we know the next six letters. We're half way there!

A Few Greek Words (for practice)

$\beta\iota\beta\lambda\iota o$	**Book:** $\beta\iota\beta\lambda\iota o$
$\iota\delta\epsilon\alpha$	**Idea:** $\iota\delta\epsilon\alpha$
$\kappa\lambda\iota\mu\alpha\xi$	**Climax:** $\kappa\lambda\iota\mu\alpha\xi$

End of Greek 102

Outline



The Thallasic Age

The period from 800 BC to AD 800.

$$\Theta\alpha\lambda\alpha\sigma\sigma\alpha$$ — the Sea.

- The first Olympic Games in 776 BC
- Homer and Hesiod lived around 700 BC
- Greek mathematics began to thrive
- First two major figures: Thales and Pythagoras.

Pythagoras (c. 570–495 BC)

Pythagoras was

- Born on the island of Samos (off Turkey).
- Philosopher, mystic, prophet and religious leader.
- Contemporary with Confucius and Lao-Tzu.

Words philosophy (love of learning) and mathematics (that which is learned) attributed to Pythagoras.

May have been first person to imagine that natural phenomena can be understood through mathematics.

Pythagoras (c. 570–495 BC)

- No contemporary documents
- Myth, legend and tradition
- Second or third hand accounts
  often written centuries later
- Aristotle's biography no longer extant.

Hardly any statement about Pythagoras uncontested.

Difficult to separate history from myth and legend.

Pythagoras (c. 570–495 BC)

- **Travelled to Egypt, Babylon and perhaps India**
- **Mathematics, astronomy and religious lore**
- **Theorem on right-angled triangles**
- **Result known to Babylonians 1000 years earlier**
- **No record of a proof by Pythagoras survives.**

The Pythagoreans

Around 530 BC Pythagoras moved to Croton in Magna Graecia (now Southern Italy).

He established an organization or school (philosophical/religious/political).

Both men and women were members of "The Pythagoreans"

Adherents were very secretive:
Bound by an oath of allegiance

Led lives of temperance; observed strict moral codes.

Pythagorean Women

“Women were given equal opportunity to study as Pythagoreans, and learned practical domestic skills in addition to philosophy.

“Women were held to be different from men, sometimes in positive ways.

“The priestess, philosopher and mathematician Themistoclea is regarded as Pythagoras’ teacher; Theano, Damo and Melissa as female disciples.”

From the Wikipedia article: *The Pythagoreans.*

Pythagorean Quotes

- "I was *Euphorbus* at the siege of Troy."
- "In anger, refrain from both speech and action."
- "Educate the children and it won't
  be necessary to punish the men."
- "Abstain from beans!"

- "There is geometry in the humming of the strings,
  There is music in the spacing of the spheres."
- "Number rules the universe."

Harmony & Discord

By tradition, Pythagoras discovered the principles of *musical harmony.*

Stringed instruments produce harmonious sounds when string lengths are ratios of small numbers.

Extended this idea to the heavens: planets emit sounds according to their speed of movement

Concept of the harmony of the spheres.

Johannes Kepler: Harmonices Mundi

All is Number

The motto of the Pythagoreans: ***All is Number.***

All natural phenomena in the universe can be expressed using whole numbers or ratios of them.

For the Pythagoreans, numbers were the essence and source of all things.

Modern physics holds that, at its deepest level, the universe is mathematical in nature.

This view is a topic of current serious discussion (*The Mathematical Universe*, by Max Tegmark).

Outline

Introduction

Set Theory II

Hilbert's Hotel

Greek 2

The Pythagoreans

Maths Week

Maths Week 2017

Next week is Maths Week Ireland.

It is the biggest event of its kind in the World.

For information, see
`http://www.mathsweek.ie/2017/`

Thank you