Sum-Enchanted Evenings

The Fun and Joy of Mathematics

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LECTURE 1

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Evening Course, UCD, Autumn 2017

Outline

Outline

Meaning and Content of Mathematics

The word Mathematics comes from Greek $\mu\alpha\theta\eta\mu\alpha$ (máthéma), meaning "knowledge" or "study" or "learning".

It is the study of topics such as

- **Quantity**
- **Structure**
- **Space**
- **Change**

Meaning and Content of Mathematics

The word Mathematics comes from Greek $\mu\alpha\theta\eta\mu\alpha$ (máthéma), meaning "knowledge" or "study" or "learning".

It is the study of topics such as

- **Quantity:** ***[Numbers. Arithmetic]***
- **Structure:** ***[Patterns. Algebra]***
- **Space:** ***[Geometry. Topology]***
- **Change:** ***[Analysis. Calculus]***

Aim of the Course

The course Sum-enchanted Evenings will run over ten (10) lectures from 25 September to 4 December.

The aim of the course is to show you

- **The tremendous *beauty* of mathematics;**
- **Its great *utility* in our daily lives;**
- **The *fun* we can have studying maths.**

Restructuring

Last year, I taught a course with the title

AweSums: The Majesty of Maths

It was well received, but the pace was too fast for some of the participants.

So, I have decided to break it into two parts:

- **Sum-enchanted Evenings (Autumn 2017)**
- **AweSums: Outstanding Problems of Maths. (Spring 2018)**

Outline

Names for the Course

- Maths for Everyone
- The Fun of Maths
- Recreational Maths
- Our Mathematical World
- The History and Development of Maths
- Mathematics: Beautiful, Useful & Fun

All of these titles have advantages, but they are generally too specific. So I settled for:

Sum-enchanted Evenings

Notes and Slides

- All the slides will be available online:
  http://mathsci.ucd.ie/~plynch/AweSums
- *No notes* are to be provided.
  Why Not? See next slide.
- Additional Reading Recommendations.
- Optional Exercises and Problems.
- No Assignments!
- No Assessments!
- No Examinations!

Why No Notes?

- Maths is NOT a Spectator Sport
- Active engagement is essential to understanding.
- *You should take your own notes:*
  - This involves repetition of what you hear.
  - This involves repetition of what you see.
  - What you write passes through your mind!
  - This process is a great help to memory.

Lectures

- Classes run from 7pm to 9pm.
- 120 minutes intensive lecturing too long
  (both for you and for me).
- Educational Theory:
  - Attention & retention both decrease with time.
- Class will be broken into short sections.

If you cannot attend a class:

- There is no need to offer reasons.
- Please do not bother to email me.
- The presentation slides will be available.

Typical Structure of a Class

1. Problem: Background and Theory (30 min)
2. *Distraction* (10 min)
3. Some History of the problem (30 min)
4. Another *Distraction* (10 min)
5. Exercises, Puzzles, History (20 min)
6. Questions & Discussion (20 min)

Total duration: about 120 minutes.

I will (normally) be available after classes to answer questions or offer clarifications.

Some Distractions

- Visual Awareness: Maths Everywhere
- Puzzles: E.g. Watermelon Puzzle
- Sieve of Eratosthenes
- The Greek Alphabet
- Lateral Thinking in Maths
- *Lecture sans paroles*
- How Cubic and Quartic Equations were cracked
- Four-colour Theorem
- Online Encyclopedia of Integer Sequences

Please ask me if you have a favorite topic!

It’s Your Course

I expect a group with a wide range of knowledge and “mathematical maturity”.

Everybody should benefit from the course.

If anything is unclear, **SHOUT OUT!** or whisper!

If something is missing, let me know.

Feedback on the course is very welcome.

It's Your Course

Classes begin at 7 pm. and run till 9 pm.

Pi Restaruant (downstairs) closes at 8:00.

There seem to be two options:

- Break at 7:50 for 15 or 20 minutes.
- Don't break at all !!!

I have no strong views but I suspect that there might be a riot if we do not have a break.

Let's have a show of hands: Who wants a break?

Popular Mathematics Books

1. John H Conway and Richard K Guy, 1996:
   The Book of Numbers. Copernicus, New York.
2. ♡ ⇒ John Darbyshire, 2004:
   Prime Obsession. Plume Publishing.
3. ♡ ⇒ William Dunham, 1991:
   Journey through Genius. Penguin Books.
4. Marcus Du Sautoy, 2004:
   The Music of the Primes. Harper Perennial.
5. ♡ ⇒ Richard Elwes, 2010:
   Mathematics 1001. Firefly Books.
6. Peter Lynch, 2016: *That's Maths.*
   Gill Books. Published in October 2016.

Outline

To Begin: An Optical Illusion

A cautionary tale:

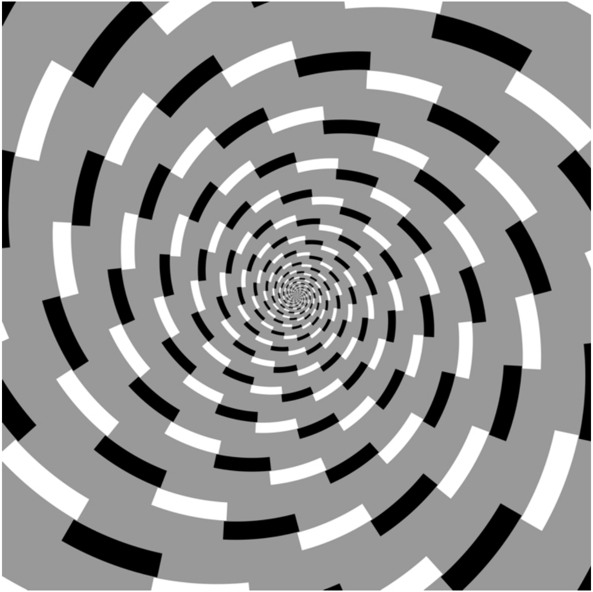
In maths we often use pictures to prove things.

This is usually very helpful.

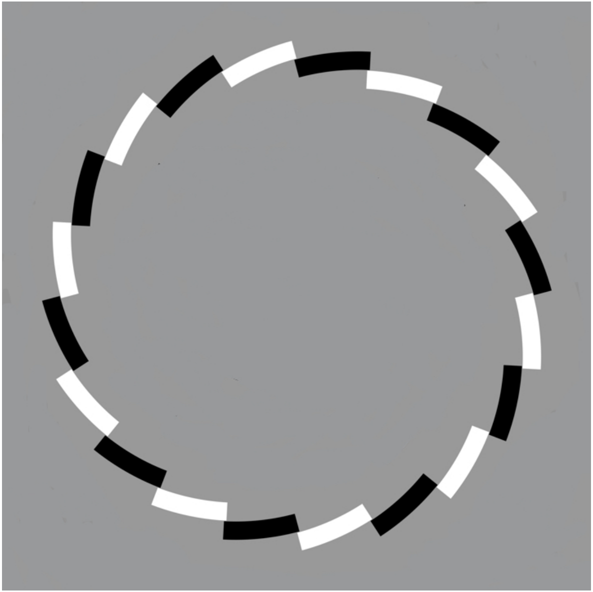
However, it can sometimes mislead us.

Let us look at the Fraser Spiral.

Fraser Spiral Illusion



Fraser Spiral Illusion



Visual Maths Proofs

Can the sum of an infinite number of quantities have a finite value?

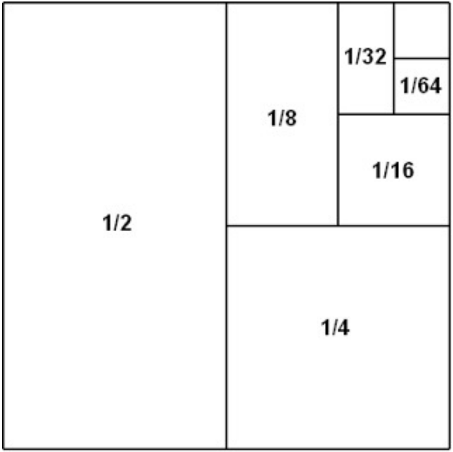
Let's look at the infinite series

$$S = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$$

Each term is half the size of the preceding one.

The terms are getting smaller but it is *not obvious* that the series converges.

A picture makes everything clear:



Unit Square: At each stage, we add *half the remainder* of the square.

Conclusion

The infinite series

$$S = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$$

has a *finite* sum:

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = 1$$

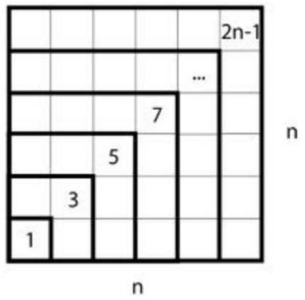
The terms are getting smaller quickly enough for the series to be convergent.

Another Simple Proof

What is the sum of the first n odd numbers?

$$1 = 1^2 \qquad 1 + 3 = 4 = 2^2 \qquad 1 + 3 + 5 = 9 = 3^2$$

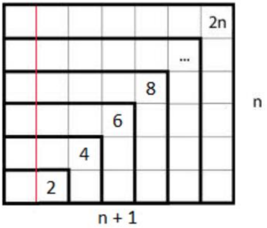
Is this pattern continued? *Can we prove it?*



$$1 + 3 + 5 + 7 + \cdots + (2n - 1) = n^2$$

What is the sum of the first n even numbers?

$$S = 2 + 4 + 6 + 8 + \cdots$$



We just add a column on the left. This increases each term of the sequence of odd numbers by 1.

$$2 + 4 + 6 + \cdots + 2n = n(n+1)$$

Now divide both sides by 2 to get:

$$1 + 2 + 3 + \cdots + n = \frac{1}{2}n(n+1)$$

Outline

Distraction 1: Remember π

To 15-figure accuracy, π is equal to

$$3.14159265358979$$

How can we remember this without much effort?

Just remember this:

How I want a drink,
Alcoholic of course,
After the heavy lectures
involving quantum mechanics.

Distraction 1: Remember π

How I want a drink,
Lemonsoda of course,
After the heavy lectures
involving quantum mechanics.

How I want a drink,
Sugarfree of course,
After the heavy lectures
involving quantum mechanics.

Repeat: To Remember π

To 15-figure accuracy, π is equal to

$$3.14159265358979$$

How can we remember this without much effort?

Just remember this:

How I want a drink,
Alcoholic of course,
After the heavy lectures
involving quantum mechanics.

Distraction 1: Remember $1/\pi$

The reciprocal of π is approximately 0.318310
Can I remember the reciprocal?

How I remember the reciprocal!
3 1 8 3 10

Now you know π and $1/\pi$ to an accuracy
greater than you are ever likely to need!

Outline

The Ancient Origins of Mathematics

Basic social living was possible without numbers

. . . but . . .

elementary comparisons and measures are needed to ensure fairness and avoid conflicts.

The need for mathematical thinking arose in problems like fair division of food.

Problem: How do you divide a woolly mammoth?

Division of Food

To divide a collection of apples, the idea of a *one-to-one correspondence* arose.

There was no direct need for *numbers* yet: the apples did not need to be counted, just broken into batches.

The problem of dividing up a slaughtered animal is more tricky: The forequarters and hindquarters of a woolly mammoth are not the same!

Fair Division: Main Idea

- **Divide a set of goods or resources between several people.**
- **Each person should receive his/her due share.**
- **Each person should be satisfied after the division (this is an *envy-free solution*).**

This problem arises in various real-world settings: auctions, divorce settlements, electronic spectrum and frequency allocation, airport traffic management.

It is an active research area in Mathematics, Economics, Conflict Resolution, and more.

I Cut and You Choose

For two people or two families, the familiar technique "I cut and you choose" could keep everyone happy.

This is the method used by children to divide a cake. It works even for an inhomogeneous cake, say half chocolate and half lemon sponge.

To divide fairly between all members of a family is *much more difficult* (as many of you know!).

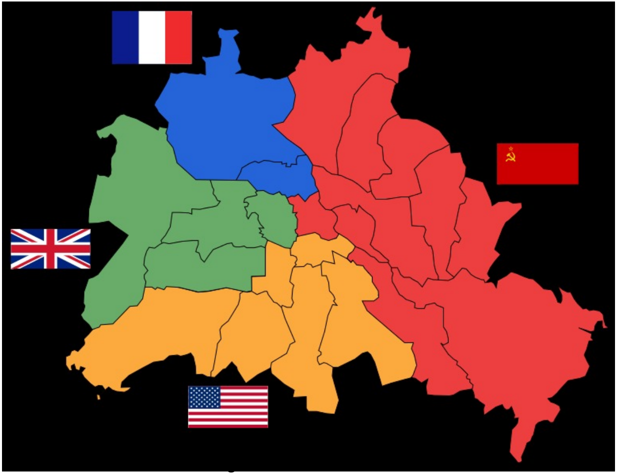
Exercise: Try to devise a generalization of the "cut-and-choose" method that works for three people . . . and one that works for four people.

This is a dfficult problem

It seems like an abstract problem, but it has practical implications:

Consider the partition of Berlin

Partition of Berlin (Potsdam Agreement, 1945)



Partition of Germany (Potsdam Agreement, 1945)

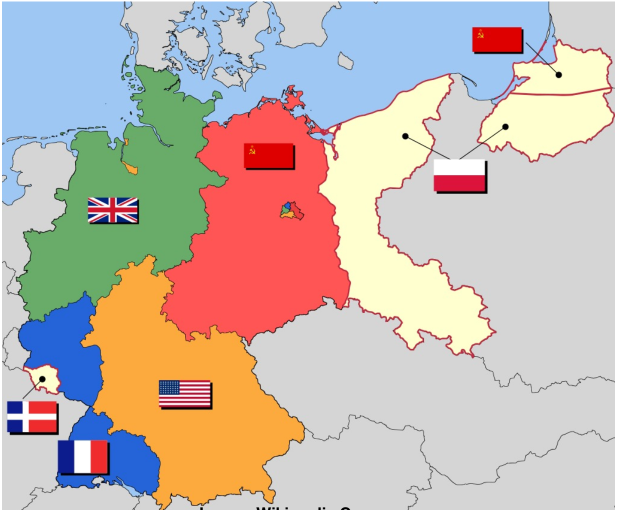
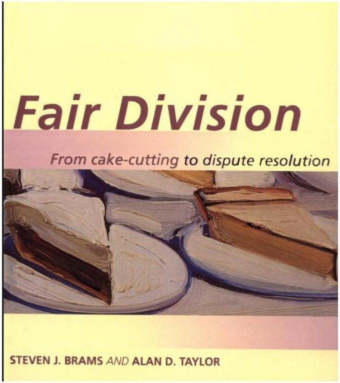
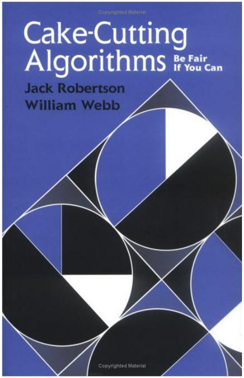


Image: Wikimedia Commons

Books on Fair Division

Two books devoted exclusively to this problem and its variations

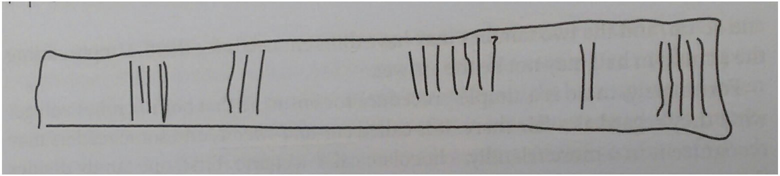




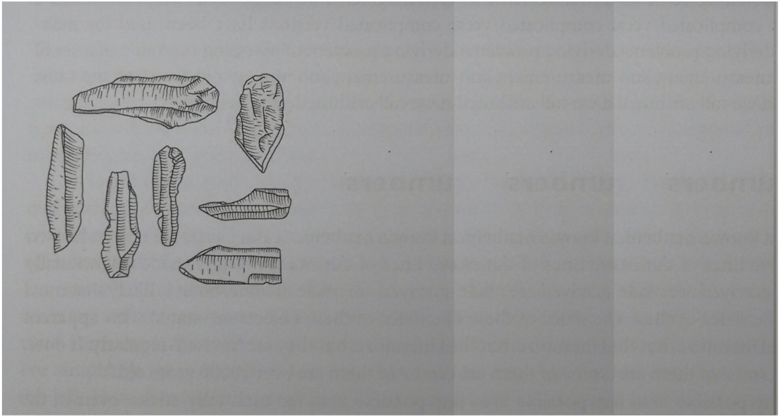
Tally Sticks

Keeping an account of sheep and such animals was done using a tally stick. The number of notches corresponded to the number of sheep.

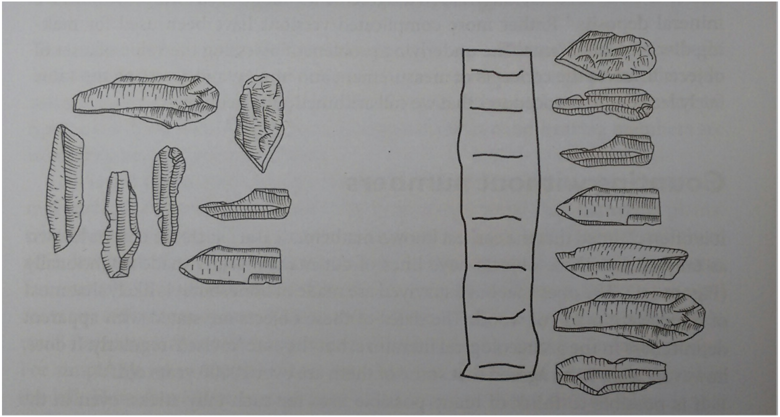
Again, for small flocks, no concept of *actual numbers* was essential.



Keeping Stock without Counting



Keeping Stock without Counting



The origin of the number line ???

Numbers

At some stage, the general notion of a number arose. Even in considering the fingers of a hand, numbers up to five would arise.

Gradually the idea of five as a concept would emerge. Placing two hands together immediately gives us the idea of a one-to-one correspondence:

Both hands have five fingers.

Through repetition and familiarity, the concept of five would become natural. Any set of objects that are in one-to-one correspondence with the fingers of the hand must have five elements.

Numerals

Gradually all the small natural numbers, at least up to about 10, came into use.

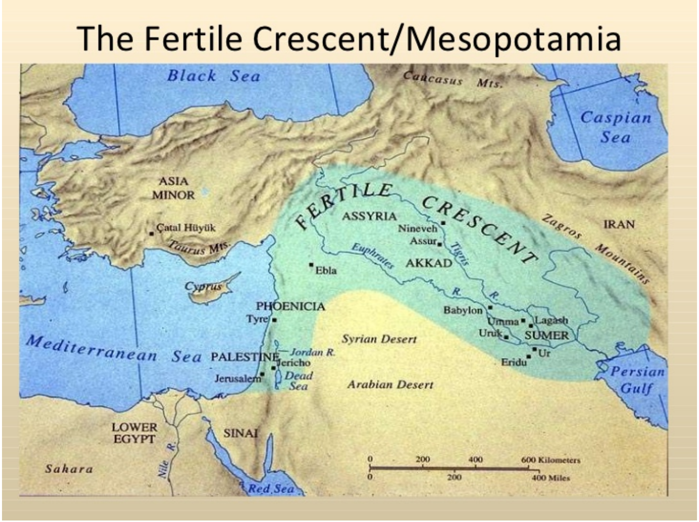
Sometimes, the connection between say two sheep and two bushels of corn was obscured.

Irish has distinct words for two apples and two people

Eventually, numerals, or symbols for the numbers, emerged.

Much numerical material is found in writings from Mesopotamia and from Ancient Egypt.

The Fertile Crescent



Mesopotamia

Loosely called the Babylonian civilisation.

A vast number of cuneiform tablets survive.

WE WILL RETURN TO BABYLON PRESENTLY AND READ A CUNEIFORM TABLET!

Bartering & Money

One group might have surplus *fish*
while another group have excess *fruit*.
Both gain by agreeing to an exchange.

A common measure was needed.
This eventually led to the idea of money.

In several cultures, objects like *cowrie shells*
were used as a medium of exchange.

In some cases, the currency had some inherent
value or at least scarcity. In others, it had not.

Exercise: Discuss the opinion of Aristotle in his
Ethics: "With money we can measure everything."

Outline

Reading a Tablet

On the next slide we will see a cuneiform tablet.
It was discovered in the Sumerian city of Nippur (in modern-day Iraq), and dates to around 1500 BC.

We're not completely sure what this is, but most scholars suspect that it is a *homework exercise.*

It is not preserved perfectly, and dealing with this is part of the challenge (and part of the fun).

If you study the picture closely, you should be able to discover a lot about Babylonian numerals.

The Nippur Tablet

The Nippur Tablet Challenge

1. How do Babylonian numerals work?
2. Describe the maths on this tablet.
3. Write the number 72 in Babylonian numerals.

Does this seem impossible? Have faith in yourself!

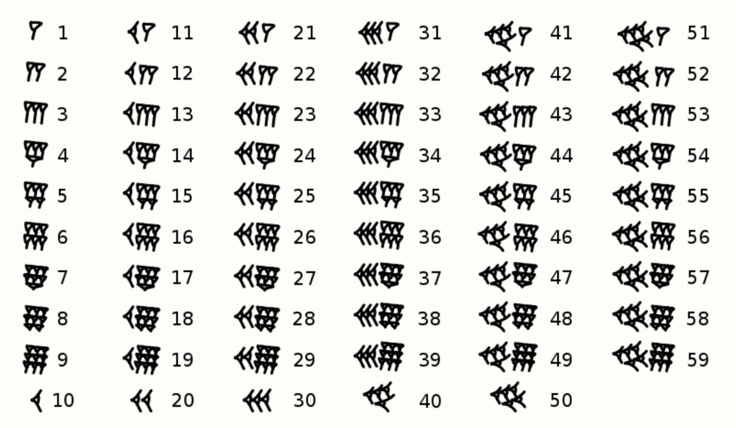
Pause to Decode the Nippur Tablet

The Sexagesimal System

The Babylonian numerical system used 60 as its base. Why?

It is uncertain why, but reasonable to speculate that, since there there are about 360 days in a year 60 was chosen to facilitate astronomical calculations.

The Babylonian Numerals



The Sexagesimal System

The great advantage is that 60 has many divisors:
1, 2, 3, 4, 5, 6, 10, 12,15, 20, 30.

This obviously facilitates all the division problems.

In Babylon, they wrote 70 = [1 | 10] and 254 = [4 | 14]

We can add these: 324 = [5 | 24].

Thus, basic arithmetic is possible with this system.

Time Measurement

Development of accurate *calendars* required mathematical development.

The relationship between days and months and years is not so simple.

Time and season could be measured by the length of shadow cast by a fixed pole.

Eventually this led to the *great obelisks* being erected in Egypt.

Exercise: Find out how high the Spire is. Using public web-cams, could it be used as a time-piece?

Time Measurement

There is a hangover from the sexagesimal system in our 'modern' units:

We have 60 seconds in a minute and 60 minutes in an hour.

We have 360 degrees in a circle so our latitude and longitude are influenced by Babylonian mathematics.

Can you think of any other examples?

Thank you